

Notes on Face, 1983 and before

10/12/82, Ec. 162 A,

Re B & P Ch. 2, "Max's Value of Firm," deals w. max. techniques.

- A. Visualizing + demystifying the rule of Max PV
- B. Totals, avgs., + ineqs.
- C. Rules of differential calc. - some intuitive understanding.

Ranking alt. invts

A. Visualizing + demystifying the Rule of Max. P.V.

1. 4 ways to deal w. functional relationships.

a. Algebra $PV = \sum_{t=0}^n \left[\frac{R_t - C_t}{(1+i)^t} \right]$

b. Tabulate

c. Visualize graphically

d. Use calc. or computer

2. All of these are useful to a more comprehensive, holistic, approach: a bird's-eye view -- or God's-eye view -- of whole system you are running (or are part of).

View each graph or table as a partial view that focuses on specified variables by making them explicit --- and leaves it implicit that other optimizing adjustments are being made. I will show how to make more of them explicit, as we go. You

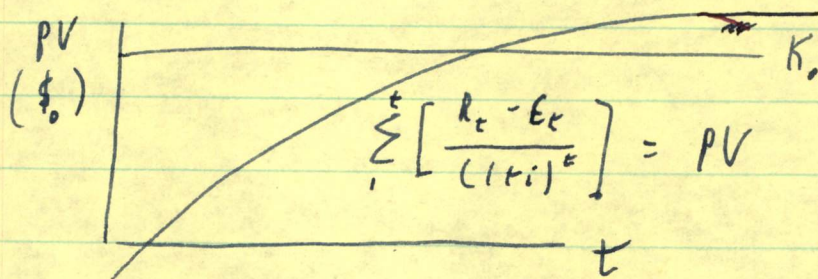
may forget that I have said this, ^{bes. now it's so general.} but I will remind you.

This is one of hardest things to get. We struggle to master one model, then want to rest, cling to it, worship it. It's our Ark in the flood.

Think of it rather as a steppingstone, helps you move on. Don't grab it like a drowning man.

3. Visualize graphically.

- a. Separate out the big initial ~~\$~~ K_0 . Deduct other costs as Expenses, E_t .



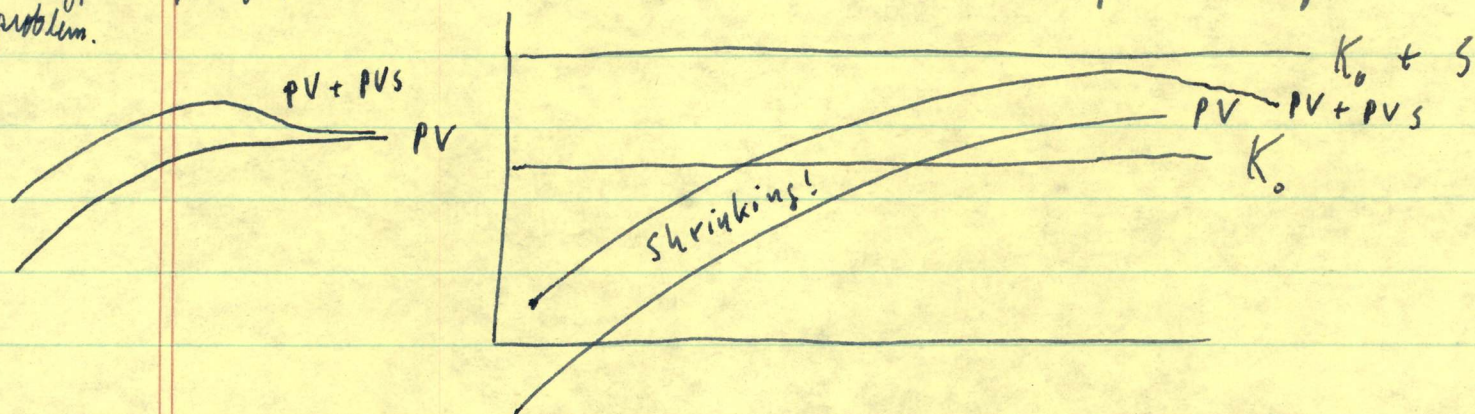
Choose to invest in things that max $PV - K_0$.

But how end life?

b. Find opt. life, L

a typical optimizing problem.

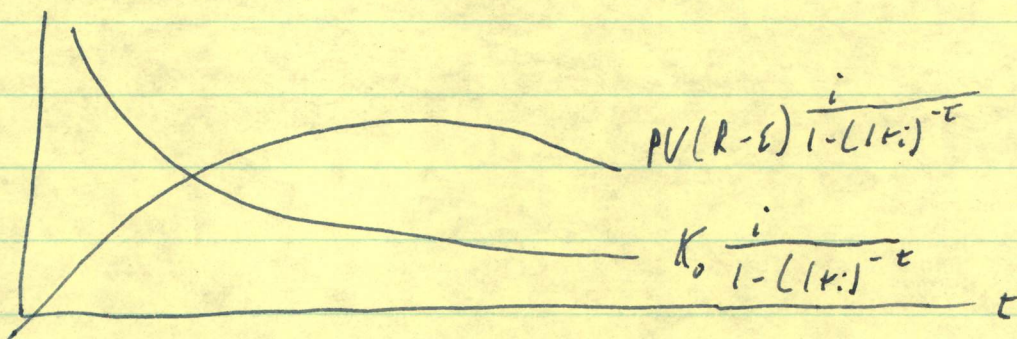
i. Add land value to both input & output



You cut off life of G_1 to release land for G_2 , etc.

"Land" cd, also mean o'head, e.g. a barn w, milk cows.

ii. Alternatively, annualize ("levelize" in Haig-speak)



(C.x.f. not covered in either text, explicitly, is implicit in PV of annuity.)

c. Max. the excess, varying the parameters you control

i. Life. This is explicit, above.

ii. K_0 , This is implicit

- Unit

- Quality & nature of process & product

Go to Disc. MP concept. Integrate w, prodn. ess. (4)

To make the implicit explicit, there are ~~the~~ 4 useful techniques.

1. Add a 3d dim.

Artists & draftsmen love this, but hard to read.

2. Use a code for 3d dim.

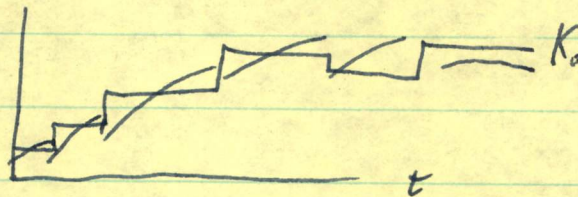
Nos.

Color

Ways of drawing lines.

3. Use a mut. mut. concept.

E.g. for each life there cd. be a diff. opt. K_0



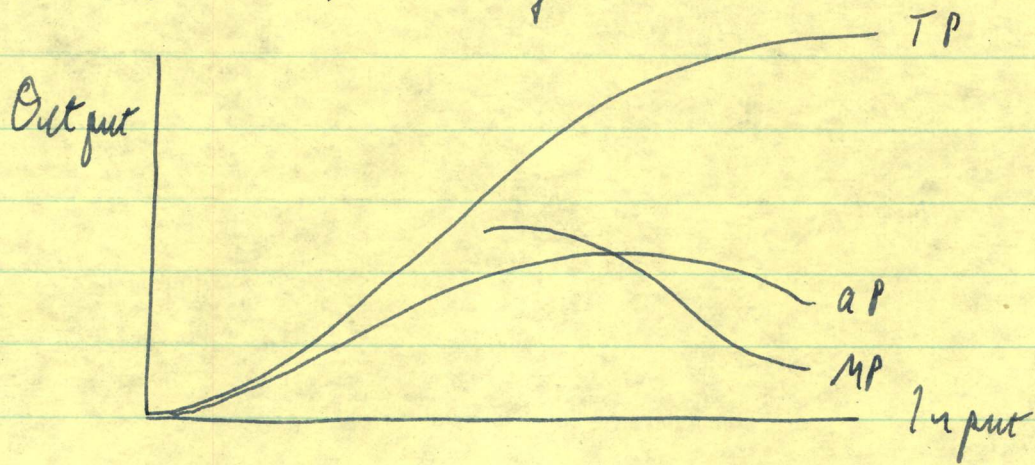
✓ 4. Extend the net product concept.

We are already using it in $(R-E)$.

It is implicit ~~that~~ ~~that~~ that $R-E$ is adjusted optimally.

All these are calc. to extend your powers of comprehension & control — to adjust many related variables together, to opt. a total result.

B. Totals, Aves. & Margs.



Quiz: Find MP from AP

<u>Input</u>	<u>AP</u>	<u>TP</u>	<u>MP</u>
11	30	330	—
12	29	348	18
13	28	364	16
14	27	378	14

1. Slopes & products.
2. Areas rep'g TP from AP & MP
3. MP/AP as elast.

$$\frac{\frac{\Delta P / \Delta X}{P / X}} = \frac{\frac{\Delta P / P}{\Delta X / X}} = \frac{\eta_P \Delta P}{\eta_X \Delta X}$$

C. Rules of diff. calc.

B + P, pp. 41 ff., give you these arbitrarily.
Math. proofs are good, but mainly for a few specialists.

We want intuitive or "heuristic" proofs, so we can be confident the math. specialists know what they are doing & aren't making fools of us. Then we can leave them to split the hairs while we move on to bigger things -- can use these as tools.

1. Power rules.

$$y = x^n$$
$$\frac{dy}{dx} = nx^{(n-1)}$$

a. $y = x^0$

$$y' = \cancel{0} 0$$

a const. doesn't change.

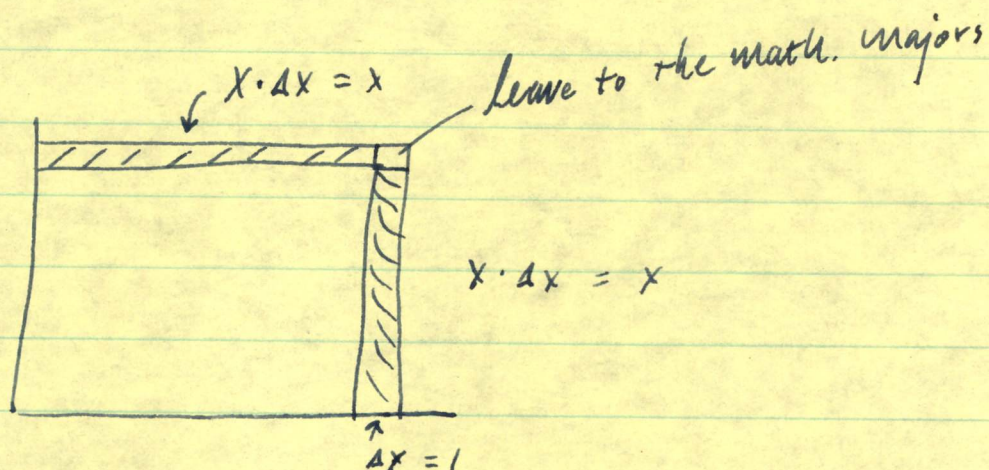
b. $y = x^1$

$$y' = \cancel{1} 1$$
$$dy = dx$$

c.

$$y = x^2$$

$$y' = 2x$$



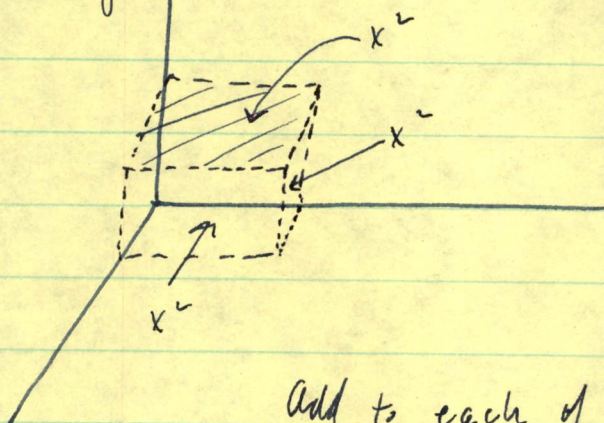
$$\frac{\Delta y}{\Delta x} = 2x. \quad (\text{Actually } 2x + 1. \quad \text{E.g., } 81 - 64 = 8 + 8 + 1$$

$$4 - 1 = 1 + 1 + 1)$$

d.

$$y = x^3$$

$$y' = 3x^2$$



Add to each of 3 faces, add a slab to ea.
Ea. slab is x^2

(Actually, $y' = 3x^2 + 3x + 1$)

LIST OF MAJOR INFERENCES DRAWN BY INSPECTING THE FORMULA
FOR PRESENT VALUE OF AN ANNUITY

Basic formula:
$$P = a \frac{1 - (1 + i)^{-n}}{i} \quad (1)$$

(Know two proofs of that; if you are good at integral calculus, know a third one, too.)

To make that simpler to read, I will make a substitution.

Let $(1 + i) \equiv I$. Now (1) becomes:

$$P = a \frac{1 - I^{-n}}{I - 1} \quad (1)$$

1. Values of P for benchmark values of i and n . Let $a = 1$, to simplify. (This gives you a general factor which you can multiply times any given value of a .)

n	P
0	0
1	$\frac{1}{1 + i}$
∞	$\frac{1}{i}$

If $i = 0$, $P = n$.

- By graphical intuition
- By l'Hôpital's Rule, for calculus whizzes.
- By your calculator. Solve for P as $i \rightarrow 0$, but doesn't quite get there. Watch $P \rightarrow n$.

2. Most of value in P comes from earlier years.

Table 1: Values of P for different n

n	$i = .01$	$i = .08$	$i = .16$
10	9.47	6.71	4.83
20	18.05	9.82	5.93
%			
% of value from years 1-10 =	52%	68%	81%

3. Elasticity of P w.r.t. n

a) An $X\%$ Δn yields ΔP of less than $X\%$. (In math notation, $\frac{\Delta n}{n} > \frac{\Delta P}{P}$;

or $\frac{d(\ln n)}{dn} > \frac{d(\ln P)}{dP} \cdot \frac{dP}{dn}$.)

E.g., in Table 1, at $i = .16$, the 100% Δn causes a $\Delta P/P$ of $(5.93/4.83) - 1 = 23\%$.

3. b) The sensitivity (or elasticity) of P w.r.t. $\underline{n}$ is a decreasing function of $\underline{n}$ (i.e. declines with $\underline{n}$).

In Table 2, $\underline{n}$ increases by doubling, i.e. in steps of 100%. $\Delta P/P$ is thus the elasticity (since $\Delta n/n = 1$).

Table 2: Elasticity (ϵ) of $\underline{P}$ w.r.t. $\underline{n}$

$i = .05$			$i = .10$	
$\underline{n}$	$\underline{P}$	$\underline{\Delta P/P}$	$\underline{P}$	$\underline{\Delta P/P}$
1	.95	---	.91	---
2	1.86	.96	1.74	.91
4	3.55	.91	3.17	.82
8	6.146	.82	5.33	.68
16	10.84	.68	7.82	.47
32	15.80	.46	9.53	.22
64	19.12	.21	9.98	.05
128	19.96	.04	9.99995	.002
∞	20.00	---	10.00000	---

Duplicate Table 2 using your own calculator. Remember the examples of shoes and houses, and Rolls Royces. Think up some more.

4. Share of interest in cash flow.

Let cash flow = \$1/yr. for $\underline{n}$ years.

Share of interest in cash flow is $\frac{n - P}{n}$.

$$\frac{n - P}{n} = 1 - \frac{P}{n} = 1 - \frac{1 - I^{-n}}{ni} \quad (2)$$

(remember: $I \equiv (1 + i)$)

Duplicate the graph used in class to illustrate (2).

Table 3: Share of interest in cash flow

n	$i =$	.05	.10	.15	.20
5		.13	.24	.33	.40
10		.23	.39	.50	.58
20		.38	.57	.69	.76
40		.57	.76	.83	.88
80		.76	.88	.92	.94

An operation is "capital-intensive" when a high share of cash flow goes to feed the banker or investor. That occurs when payments are long-deferred, or strung out in the future. Normally, that occurs when capital is durable, heavy and expensive, like a Rolls Royce (life = 40 or 50 years; $P = \$95,000 + \text{tax}$. Is it worth it?).

5. The increase of P w.r.t. n is a decreasing function of life (n).
 - a) Explain.
 - b) Demonstrate.
6. A manager can control the ratio of capital to labor in his investments, and does, in response to changing wage and interest rates. Explain how.
 What operations are more sensitive to interest rates? To wage rates?
7. Depreciation, interest, and capital recovery for recycling.

A. Depreciation

Normally, "in equilibrium," depreciation = capital recovery.
 "In equilibrium" means market forces always push in this direction--not that they always get there. You as a manager want your capital recovery to exceed depreciation. If you can do so, good for you--but that makes you one of the market forces pushing toward equilibrium.

Duplicate the graphic demonstration in class that depreciation[†] of an asset yielding a level cash flow of \$1 is:

$$\text{Depreciation} \equiv \Delta P = I^{-n} \quad (3)$$

(Your last Reminder: $I \equiv (1 + i)$)

(Remember that depreciation is a negative change in P , expressed as a positive number. This lets us omit the minus sign.)

[†] When you are repaying a debt, the decline in the balance due is called "amortization." The math is the same.

7. A. Depreciation (continued)

Algebraic proof of (3):

$$\begin{aligned}\Delta P &= P_n - P_{n-1} = \frac{1 - I^{-n}}{i} - \frac{1 - I^{-(n-1)}}{i} \\ &= \frac{I^{1-n} - I^{-n}}{i} = \frac{I^{-n}(1+i-1)}{i} = I^{-n}\end{aligned}\tag{4}$$

Calculus proof of (3):

$$\frac{\delta P}{\delta n} = \frac{\delta}{\delta n} \left[\frac{1 - I^{-n}}{i} \right] = - \frac{\ln I \cdot I^{-n}}{i}$$

$$\text{But } \frac{\ln I}{i} \approx 1\tag{5}$$

$$\text{Therefore } \frac{\delta P}{\delta n} \approx I^{-n}\tag{6}$$

The calculus answer is less exact because it is designed for continuous functions, and we are dealing with a step-function, where interest is compounded yearly. Calculus is more accurate when interest is compounded continuously. For most management decisions the difference is negligible--you have bigger worries.

B. Interest

Knowing depreciation, we also know interest, because cash flow is divided between the two. Interest is the complement of depreciation:

$$\text{Interest} = 1 - I^{-n}\tag{7}$$

(7) also comes directly from (1), by solving for P_i :

$$P_i = 1 - I^{-n} \quad (a = 1)\tag{1.A}$$

7. B. Interest (continued)

This is how banks tell you each year how much of what you paid on your loan was interest (which is deductible from taxable income) and how much was repayment of principle (which is not).

In the early years of a long loan, repayment is shockingly small. E.g. at 11.5%, with a loan amortized over 40 years, what % of the debt is repaid in year one?

$$-\frac{\Delta P}{P} = \frac{\frac{I^{-n}}{1 - I^{-n}}}{\frac{i}{I^n - 1}} = \frac{i}{I^n - 1} = \frac{.115}{76.8} = .0015 \quad (8)$$

On a debt of \$100,000, you've worked it down by \$150. Pretty slow going! You've paid them \$11,650; all but \$150 was interest.

C. Capital Recovery for Recycling

But look at it from the banker's viewpoint. He has only recovered \$150 of his loan. The rest is committed to you--you're using his money. Assuming that the interest rate is fair, and the bank has to pay to attract money, and interest in general is either just or inevitable, then he cannot treat the interest portion as capital recovery. It is income, and in law, custom and logic it has a prior claim on cash flow. Recovery is just what's left over after paying interest.

Why should you as a manager worry about the banker's viewpoint? You might be managing a bank, but otherwise, every manager supervises investments. That's what it's all about. You're throwing down nickels and picking up dimes--later, and you hope not much later. Throwing down those nickels is the motive force that drives our economy, and any economy, so it's important socially, too.

Your investment in real capital is like the banker's investment in a loan to you. He must recover the principal before he can relend it; you must recover your capital before you can recycle it by reinvesting it. That recycled capital is most of what managers have to work with. It is "the ready." It is what lets you "get there firstest with the mostest" (in Stonewall Jackson's egregious English) and seize prizes before equilibrium sets in. It is what keeps your organization going without infusion of outside capital, and without any new saving. It's what lets you achieve high volume without having a lot of assets. It is your working capital. Treasure it, nurture it, recycle it, it's what keeps you alive.

7. C. Capital Recovery for Recycling (continued)

But if you bury it in assets with 40-year recovery periods, you won't remobilize most of it for a long time; half your life, perhaps.

You might think you recover $1/40$ of it, or $2-1/2\%$, each year, which would be slow enough. That is "straight-line" depreciation. But we saw in (8) you recover only $.15\%$, which is 6% of $2-1/2\%$.

8. Discounted Cash Flow (DCF) and marginal analysis.

DCF is the bottom line after forecasting revenues and then:

- a) Revenue - Expenses = Cash Flow
- b) Cash Flow - Interest = DCF

DCF is the most you can pay a contractor to build a thing, turnkey basis.

Marginal DCF is the discounted marginal product from adding inputs at time 0.

... of Capital Theory for use in planning.
by M. Geoffrey, 6/27

1. a) Compound interest, growth of money at a rate of interest.

n = number of years from deposit of money in bank or other investment

i = interest rate per year

F_n = Future value, after lapse of n years

P_0 = Present value (or Principal)

n	F_n
0	1
1	$1 + i$
2	$(1+i) + i(1+i) = (1+i)(1+i) = (1+i)^2$
3	$(1+i)^2 + i(1+i)^2 = (1+i)^2(1+i) = (1+i)^3$
...	You see the repeating pattern
n	$(1+i)^{n-1} + i(1+i)^{n-1} = (1+i)^{n-1}(1+i) = (1+i)^n$

Rule: $F_n = P_0 (1+i)^n$ (1)

That is "The Compound Interest Formula."

- b) Doubling Time.

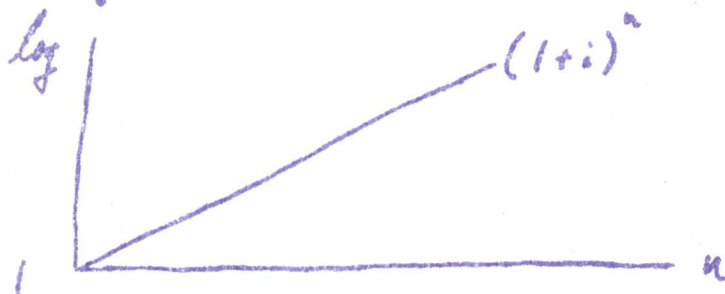
Rule: Doubling time = $\frac{.72}{i}$

(This is derived by setting $(1+i)^n = 2$ and solving for n .)

$$n = \frac{\ln 2}{\ln(1+i)} \approx \frac{.72}{i} \quad (2)$$

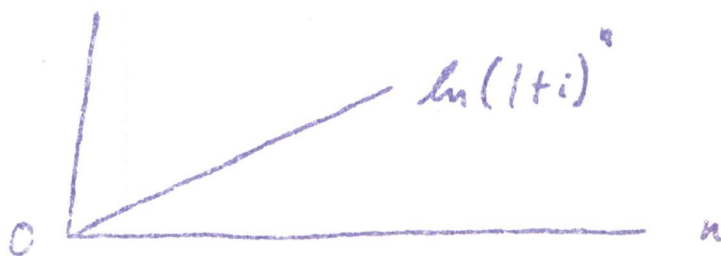
c) Semi-log scale

On a semi-log scale, a constant % growth rate is a straight line.



(Alternatively, on a common scale, the log of a constant growth rate is a straight line, as below:

$$\ln[(1+i)^n] = n \ln(1+i) \approx ni$$



We will use semi-log scale here.)

On either scale, the slope of the curve = i

2. Discounting, the opposite of compounding.

From (1)

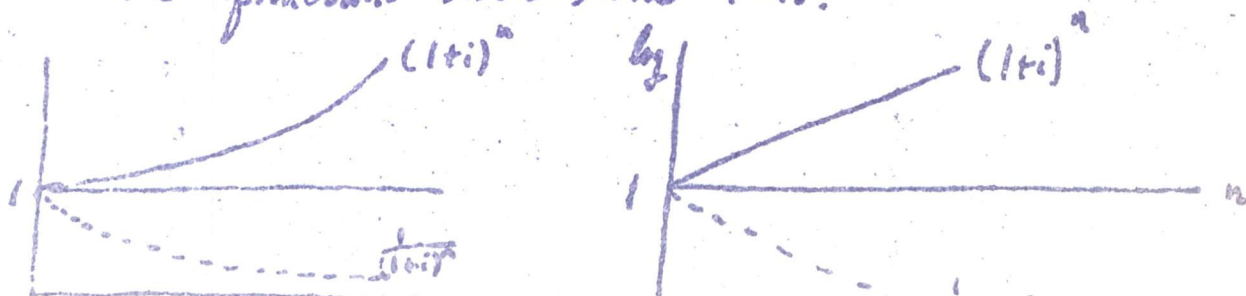
$$F_n = P_0 (1+i)^n$$

$$\therefore P_0 = \frac{F_n}{(1+i)^n} \quad (3)$$

Dividing by the compound interest expression is called "discounting". It gives you the present value which would grow, at i compounded, to equal F_n after lapse of n years.

By compounding and discounting, $(1+i)^n$ is the rate of exchange between points in time. It makes values commensurable, ~~and~~ when they have different dates.

The functions look like this:



The compounding curve shows "exponential growth"; the discount curve shows "exponential decay".

P_0 in (3) is also called PV, for Present Value. A sum of PVs from a flow of future sums is DCF, for Discounted Cash Flow.

3. Capitalization

A dollar in the bank yields $\$i$ per year, and $\$100$ yields $\$100 \cdot i$ --- at 5%, that is $\$5$.

Therefore, an income of $\$5$ per year is worth $\$100$.

Generally, let a = annual income.

$$a = P_0 \cdot i \quad (4)$$

$$P_0 = \frac{a}{i} \quad (5)$$

(5) is called "capitalizing". It converts a flow of cash into a fund of capital.

(5) applies only to a perpetual flow. For finite flows we modify it, as below.

4. Capitalizing for n years: the PV formula

a) Simple derivation

If a cash flow ends after lapse of n years, the PV is a/i less the discounted value of a/i after n years:

$$PV = \frac{a}{i} - \frac{a}{i} (1+i)^{-n} \quad (6)$$

(Note: $(1+i)^{-n}$ is shorthand for $\frac{1}{(1+i)^n}$)

That is, you get the perpetuity for n years, then lose the rest of it after that.

Simplifying,

$$PV = \frac{a}{i} [1 - (1+i)^{-n}] = a \frac{1 - (1+i)^{-n}}{i} \quad (6a)$$

(6a) capitalizes an annuity for n years. It is called the PV formula. Memorize it. Everything else derives from it.

b) Derivation as sum of progression - an alternative derivation (You may skip this if rushed, but it adds to your understanding.)

i) PV is the sum of a ~~cash~~ series of payments each one discounted one more year. Let the first

payment be after lapse of one year, ^{and $a \neq 1$,} then:

$$PV = (1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-n} \quad (7)$$

(7) is a geometric progression, whose first term is $(1+i)^{-1}$ and whose ratio is $(1+i)^{-1}$.

ii) Sum of series formula, derivation.

$$S = 1 + r + r^2 + \dots + r^{n-1}$$

$$rS = r + r^2 + \dots + r^{n-1} + r^n$$

$$\therefore S - rS = 1 - r^n$$

$$S = \frac{1 - r^n}{1 - r} \quad (8)$$

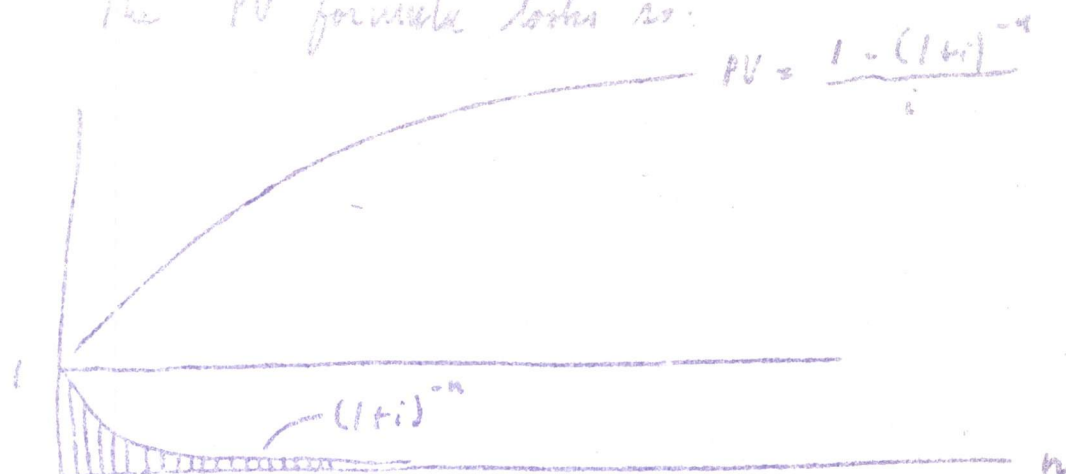
iii) Applying (8) to (7)

$$r = (1+i)^{-1}$$

$$S = (1+i)^{-1} \frac{1 - (1+i)^{-n}}{1 - (1+i)^{-1}} = \frac{1 - (1+i)^{-n}}{1+i - 1}$$

$$= \frac{1 - (1+i)^{-n}}{i} \quad (6b)$$

The PV formula looks so:



The PV curve is the sum of the shaded areas under the $(1+i)^{-n}$ curve, for any value of n .

PV is the cumulative total; $(1+i)^{-n}$ is the marginal increment as n grows.

d) Interest and depreciation (preliminary)

If you have an asset for which you paid \$ PV how do you divide its cash flow between interest and depreciation? (6b) gives our easy answer, for any year, Interest is $PV \cdot i$. (It has to be, to justify tying up the capital, and/or to meet interest on debt.)

Box:

$$PV \cdot i = 1 - (1+i)^{-n}$$

(6.2)

recall that cash flow = 1

$\therefore$ depreciation is $(1+i)^{-n}$, the complement.

Note how the depreciation path follows the two curves backwards on p. 7. Do you understand? Think about it.

5. A simpler notation for future use.

It can be shown that:

$$(1+i)^n \cong e^{in} \quad (9)$$

where $e = 2.718281828\dots$, the base of natural logs. This technically implies that interest is computed "continuously," but we are not here concerned with this aspect. We are merely concerned with a notation easier to write, and read; also to acquaint you with its use.

(The right side of (9) is actually a trifle higher than the left. To make them absolutely equal, a smaller rate of interest would be required on the right, called the "force of interest.")

For those interested in math. proof:

$$e \equiv \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Let $\frac{f}{i} = n$, where f is frequency of conversion per year (e.g. if interest is compounded monthly, $f=12$)

$$\lim_{f \rightarrow \infty} \left(1 + \frac{i}{f}\right)^{f \cdot \frac{f}{i}} = \lim_{f \rightarrow \infty} \left(1 + \frac{i}{f}\right)^{\frac{f}{i} \cdot i} = e^i$$

6. Capital Recovery Factor, CRF.

With our new notation,

$$PV = a \frac{1 - e^{-in}}{i} \quad (6, d)$$

Now suppose we know the PV, and want to find the corresponding annuity for n years. Solve (6, d) for a :

$$a = PV \frac{i}{1 - e^{-in}} \quad (10)$$

The fraction is the "CRF". It is also called "left service"; the instalment plan payment; and

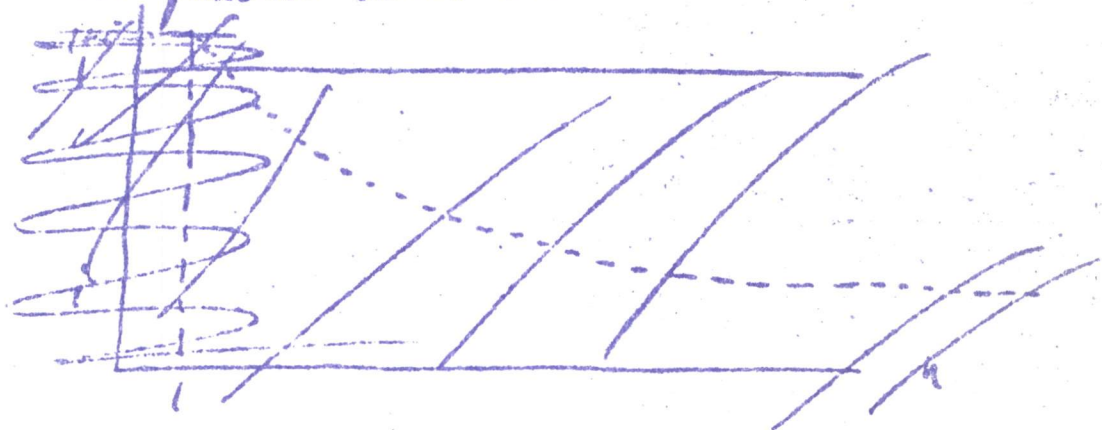
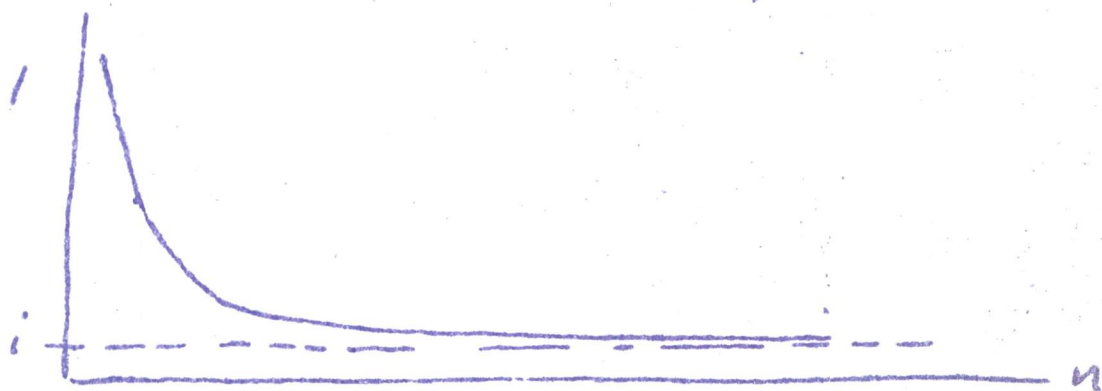
"the annuity where $PV = 1$."

It converts an occasional lump sum capital outlay into an equivalent annual amount, comparable to other current outlays.

It varies from a value of $(1+i)$ when $n=1$, down to i , when $n=\infty$.

For values of $n > 30$ and $i > .08$ it is so close to i that the difference is trifling.

The function looks so:



Notes:

a) Making something last longer has been called "spreading the overhead through time". Note that this

ceases to help much after a few years.

b) For a given income stream, whether and how fast you recover principal depends mainly on the rate of interest.

E.g. at ± 50 years, if your cash flow is 8.2% of principal, you recover P in 50 years at an 8% rate of interest. But if the cash flow is 8.0% of P , you recover no P at all, even. It all goes for interest.

So when investors say they are able to recoup their investments, it may be double talk. They may be figuring 10% interest off the top, leaving nothing for amortization. Be skeptical and analytical. Get the facts, and feed the formula.

7. Linking fund formula, and "Temporal mean" values.

$$\text{If } a = PV \frac{i}{1 - e^{-in}}$$

$$\text{and } FV = PV e^{in}$$

then

$$a = \frac{FV}{e^{in}} \frac{i}{1 - e^{-in}} = FV \frac{i}{e^{in} - 1} \quad (11)$$

Thus a tree to be harvested n years in the future is worth the same as an annuity of a , from (11).

Since e^{in} doubles every 10 years at 7.2%, a tree to mature in 80 years isn't worth much per year:

$$a = R_{80} \frac{.072}{2^8 - 1} = R_{80} \frac{.072}{256 - 1}$$

$$= R_{80} \cdot \cancel{.00117} \cdot .000282$$

If one acre would yield \$1000. after 80 years, then

is no better than sheep whose owner will rent the land for ~~44¢~~ 28¢ per acre per year.

$80 \times \frac{28¢}{14¢} = \frac{\$22.51}{\cancel{\$14.00}}$. The rent is compound interest.

The mean annual increment of the trees is $\$1,000/80 = \12.50 a year. But the true annual rent is only ~~44¢~~ 28¢

The sinking fund formula is a new kind of average, a "temporal mean". If $i = 0$, it is a simple mean.

$\lim_{i \rightarrow 0} \frac{i}{e^{in} - 1} = \frac{0}{0}$. This is not indeterminate, by l'Hospital's Rule, but is:

$$\frac{\frac{di}{di}}{n e^{in} - 0} = \frac{1}{n} \quad (12)$$

Thus, (11) becomes a simple mean when $i = 0$. For positive i , (11) is less than $\frac{1}{n}$.

In planning investments of variable life, a is something to maximize. I.e., select the life that maximizes a.

Since the denominator keeps growing with time, the numerator must grow faster to justify extending life.

This formula, slightly modified, is the way to plan the lives of all repeating investments.

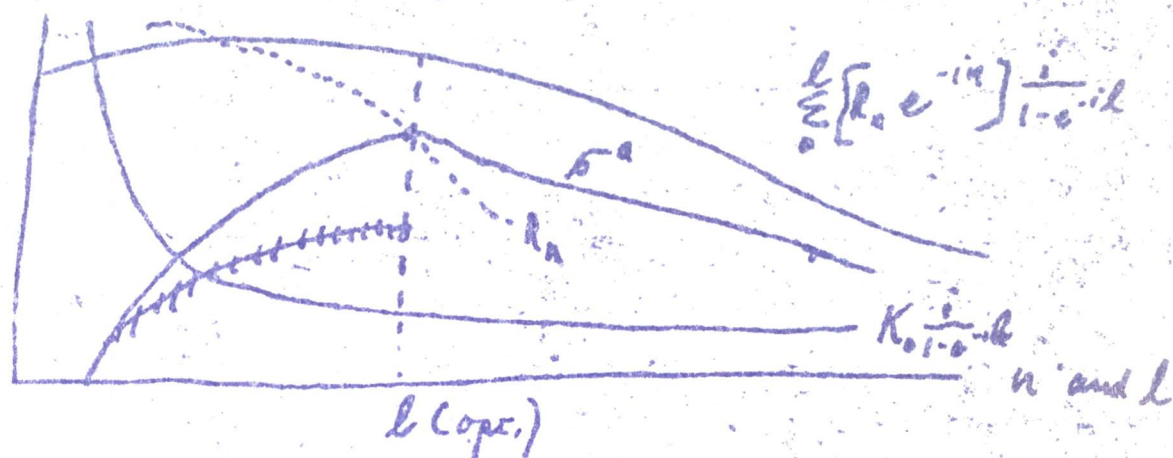
Not just trees. Take each year's cash flow, and compound it forward to L, the year of end of life.

$$a = \frac{\sum_0^L [R_n e^{i(L-n)}]}{e^{iL} - 1} \quad (13)$$

Alternatively, divide through by e^{iL} (top and bottom), discounting all to time 0 and apply CRF:

$$a = \frac{\sum_0^L [R_n e^{-in}]}{1 - e^{-iL}} \quad (14)$$

Graphically, a typical pattern is:



Lower line shows spreading capital cost over time.
Upper line shows "temporal mean" revenue.

As a marginal solution, since the pattern repeats, no marginal year is good if it yields less than $\underline{a}$. You may see the wisdom of letting R_n fall below $\underline{a}$ in two ways:

- a) If $R_n < \underline{a}$, it will lower $\underline{a}$
- b) You can start a new cycle any time, whose "temporal mean" net yield is $\underline{a}$. There is no need to accept any year whose yield is lower (unless it leads to and is an integral part of a new cycle whose overall yield $\geq \underline{a}$)

Returning to (a), p.15. how do we know that $R_n < a$ with lower a ?

Basically, because $R_n = a$ will leave a unchanged. When you squeeze out more years service and income from an old, nearly depreciated building or machine, that increases two things at once: i) The sum of DCF; and ii) The number of years over which the sum is spread to get the "temporal mean." In finding a , the temporal mean, these two increases offset each other; and if $R_n = a$, they exactly offset each other.

The simplest case is when R is constant for all n .

$$\text{Then, } \sum_{n=0}^{\infty} [R_n e^{-in}] = R \frac{1 - e^{-iL}}{i}$$

$$\therefore a = \frac{R \frac{1 - e^{-iL}}{i}}{\frac{i}{1 - e^{-iL}}} = R \quad (15)$$

To hold a constant, we go on adding $R = a$.

If we could hold R constant in perpetuity, we need never replace. Normally, of course, R deteriorates down

Returning to (b), the opportunity cost of holding land in an old use is the annual value of the land in the succeeding use. The "annual value of land" is seldom anything as simple as a fixed yearly payment, because to use land you invest capital to improve it, and then recover the capital plus interest. There is a cycle of investment and liquidation, which one must evaluate as a whole and smooth out. This is what our formulas do.

The way that decision-makers more commonly perceive this principle is in terms of making a return on the value of their investment.

Since a is from a repeating cycle, you may receive it in perpetuity. You may therefore find the DCF or lump-sum equivalent by capitalizing in perpetuity. (See p. 4, #3). Its value, S , is:

$$S = \frac{a}{i}$$

and $\therefore$

$$a = S \cdot i$$

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a is the cash flow required to earn i on S .

So if $R_n < a$, you are not earning a return on the value of your investment in land.

8. What to do when cycles recur but do not repeat.

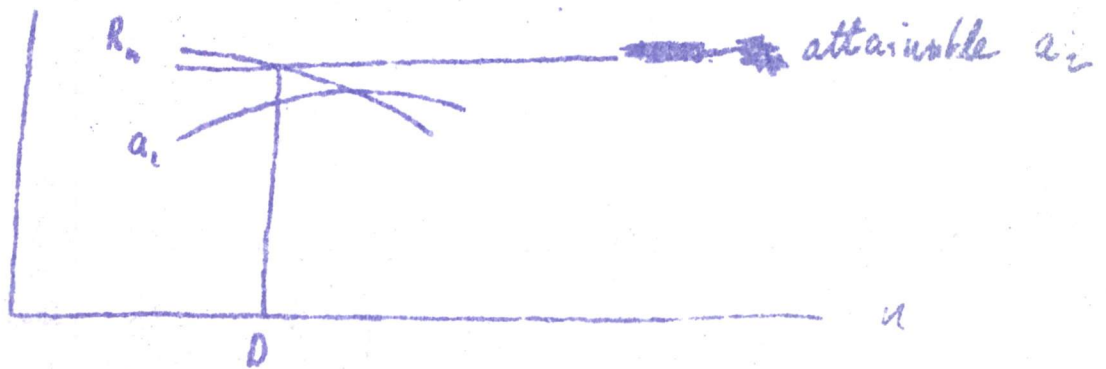
Commonly, we expect progressive improvement in techniques and knowledge and materials and skills, so future real costs are lower. Equally or more important, we expect increasing demand for renewable resources (lands) and their products. In some cases the higher demand further lowers costs by allowing for economies of scale.

The simple rule of maximizing a holds good when a) cycles repeat and b) future cycles are the same as the first. When future cycles are different, there are two other rules to follow. Both of them are also good when future cycles are identical — i.e., they are good in all circumstances. And they are mutually consistent. (Indeed, a mathematical wizard would regard them as two ways of saying the same thing.)

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when ~~any~~ ^{any} $R_n^{\dagger}$ of G_1 falls below a_2 (of G_2).

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How important this is depends on the length of generations we are dealing with. Trees are about the longest cycle going, and even in forestry it bears some weight. Getting down to hogs, cattle, chickens, Christmas Trees, nursing stock, and other common items of growing capital, it bears much weight. With commercial buildings of short life it is weighty. With rapidly evolving technology like computers, the succeeding generations come very fast. Likewise with fabrics, and military hardware. You may supply your own additional examples.

To determine a_2 , we must look ahead to a_3 , a_4 , a_5 , etc. We may often decide that the generations come so fast, we will choose to skip one in order to reduce capital outlays. Or we may skip in a skip, fast one — an interior one of land like a fast-food franchise or drive-in theatre.

To make this choice, we must plan several generations ahead. This leads us into:

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Site value, S , now equals the sum of a series. Let:

PV = Present value of G_1

g = a % growth factor by which each PV surpasses the prior one.

$$S = PV \left[1 + \frac{1}{1+i} e^{g-i} + e^{2(g-i)} + \dots + e^{n(g-i)} \right]$$

$$S = PV \frac{1}{1 - e^g - iL}$$

(from (8), p. 6) (17)

The effect of g in the exponent in the denominator

is:

- a) To raise S
- b) To shorten the value of L (life) which maximizes S .

(a) is algebraically obvious.

(b) follows because g raises the % growth rate of the denominator, thus causing S to top out when the % growth rate of PV is higher.

This is evident if g is nearly as high as i . Then when $L = 1$, the denominator $\rightarrow 0$, as $S \rightarrow \infty$. This is a hard act to follow when $L = 2$, requiring a large rise of PV to keep S growing.

9 How much do succeeding generations add to value?

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of renewal and reinvestment. How much do they change things?

Let's see first what they add to site value.

$$S = PV \frac{1}{1 - e^{-il}}$$

The share of S derived from generations after the first one is:

$$\frac{S - PV}{S} = 1 - \frac{PV}{S} = 1 - (1 - e^{-il}) = e^{-il} \quad (18)$$

$$\text{If } l = 1, e^{-il} = \frac{1}{e^i} \approx \frac{1}{1+i}$$

$$\text{If } i = .05, \frac{1}{1+i} = .95.$$

This is a 95% share.

For longer lives, the share falls. When $il = .71$, the share is $\frac{1}{2}$; and so on.

Now recall that an alternative way of defining

the replacement date is when $R_n \leq S \cdot i$. (pp. 17-18).

If future generations add 50% or more to S , then obviously they have a large impact on the date of optimal replacement. You should not ignore them.

What the math is telling us is that by replacing one year earlier, you are also advancing all later generations by one year. If each generation is short, there are many of them coming in the next, say, 50 years, so the cumulative value is material.

Note that (18), p. 25, assumes $g = 0$. If $g > 0$, repeating the same steps shows that:

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10. Application to growing capital, where the capital is also the product (like cattle, trees, maturing cheese and liquors, poultry, inventories.)

Inventories fit the pattern because they add value while being held, and occupy storage space. "Growth" need not be physical.

Minerals in the ground do not fit the pattern, because the site is not normally renewable. (However, we will see later a lot of resemblance between virgin timber and mines, and combine the mine and timber models to handle old-growth harvesting.)

Growing capital is a simple model called P/PD (point-input, point-output). All intermediate revenues and costs can all be moved to the beginning by discounting, so if we get a good model for this case we can easily make it universal.

From (1), p. 12, the annual value of a revenue in year n , R_n , is:

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From (13), p.14, capital costs in year 0 are compounded forward to n and subtracted:

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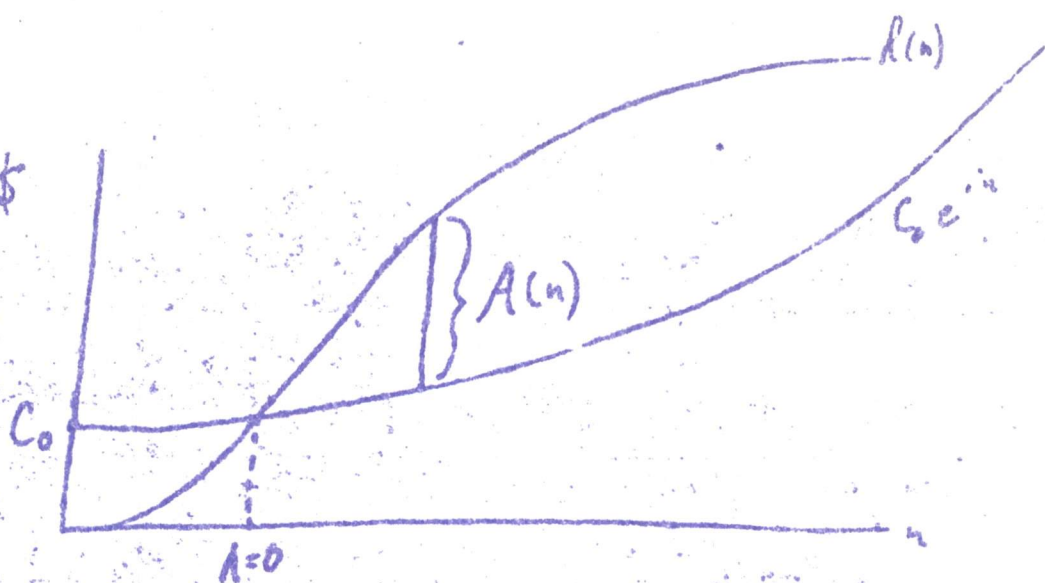
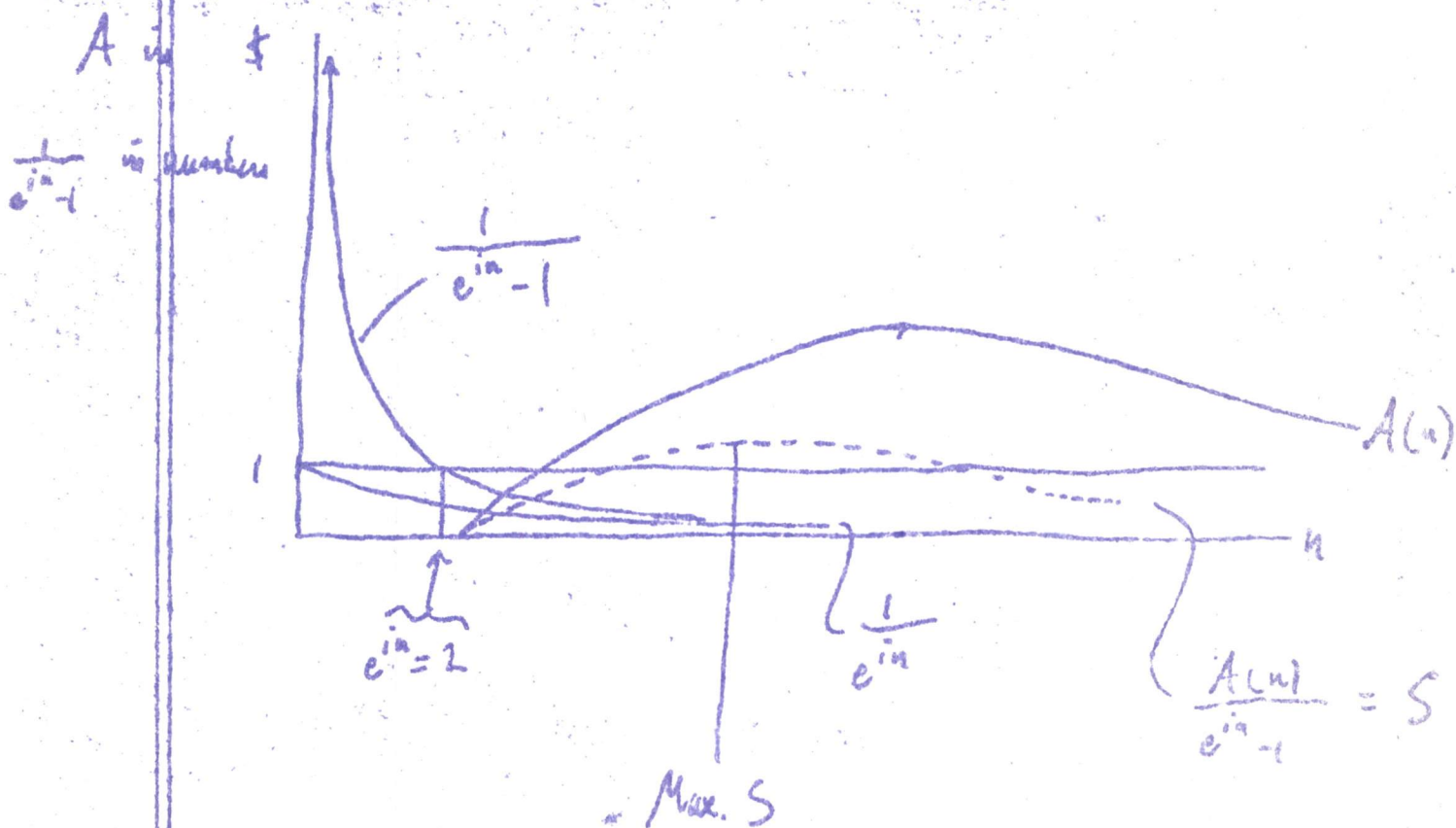
Optimal life is the value of n that maximizes a . This is the same as maximizing a/i (you divide both sides by i : this is capitalizing in perpetuity.)

It helps to visualize this graphically. First, visualize it algebraically. Finding optimal n is an optimizing process, which means two contrary forces are at work as you let n increase, and the solution is to find the best balance.

As n rises, both the numerator and denominator of (20) rise, the first tending to raise a , the second to lower a . We will graph these two forces separately.

For simpler notation, let:

$$R_n - C_0 e^{in} \equiv A(n) \quad (21)$$



The payoff is finding max. S , as above.

As a marginal condition, this maximum is found where annual growth, R' , equals interest on R plus S .

To show this, we find dS/dn and set it equal to zero.

$$\frac{dS}{dn} = \frac{(e^{in} - 1)(R' - Cie^{in}) - (R - Cie^{in})(ie^{in})}{(e^{in} - 1)^2} = 0 \quad (22)$$

$$R'(1 - e^{-in}) - \cancel{Cie^{in}} + Ci = Ri - \cancel{Cie^{in}}$$

$$R' = (R - C) \frac{i}{1 - e^{-in}} = Ri - Ri + \frac{Ri}{1 - e^{-in}} - \frac{Ci}{1 - e^{-in}}$$

$$= Ri + R \left[\frac{i - i + ie^{-in}}{1 - e^{-in}} \right] - \frac{Ci}{1 - e^{-in}}$$

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Q.E.D.

Thus, we max. S by letting the capital grow as long as it yields i on $(R + S)$, or better.

In the same simultaneous process, we determine the value of $\underline{S}$.

If there are intermediate revenues or costs, we handle them as in (13), p. 14. Use the Σ sign. This gets a little lengthy, but can all be done with 3 or 4 columns on one sheet of tabulating paper, and even faster on a programmable calculator, using the key marked "sum", which cumulates in a memory.

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- a. Mines. Optimal life; optimal time to begin.
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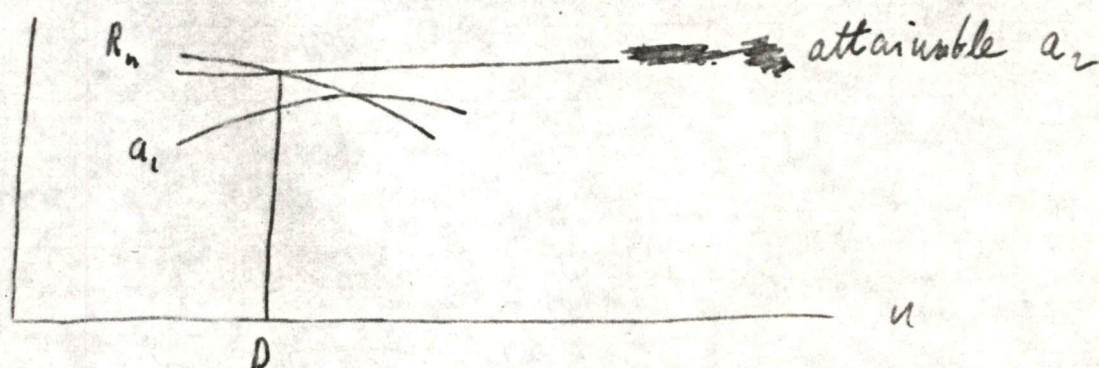
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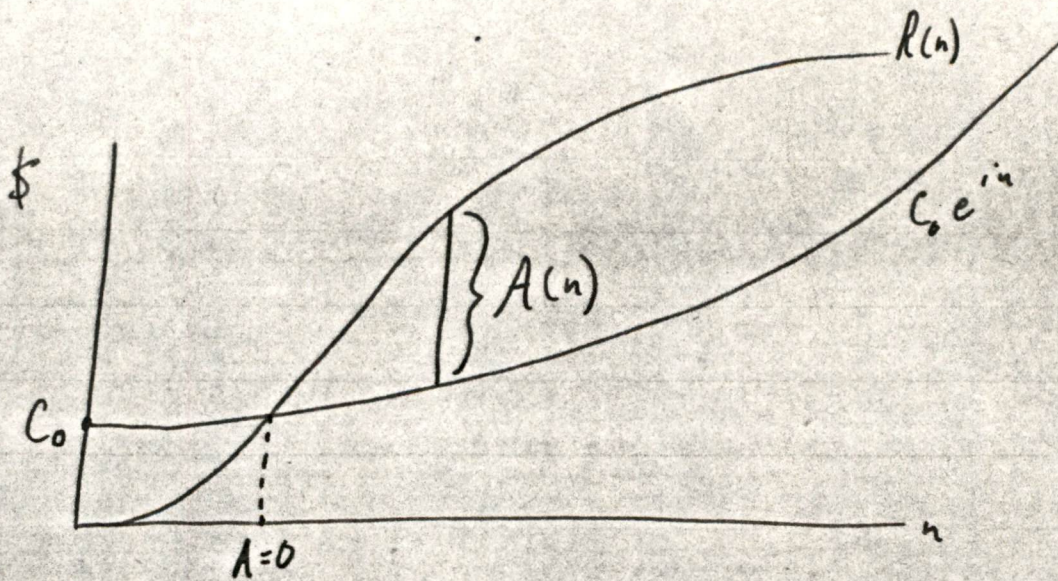
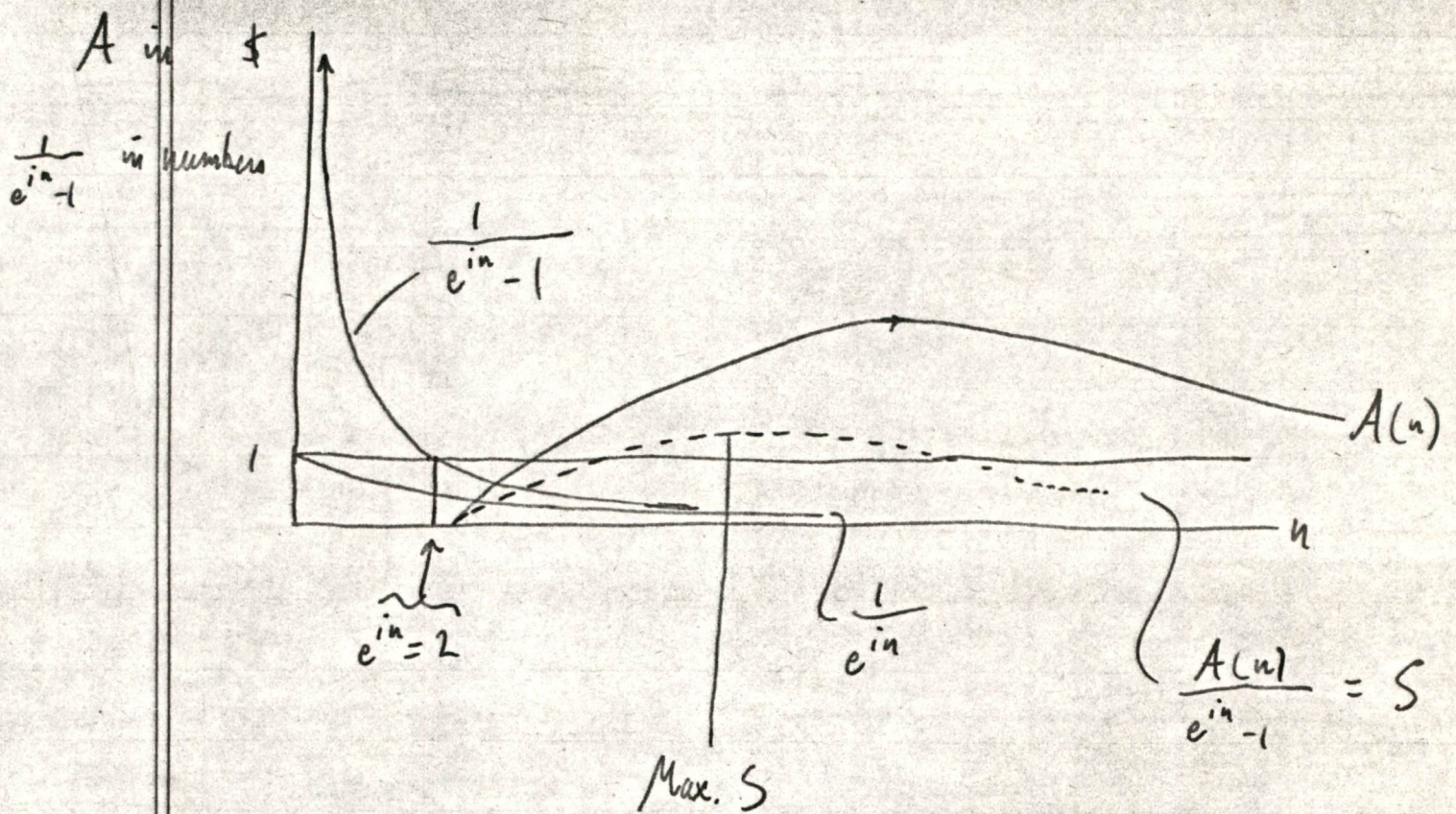
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pp. 12-29 are a super job, article on replacement tripping.
Some elements of capital theory for use in planning.
by M. Gaffney, 6/77

1. a) Compound interest, growth of money at a rate of interest.

n = number of years from deposit of money in bank, or other investment

i = interest rate per year

F_n = Future value, after lapse of n years

P_0 = Present value (or Principal)

n	F_n
0	1
1	$1 + i$
2	$(1+i) + i(1+i) = (1+i)(1+i) = (1+i)^2$
3	$(1+i)^2 + i(1+i)^2 = (1+i)^2(1+i) = (1+i)^3$
...	You see the repeating pattern
n	$(1+i)^{n-1} + i(1+i)^{n-1} = (1+i)^{n-1}(1+i) = (1+i)^n$

Rule: $F_n = P_0 (1+i)^n$ (1)

That is "The Compound Interest Formula."

- b) Doubling time.

Rule: Doubling time = $\frac{.72}{i}$

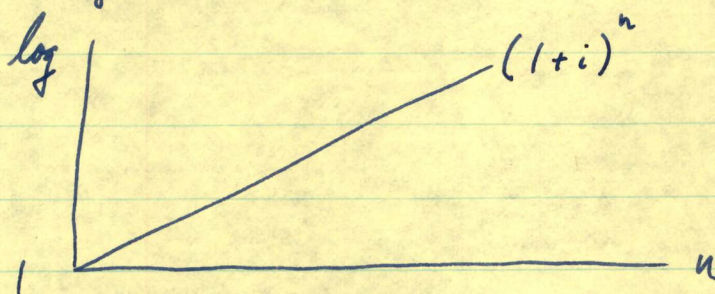
(This is derived by setting $(1+i)^n = 2$ and solving for n .)

$$n = \frac{\ln 2}{\ln(1+i)} \approx \frac{.69}{i} = \frac{.69}{i} \cdot \frac{i}{\ln(1+i)} \quad (2)$$

$$\approx \frac{.69(1.04)}{i} = \frac{.72}{i}$$

c) Semi-log scale

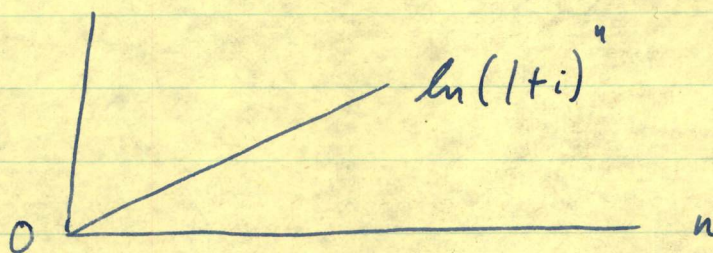
On a semi-log scale, a constant % growth rate is a straight line.



(Alternatively, on a common scale, the log of a constant growth rate is a straight line, as below:

$$\ln[(1+i)^n] = n \ln(1+i) \approx ni \frac{\ln(1+i)}{i} = \delta ni \approx ni$$

$$\text{where } \delta \equiv \frac{\ln(1+i)}{i} \approx 1$$



We will use semi-log scale here.)

On either scale, the slope of the curve = i . (Deriv. needed)

2. Discounting, the opposite of compounding.

From (1)

$$F_n = P_0 (1+i)^n$$

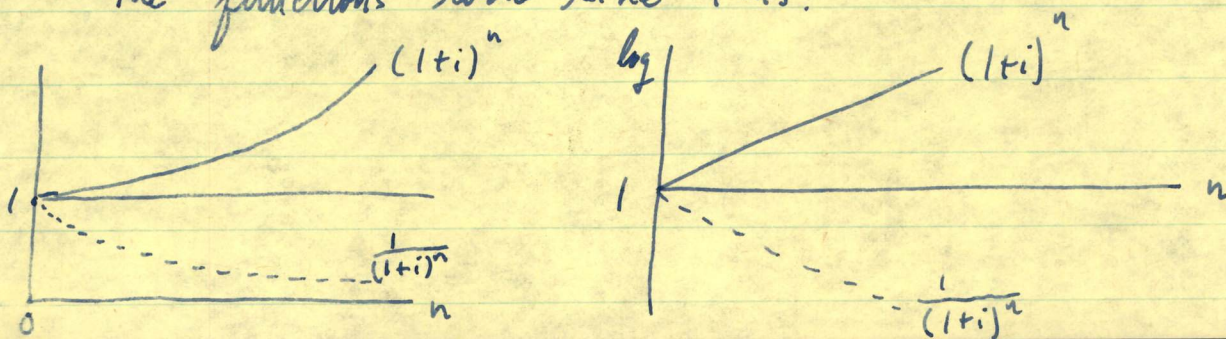
$\therefore$

$$P_0 = \frac{F_n}{(1+i)^n} \quad (3)$$

Dividing by the compound interest expression is called "discounting". It gives you the present value which would grow, at i compounded, to equal F_n after lapse of n years.

By compounding and discounting, $(1+i)^n$ is the rate of exchange between points in time. It makes values commensurable, ~~and~~ when they have different dates.

The functions look like this:



The compounding curve shows "exponential growth"; the discount curve shows "exponential decay".

P_0 in (3) is also called PV, for Present Value. A sum of PVs from a flow of future sums is DCI, for Discounted Cash Flow.

3. Capitalization

a dollar in the bank yields $\$i$ per year, and $\$100$ yields $\$100 \cdot i$ --- at 5%, that is $\$5$.

Therefore, an income of $\$5$ per year is worth $\$100$.

Generally, let a = annual income.

$$a = P_0 \cdot i \quad (4)$$

$$P_0 = \frac{a}{i} \quad (5)$$

(5) is called "capitalizing". It converts a flow of cash into a fund of capital.

(5) applies only to a perpetual flow. For finite flows, we modify it, as below.

4. Capitalizing for n years: the PV formula

a) Simple derivation

If a cash flow ends after lapse of n years, the PV is a/i less the discounted value of a/i after n years:

$$PV = \frac{a}{i} - \frac{a}{i} (1+i)^{-n} \quad (6)$$

(Note: $(1+i)^{-n}$ is shorthand for $\frac{1}{(1+i)^n}$)

That is, you get the perpetuity for n years, then lose the rest of it after that.

Simplifying,

$$PV = \frac{a}{i} [1 - (1+i)^{-n}] = a \frac{1 - (1+i)^{-n}}{i} \quad (6a)$$

(6a) capitalizes an annuity for n years. It is called the PV formula. Memorize it. Everything else derives from it.

b) Derivation as sum of progression - an alternative derivation. (You may skip this if rushed, but it adds to your understanding.)

i) PV is the sum of a ~~cash~~ series of payments each one discounted one more year. Let the first

payment be after lapse of one year, ^{and $a=1$,} then:

$$PV = (1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-n} \quad (7)$$

(7) is a geometric progression, whose first term is $(1+i)^{-1}$ and whose rate is $(1+i)^{-1}$

ii) Sum of series formula, derivation

$$S = 1 + r + r^2 + \dots + r^{n-1}$$

$$rS = r + r^2 + \dots + r^{n-1} + r^n$$

$$\therefore S - rS = 1 - r^n$$

$$S = \frac{1 - r^n}{1 - r} \quad (8)$$

iii) Applying (8) to (7)

$$r = (1+i)^{-1}$$

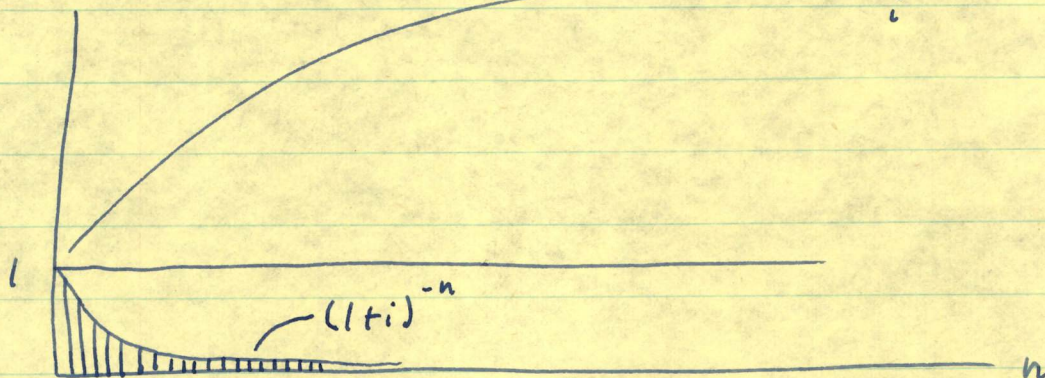
$$S = (1+i)^{-1} \frac{1 - (1+i)^{-n}}{1 - (1+i)^{-1}} = \frac{1 - (1+i)^{-n}}{1+i - 1}$$

$$= \frac{1 - (1+i)^{-n}}{i} \quad (6b)$$

c) Graphics

The PV formula looks so:

$$PV = \frac{1 - (1+i)^{-n}}{i}$$



The PV curve is the sum of the shaded areas under the $(1+i)^{-n}$ curve, for any value of n .

PV is the cumulative total; $(1+i)^{-n}$ is the marginal increment as n grows.

d) Interest and depreciation (preliminary)

If you have an asset for which you paid \$ PV, how do you divide its cash flow between interest and depreciation? (6b) gives an easy answer, for any year, Interest is $PV \cdot i$. (It has to be, to justify tying up the capital, and/or to meet interest on debt.)

But:

$$PV \cdot i = 1 - (1+i)^{-n}$$

(6.c)

Recall that cash flow = 1

$\therefore$ depreciation is $(1+i)^{-n}$, the complement.

Note how the depreciation path follows the two curves backwards on p. 7. Do you understand? Think about it.

5. A simpler notation for future use.

It can be shown that:

$$(1+i)^n \cong e^{in} \quad (9)$$

where $e = 2.718281828 \dots$, the base of natural logs. This technically implies that interest is computed "continuously," but we are not here concerned with this aspect. We are merely concerned with a notation easier to write and read; also to acquaint you with its use.

(The right side of (9) is actually a trifle higher than the left. To make them absolutely equal, a smaller rate of interest would be required on the right, called the "force of interest").

For those interested in math. proof:

$$e \equiv \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m$$

Let $\frac{f}{i} = m$, where f is frequency of

conversion per year (e.g. if interest is compounded monthly, $f=12$)

$$\lim_{f \rightarrow \infty} \left(1 + \frac{i}{f}\right)^{f \cdot t} = \lim_{f \rightarrow \infty} \left(1 + \frac{i}{f}\right)^{\frac{f}{i} \cdot i \cdot t} = e^{i \cdot t}$$

6. Capital Recovery Factor, CRF.

With our new notation,

$$PV = a \frac{1 - e^{-in}}{i} \quad (6, d)$$

Now suppose we know the PV, and want to find the corresponding annuity for n years. Solve (6, d) for a :

$$a = PV \frac{i}{1 - e^{-in}} \quad (10)$$

The fraction is the "CRF". It is also called "debt service"; the instalment plan payment; and

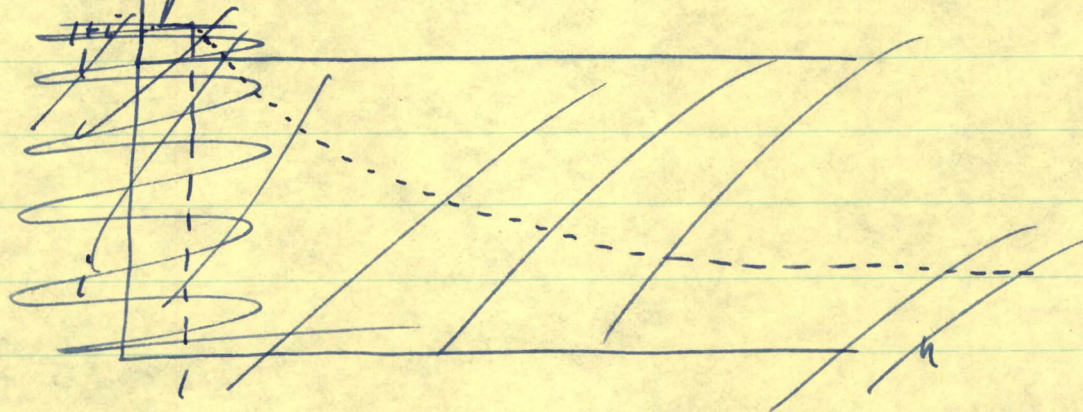
"the annuity whose $PV = 1$."

It converts an occasional lump sum capital outlay into an equivalent annual amount, comparable to other current outlays.

It varies from a value of $(1+i)$ when $n=1$, down to i , when $n=\infty$.

For values of $n > 30$ and $i > .08$ it is so close to i that the difference is trifling.

The function looks so:



Morals: a) Making something last longer has been called "spreading the overhead through time". Note that this

ceases to help much after a few years.

b) For a given income stream, whether and how fast you recover principal depends mainly on the rate of interest.

E.g. at ~~£~~ 50 years, if your cash flow is 8.2% of principal, you recover P in 50 years at an 8% rate of interest. But if the cash flow is 8.0% of P , you recover no P at all, even. It all goes for interest.

So when investors say they aren't able to recoup their investments, it may be double talk. They may be figuring 10% interest off the top, leaving nothing for amortization. Be skeptical and analytical. Get the facts, and feed the formula.

7. Linking fund formula, and "Temporal mean" values.

$$\text{If } a = PV \frac{i}{1 - e^{-in}}$$

and $FV = PV e^{in}$

then

$$a = \frac{FV}{e^{in}} \frac{i}{1 - e^{-in}} = FV \frac{i}{e^{in} - 1} \quad (11)$$

Thus a tree to be harvested n years in the future is worth the same as an annuity of a , from (11).

Since e^{in} doubles every 10 years at 7.2%, a tree to mature in 80 years is worth much per year:

$$a = R_{80} \frac{.072}{2^8 - 1} = R_{80} \frac{.072}{256 - 1}$$

$$= R_{80} \times \cancel{.000141} ,000282$$

If one acre would yield \$1000. after 80 years, that

is no better than sheep whose owner will rent the land for $\frac{44¢}{28¢}$ per acre per year.

$$80 \times \frac{28¢}{\cancel{44¢}} = \frac{\$22.54}{\cancel{\$11.20}}$$

interest.

The rent is compound

The mean annual increment of the trees is $\$1,000/80 = \12.50 a year. But the true annual rent is only $\cancel{44¢}$ 28¢

The sinking fund formula is a new kind of average, a "temporal mean". If $i = 0$, it is a simple mean.

$$\lim_{i \rightarrow 0} \frac{i}{e^{in} - 1} = \frac{0}{0} \quad \text{This is not}$$

indeterminate, by l'Hospital's Rule, but is:

$$\frac{di/di}{ne^{in} - 0} = \frac{1}{n} \quad (12)$$

Thus, (11) becomes a simple mean when $i = 0$. For positive i , (11) is less than $\frac{1}{n}$.

In planning investments of variable life, a is something to maximize. i.e., select the life that maximizes a.

Since the denominator keeps growing with time, the numerator must grow faster to justify extending life.

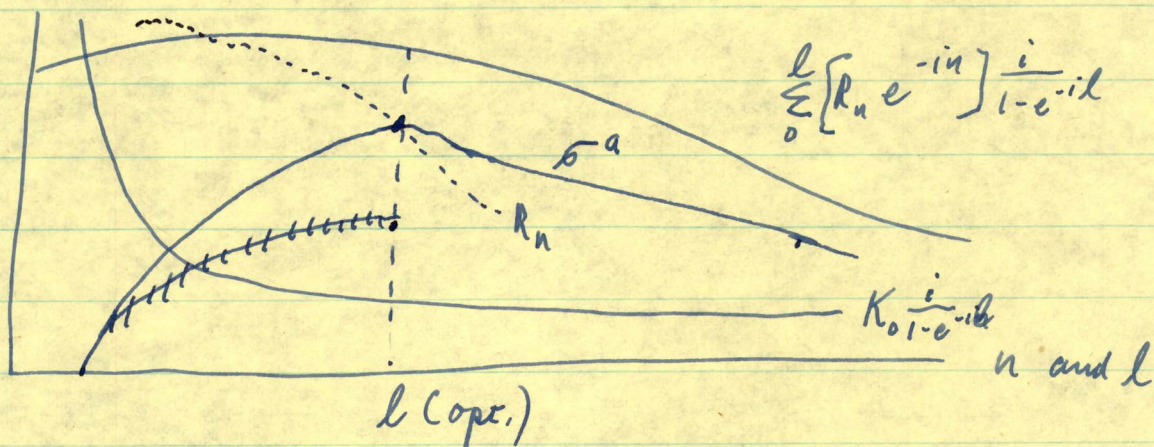
This formula, slightly modified, is the way to plan the lives of all repeating investments. Not just trees. Take each year's cash flow, and compound it forward to L, the year of end of life.

$$a = \sum_0^L [R_n e^{i(L-n)}] \frac{i}{e^{iL} - 1} \quad (13)$$

Alternatively, divide through by e^{iL} (top and bottom), discounting all to time 0 and apply C.F.:

$$a = \sum_0^L [R_n e^{-in}] \frac{i}{1 - e^{-iL}} \quad (14)$$

Graphically, a typical pattern is:



Lower line shows spreading capital cost over time.
Upper line shows "temporal mean" revenues.

As a marginal solution, since the pattern repeats, no marginal year is good if it yields less than a. You may see the wrongness of letting R_n fall below a in two ways:

- a) If $R_n < a$, it will lower a
- b) You can start a new cycle any time, whose "temporal mean" net yield is a. There is no need to accept any year whose yield is lower (unless it leads to and is an integral part of a new cycle whose overall yield $\geq \underline{a}$)

Returning to (a), p. 15, how do we know that $R_n < a$ will lower a?

Basically, because $R_n = a$ will leave a unchanged. When you squeeze one more year's service and income from an old, nearly depreciated building or machine, that increases two things at once: i) The sum of DCF; and ii) The number of years over which the sum is spread to get the "temporal mean." In finding a, the temporal mean, these two increases offset each other; and if $R_n = a$, they exactly offset each other.

The simplest case is when R is constant for all n.

$$\text{Then, } \sum_0^L [R_n e^{-in}] = R \frac{1 - e^{-iL}}{i}$$

$$\therefore a = \frac{R \frac{1 - e^{-iL}}{i}}{\frac{1 - e^{-iL}}{i}} = R \quad (15)$$

To hold a constant, we go on adding $R = a$.

If we could hold R constant in perpetuity, we need never replace. Normally, of course, R dwindles down (just as the days do in "September Song.")

Returning to (b), the opportunity cost of holding land in an old use is the annual value of the land in the succeeding use. The "annual value of land" is seldom anything as simple as a fixed yearly payment, because to use land you invest capital to improve it, and then recover the capital plus interest. There is a cycle of investment and liquidation, which one must evaluate as a whole and smooth out. This is what our formulas do.

The way that decision-makers more commonly perceive this principle is in terms of making a return on the value of their investment.

Since a is from a repeating cycle, you may receive it in perpetuity. You may therefore find the DC7 or lump-sum equivalent by capitalizing in perpetuity. (See p. 4, #3). Site value, S , is:

$$S = \frac{a}{i}$$

and $\therefore$

$$a = S \cdot i$$

(16)

a is the cash flow required to earn i on S .

So if $R_n < a$, you are not earning a return on the value of your investment in land.

8. What to do when cycles recur but do not repeat.

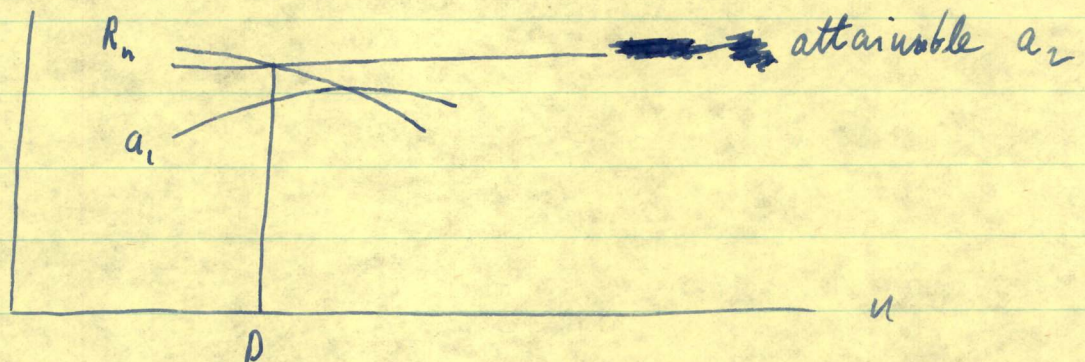
Commonly, we expect progressive improvement in techniques and knowledge and materials and skills, so future real costs are lower. Equally or more important, we expect increasing demand for renewable resources (lands) and their products. In some cases the higher demand further lowers costs by allowing for economies of scale.

The simple rule of maximizing a holds good when a) Cycles repeat and b) Future cycles are the same as the first. When future cycles are different, there are two other rules to follow. Both of them are also good when future cycles are identical — i.e., they are good in all circumstances. And they are mutually consistent. (Indeed, a mathematical wizard would regard them as two ways of saying the same thing.)

Rule A: End first cycle (G_1 , for Generation #1)
when ~~any~~ ^{any} $R_n^\#$ of G_1 falls below a_2 (of G_2).

Do this even tho a_1 is still rising.

Rule A is a marginal rule, and especially useful in the short run. Graphically:



D is D-date, the date to terminate G_1 for G_2 .

The difficulty with Rule A is that of determining "attainable a_2 ". The difficulty is that a_2 will also have to be foreshortened, most likely because of the coming of a_3 from G_3 ; and so on.

This means any R_n after G_1 is over the hump of its early construction phase of negative cash flow.

How important this is depends on the length of generations we are dealing with. Trees are about the longest cycle going, and even in forestry it bears some weight. Getting down to hogs, cattle, chickens, Christmas trees, nursing stock, and other common items of growing capital, it bears much weight. With commercial buildings of short life it is weighty. With rapidly evolving technology like computers, the succeeding generations come very fast. Likewise with fabrics, and military hardware. You may supply your own additional examples.

To determine a_2 , we must look ahead to a_3 , a_4 , a_5 , etc. We may often decide that the generations come so fast, we will choose to skip one in order to reduce capital outlays. Or we may skip in a cheap, fast one — an interim use of land like a fast-food franchise or drive-in theatre.

To make this choice, we must plan several generations ahead. This leads us into:

Rule B: Maximize the present value of the site in perpetuity.

It is an overwhelming thought that, to make perfect choices today, we need to look ahead forever. We know we cannot really do so, in any detail. Fortunately, discounting the future also means that remote future facts bear only lightly on current choices.

What we need then are good forecasts of the next few years, plus a general feel for the more remote future, plus accurate mathematical tools for ~~some~~ a) Telling us what kinds of things we have to forecast, and b) Translating them into numbers for current decisions.

The simplest rule, to get a feel for the matter, is derived by assuming that each succeeding generation outperforms G_1 by a constant percentage.

Site value, S , now equals the sum of a series. Let:

PV = Present value of ~~the~~ G_1

g = a % growth factor by which each PV surpasses the prior one.

$$S = PV \left[1 + \cancel{PV} e^{g-il} + e^{2(g-il)} + \dots + e^{\infty(g-il)} \right]$$

$$S = PV \frac{1}{1 - e^{g - il}} \quad (\text{from (8), p. 6}) \quad (17)$$

The effect of g in the exponent in the denominator

- is:
- a) To raise S
 - b) To shorten the value of l (life) which maximizes S .

(a) is algebraically obvious.

(b) follows because g raises the % growth rate of the denominator, thus causing S to top out when the % growth rate of PV is higher.

This is evident if g is nearly as high as i . Then when $\underline{l} = 1$, the denominator $\rightarrow 0$, so $S \rightarrow \infty$. This is a hard act to follow when $l = 2$, requiring a large rise of PV to keep S growing.

9 How much do succeeding generations add to value?

It is tempting to choose the simplest policy to understand, and therefore to ignore future cycles

of renewal and reinvestment. How much do they change things?

Let's see first what they add to site value.

$$S = PV \frac{1}{1 - e^{-il}}$$

The share of S derived from generations after the first one is:

$$\frac{S - PV}{S} = 1 - \frac{PV}{S} = 1 - (1 - e^{-il}) = e^{-il} \quad (18)$$

$$\text{If } l = 1, \quad e^{-il} = \frac{1}{e^i} \approx \frac{1}{1+i}$$

$$\text{If } i = .05, \quad \frac{1}{1+i} = .95.$$

This is a 95% share.

For longer lives, the share falls. When $il = .72$, the share is $\frac{1}{2}$; and so on.

Now recall that an alternative way of defining

the replacement date is when $P_n \leq S \cdot i$. (pp. 17-18).

If future generations add 50% or more to S , then obviously they have a large impact on the date of optimal replacement. You should not ignore them.

What the math is telling us is that by replacing one-year earlier, you are also advancing all later generations by one year. If each generation is short, there are many of them coming in the next, say, 50 years, so the cumulative value is material.

Note that (18), p. 25, assumes $g = 0$. If $g > 0$, repeating the same steps shows that:

$$\frac{S - PV}{S} = e^{g - iL} \quad (19)$$

a larger, and perhaps much larger figure.

10. Application to growing capital, where the capital is also the product (like cattle, trees, maturing cheese and liquors, poultry, inventories.)

Inventories fit the pattern because they add value while being held, and occupy storage space. "Growth" need not be physical.

Minerals in the ground do not fit the pattern, because the site is not normally renewable. (However, we will see later a lot of resemblance between virgin timber and mines, and combine the mine and timber models to handle old-growth harvesting.)

Growing capital is a simple model called PIPD (point-input, point-output). Intermediate revenues and costs can all be moved to the beginning by discounting, so if we get a good model for this case we can easily make it universal.

From (11), p. 12, the annual value of a revenue in year n , R_n , is:

$$a = R_n \frac{i}{e^{in} - 1} \quad (11)$$

From (13), p. 14, capital costs in year 0 are compounded forward to $\underline{n}$ and subtracted:

$$a = (R_n - C_0 e^{in}) \frac{i}{e^{in} - 1} \quad (20)$$

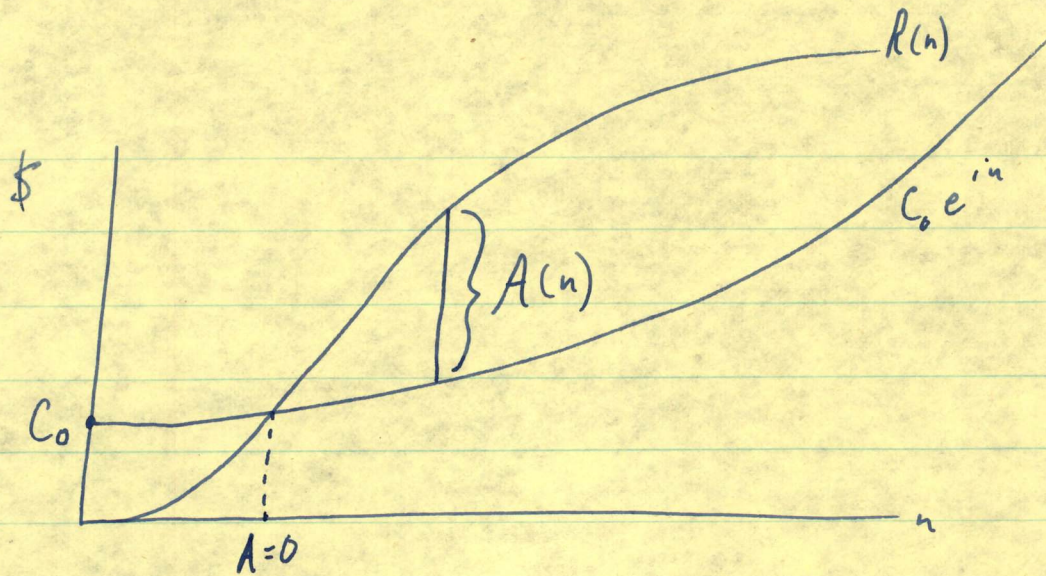
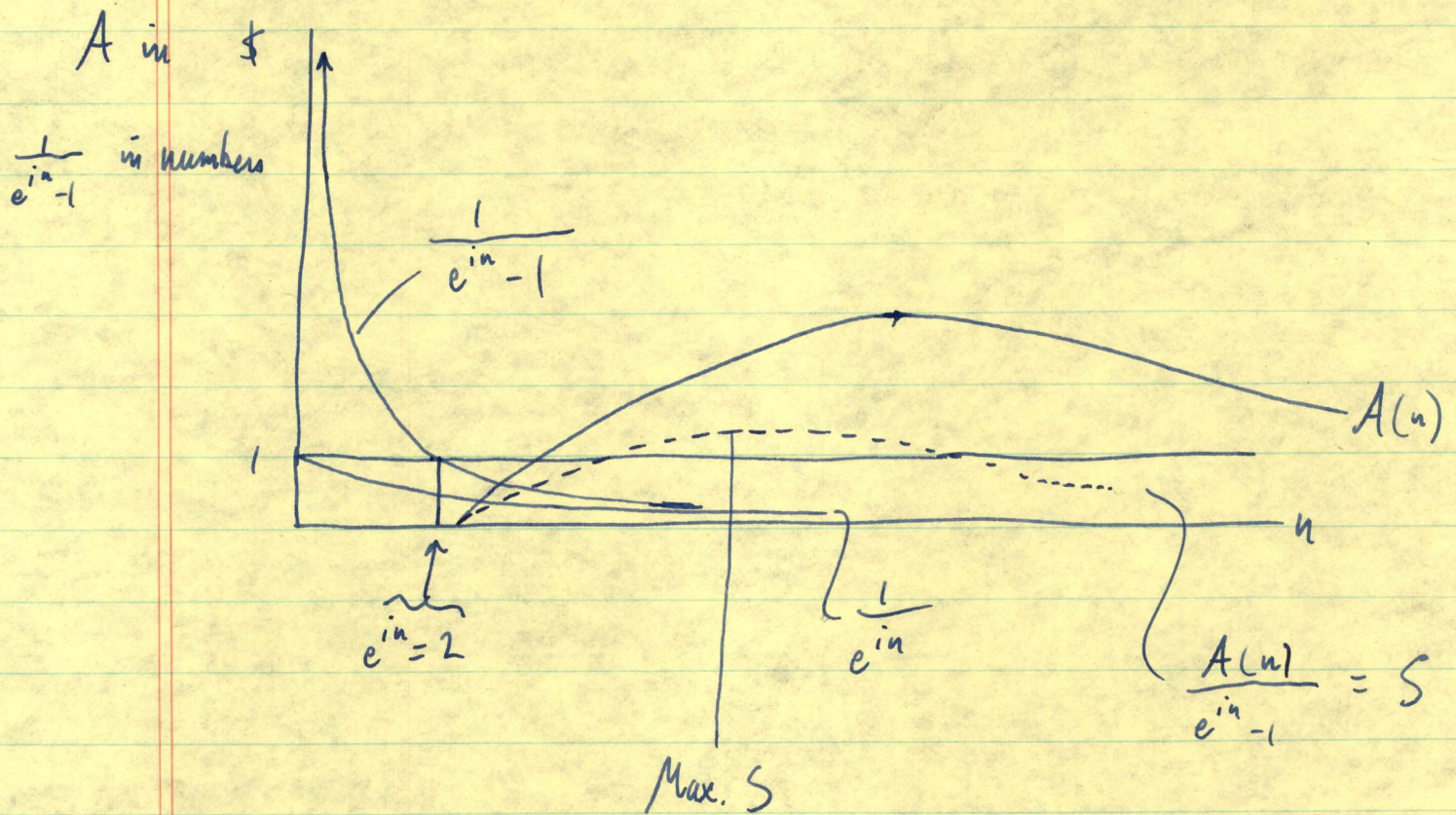
Optimal life is the value of $\underline{n}$ that maximizes $\underline{a}$. This is the same as maximizing a/i (you divide both sides by $\underline{i}$: this is capitalizing in perpetuity.)

It helps to visualize this graphically. First, visualize it algebraically. Finding optimal $\underline{n}$ is an optimizing process, which means two contrary forces are at work as you let $\underline{n}$ increase, and the solution is to find the best balance.

As $\underline{n}$ rises, both the numerator and denominator of (20) rise, the first tending to raise $\underline{a}$, the second to lower $\underline{a}$. We will graph these two forces separately.

For simpler notation, let:

$$R_n - C_0 e^{in} \equiv A(n) \quad (21)$$



The payoff is finding max. S , as above.

As a marginal condition, this maximum is ~~found~~ where annual growth, R' , equals interest on R plus S .

To show this, we find dS/dn and set it equal to zero.

$$\frac{dS}{dn} = \frac{(e^{in} - 1)(R' - Cie^{in}) - (R - Ce^{in})(ie^{in})}{(e^{in} - 1)^2} = 0 \quad (22)$$

$$R'(1 - e^{-in}) - \cancel{Cie^{in}} + Ci = Ri - \cancel{Cie^{in}}$$

$$R' = (R - C) \frac{i}{1 - e^{-in}} = Ri - Ri + \frac{Ri}{1 - e^{-in}} - \frac{Ci}{1 - e^{-in}}$$

$$= Ri + R \left[\frac{i - i + ie^{-in}}{1 - e^{-in}} \right] - \frac{Ci}{1 - e^{-in}}$$

$$= Ri + \left[\frac{Ri - Cie^{in}}{e^{in} - 1} \right] = Ri + Si$$

Q.E.D.

Thus, we max. S by letting the capital grow so long as it yields i on $(R + S)$, or better.

In the same simultaneous process, we determine the value of S .

If there are intermediate revenues or costs, we handle them as in (13), p. 14. Use the Σ sign. This gets a little lengthy, but can all be done with 3 or 4 columns on one sheet of tabulating paper, and even faster on a programmable calculator, using the key marked "sum", which cumulates in a memory.

II. Other applications. These notes will be extended next year to include at least the following.

- a. Mines. Optimal life; optimal time to begin.
- b. Ripening suburban land - when to develop, and how intensively.
- c. Normalizing or staggering - how to shift from periodic models, like those used so far, to simultaneous models where replacements are staggered.
- d. Capital conservation - how to forecast capital needs and get along on what you have, so you can always meet the next payroll.
- e. Factor proportions - how much of the product is really going for interest, payroll, and rent? How ~~is~~ are the shares changed?

f. Depreciation (and depletion and obsolescence).
How fast is your wealth declining? How much
need you set aside to offset the decline?
How does inflation affect this?

g. Taxation. Taxes deferred are taxes denied.
What does this imply for public policy - how to create a
neutral tax system?

What does this imply for private policy - tax
planning.

h. Risk analysis and time. What is the PV
of a 2% annual chance of flood or fire?

i. Cash flow analysis

Max NPV

Ec. 162, Wed, 11/4/81

1. Review, some ends.

a. Find n . $P = a \frac{1 - (1+i)^{-n}}{i}$

$$\frac{Pi}{a} - 1 = -(1+i)^{-n}$$

~~$$-n \ln(1+i) = \ln\left(\frac{Pi}{a} - 1\right)$$~~

~~$$n = \frac{-\ln\left(\frac{Pi}{a} - 1\right)}{\ln(1+i)}$$~~

$$(1+i)^{-n} = 1 - \frac{Pi}{a}$$

$$-n = \frac{\ln\left(1 - \frac{Pi}{a}\right)}{\ln(1+i)}$$

If $\frac{a}{P} = i$, $n = \infty$.

(What actually happens is, your debt, or "exposure", rises, Payoff is negative.)

If $a = P$, $n = \frac{-\ln(1-i)}{\ln(1+i)} \approx 1+i$

(bes. you must keep going another $i\%$ of yr. 2 to earn int.)

b. n is no mtr. of how long yr. money is tied up, bes. you get most of it back earlier. However, you only get $17\% - 25\%$ or so back in $1/2$ yr.

b. IRR

i)

$$P = a \frac{1 - (1+i)^{-n}}{i} + R(1+i)^{-n}$$

$$Pi - Ri(1+i)^{-n} = a[]$$

$$i = \frac{a [1 - (1+i)^{-n}]}{P - R(1+i)^{-n}}$$

cf. results w. Carter method, where

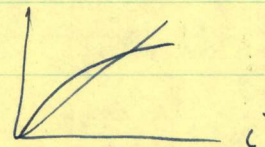
$$i = \frac{a}{P} + \left(\frac{R}{P}\right)^{\frac{1}{n}} - 1$$

P	a	n	R	IRR %	$\frac{a}{P}$ %	$\left(\frac{R}{P}\right)^{\frac{1}{n}} - 1$
75	7.5	5	100	14.95	10	5.9
25	7.5	30	100	30.03	30	4.7

ii) 2 solutions? Esoteric twaddle — a riddle for mathematicians.

0 is always a solution, but not of every relevance.

$$i = \frac{a}{P} [1 - (1+i)^{-n}]$$



iii) Money reinvested at IRR? No bearing.
Unrecovered money is what is earning IRR.

11/4/81

3

2. Max NPV

(a.) Be sure it is Net.

- i. Subtract costs
- ii. Apples & oranges. Discount all to PV
if C_0 all at $t=0$, no discount.
if C_n all at last yr, full discount.
if intermediate, yr. x , disc. x yrs.

- iii. Note, Discounting is a way to subtract a cost.
Interest cost.

$$PV = \frac{F}{(1+i)}$$

$$P + Pi = F$$

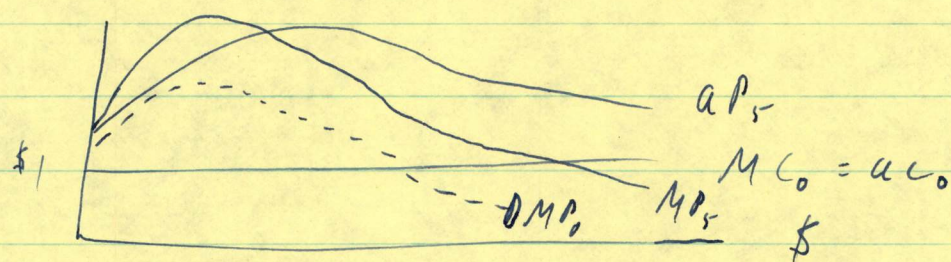
$$P = F - Pi$$

net product.
So Disc. products are net of int. cost. A sort of
Disc. net products takes care of all factors — a
complete analysis. Max. ^{net} return to fixed input.

(b.) Be sure you remember to stick w.
input analysis. Consider different interest rates.
E.g. the exam qn. on Xmas Tree intensity.
A great failing in input. is to fail to integrate
basic input analysis of prodn. eqs. To avoid this, use
Disc. MP. Many say, "If $NPV > 0$, do it." Even some books

say this. This overlooks that ~~inv.~~ invs. may be mutually exclusive (4)

(c) Put it all together in a regular procedure.



Graph helps you picture the Procedure you follow.
You need a set procedure for initial ~~security~~ confidence. (Need to practise varying procedures for flexibility + creativity.)

(d) ~~is~~ Be sure you consider the time period as a variable.

We have had
$$i = \left(\frac{F + L}{C + L} \right)^{\frac{1}{n}} - 1$$

Solve for
$$L = \frac{F - C_0 (1+i)^n}{(1+i)^n - 1}$$

- Same as $\frac{a}{i}$

- Same as $\frac{F}{(1+i)^n} \times \frac{1}{1-r}$ where $r = (1+i)^{-n}$

Given different results.

In max PV, where n is variable, you get different results.

n	F	PV	a	a/i
1	100	91	100	1000
2	150	130	71	714
—				
5	300	186	49	491
—				
10	500	193	31	314

In general. This is bc, you max. $\frac{n}{v}$ when % growth rates are =. And % growth rate of denom. is highly leveraged by subtracting 1.

E.g. from $n=1$ to $n=2$

$$\frac{d}{dn} = i(1+i)^n, \text{ within case}$$

$$\frac{i(1+i)^n}{(1+i)^n} = i$$

$$\frac{i(1+i)^n}{(1+i)^n - 1} = \frac{i}{1 - (1+i)^{-n}} > i$$

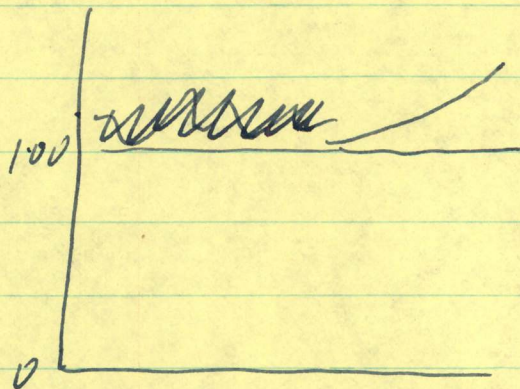
Note $\frac{i(1ki)^n}{(1ki)^n - 1}$

If $n = 1$, denom. = i

$n = 2$, = $1 + 2i ki^2$

More than doubles.

Illustrate
effect on
% change.



$$\frac{d \frac{u}{v}}{dx} = \frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2} = \frac{\frac{du}{dx}}{v} - \frac{\frac{dv}{dx}}{v} \cdot \frac{u}{v}$$

$$= \frac{\frac{du}{dx}}{u} \cdot \frac{u}{v} - \frac{\frac{dv}{dx}}{v} \cdot \frac{u}{v}$$

$$= \frac{u}{v} \left(\frac{\frac{du}{dx}}{u} - \frac{\frac{dv}{dx}}{v} \right)$$

Max when $\frac{\frac{du}{dx}}{u} = \frac{\frac{dv}{dx}}{v}$

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{u}{v}$$

or

$$\frac{F' - C i (1+i)^n}{i (1+i)^n} = \frac{F - C (1+i)^n}{(1+i)^n - 1}$$

or

$$\frac{F' - C i (1+i)^n}{F - C (1+i)^n} = \frac{i}{1 - (1+i)^{-n}}$$

Thurs. 10-22

Topics

2. Cap. recovery. Cap. Conservation.

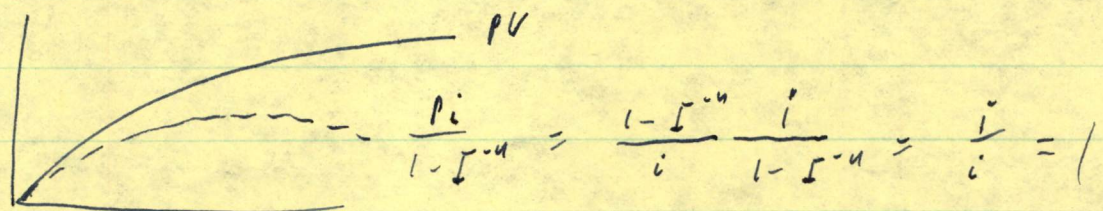
Half-lives.

Naive payback

100%

$$n = \frac{-\ln(1 - \frac{p_i}{a})}{\ln I} \quad (p_i < a)$$

1. 2.



Declining c.f. required, e.g., $\frac{1 - I^{-10}}{1 - I^{-n}}$

3. ~~Triple~~ ^{Double} sensitivity of L.V. to i

$$L = \frac{S - C I^{\times -n}}{I^n - 1}$$

or

$$L = \frac{-C_0 + \frac{1 - I^{-n}}{i}}{1 - I^{-n}}$$

If g expected, triply sensitive.

$$1 - I^{-n} \text{ becomes } 1 - (1 + i - g)^{-n}$$

4. Real int.

$$\bar{i} \approx i - g$$

Rule of thumb.

$$P_0 = \frac{F_1}{(1+g)(1+i)}$$

see p. 15, math. Techniques

5. IRR is fixable

a. "Multiple solutions" is ~~an~~ an esoteric problem for mathematicians.

b. Is money reinvested at IRR?

Not relevant - that is another issue.

E.g. see. S^1 is plowed back.

6. Integrate prodn costs & discounting

a. Max NPV

b. Max IRR

This does assume you can reinvest at $\hat{i}$, the max $\bar{i}$.