



List of Major Inferences Drawn by Inspecting the Formula for Present Value of an Annuity

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Article citation:

Gaffney, Mason. "List of Major Inferences Drawn by Inspecting the Formula for Present Value of an Annuity." Econ 162A, Fall 1983. Study guide.

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LIST OF MAJOR INFERENCES DRAWN BY INSPECTING THE FORMULA
FOR PRESENT VALUE OF AN ANNUITY

Basic formula:

$$P = a \frac{1 - (1 + i)^{-n}}{i} \quad (1)$$

(Know two proofs of that; if you are good at integral calculus, know a third one, too.)

1. Values of P for benchmark values of i and n . Let $a = 1$, to simplify. (This gives you a general factor which you can multiply times any given value of a .)

n	P
0	0
1	$\frac{1}{1+i}$
∞	$\frac{1}{i}$

If $i = 0$, $P = n$.

- By graphical intuition.
- By l'Hôpital's Rule, for calculus whizzes.
- By your calculator. Solve for P as $i \rightarrow 0$, but doesn't quite get there. Watch $P \rightarrow n$.

2. Most of value in P comes from earlier years.

Table 1: Values of P for different n

n	$i = .01$	$i = .08$	$i = .16$
10	9.47	6.71	4.83
20	18.05	9.82	5.93
%			
% of value from years 1-10 =	52%	68%	81%

3. Elasticity of P w.r.t. n

a) An $\% \Delta n$ yields ΔP of less than $\% \Delta n$. (In math notation, $\frac{\Delta P}{P} > \frac{\Delta n}{n}$;

or $\frac{d(\ln n)}{dn} > \frac{d(\ln P)}{dP} \cdot \frac{dP}{dn}$.)

E.g., in Table 1, at $i = .16$, the 100% Δn causes a $\Delta P/P$ of $(5.93/4.83) - 1 = 23\%$.

3. b) The sensitivity (or elasticity) of P w.r.t. n is a decreasing function of n (i.e. declines with n).

In Table 2, n increases by doubling, i.e. in steps of 100%. $\Delta P/P$ is thus the elasticity (since $\Delta n/n = 1$).

Table 2: Elasticity (ϵ) of P w.r.t. n

$i = .05$			$i = .10$	
n	P	$\Delta P/P$	P	$\Delta P/P$
1	.95	—	.91	—
2	1.86	.96	1.74	.91
4	3.55	.91	3.17	.82
8	6.146	.82	5.33	.68
16	10.84	.68	7.82	.47
32	15.80	.46	9.53	.22
64	19.12	.21	9.98	.05
128	19.96	.04	9.99995	.002
∞	20.00	—	10.00000	—

Duplicate Table 2 using your own calculator. Remember the examples of shoes and houses, and Rolls Royces. Think up some more.

4. Share of interest in cash flow.

Let cash flow = \$1/yr. for n years.

Share of interest in cash flow is $\frac{n - P}{n}$.

$$\frac{n - P}{n} = 1 - \frac{P}{n} = 1 - \frac{1 - (1 + i)^{-n}}{ni} \quad (2)$$

Duplicate the graph used in class to illustrate (2).

Table 3: Share of interest in cash flow

n	$i =$	.05	.10	.15	.20
5		.13	.24	.33	.40
10		.23	.39	.50	.58
20		.38	.57	.69	.76
40		.57	.76	.83	.88
80		.76	.88	.92	.94

An operation is "capital-intensive" when a high share of cash flow goes to feed the banker or investor. That occurs when payments are long-deferred, or strung out in the future. Normally, that occurs when capital is durable, heavy and expensive, like a Rolls Royce (life = 40 or 50 years; $P = \$95,000 + \text{tax}$. Is it worth it?).

5. The increase of P w.r.t. n is a decreasing function of life (n).
 - a) Explain.
 - b) Demonstrate.
6. A manager can control the ratio of capital to labor in his investments, and does, in response to changing wage and interest rates. Explain how.
 What operations are more sensitive to interest rates? To wage rates?
7. Depreciation, interest, and capital recovery for recycling.

A. Depreciation

Normally, "in equilibrium," depreciation = capital recovery. "In equilibrium" means market forces always push in this direction--not that they always get there. You as a manager want your capital recovery to exceed depreciation. If you can do so, good for you--but that makes you one of the market forces pushing toward equilibrium.

Duplicate the graphic demonstration in class that depreciation[†] of an asset yielding a level cash flow of \$1 is:

$$\text{Depreciation} \equiv \Delta P = (1 + i)^{-n} \quad (3)$$

(Remember that depreciation is a negative change in P , expressed as a positive number. This lets us omit the minus sign.)

[†] When you are repaying a debt, the decline in the balance due is called "amortization." The math is the same.

7. A. Depreciation (continued)

Algebraic proof of (3):

$$\begin{aligned}\Delta P &= P_n - P_{n-1} = \frac{1 - (1+i)^{-n}}{i} - \frac{1 - (1+i)^{-(n-1)}}{i} \\ &= \frac{(1+i)^{1-n} - (1+i)^{-n}}{i} = \frac{(1+i)^{-n}(1+i-1)}{i} = (1+i)^{-n}\end{aligned}\tag{4}$$

Calculus proof of (3):

$$\frac{\delta P}{\delta n} = \frac{\delta}{\delta n} \left[\frac{1 - (1+i)^{-n}}{i} \right] = - \frac{\ln(1+i) \cdot (1+i)^{-n}}{i}$$

$$\text{But } \frac{\ln(1+i)}{i} \approx 1\tag{5}$$

$$\text{Therefore } \frac{\delta P}{\delta n} \approx (1+i)^{-n}\tag{6}$$

The calculus answer is less exact because it is designed for continuous functions, and we are dealing with a step-function, where interest is compounded yearly. Calculus is more accurate when interest is compounded continuously. For most management decisions the difference is negligible--you have bigger worries.

B. Interest

Knowing depreciation, we also know interest, because cash flow is divided between the two. Interest is the complement of depreciation:

$$\text{Interest} = 1 - (1+i)^{-n}\tag{7}$$

(7) also comes directly from (1), by solving for P_i :

$$P_i = 1 - (1+i)^{-n} \quad (a = 1)\tag{1.A}$$

7. B. Interest (continued)

This is how banks tell you each year how much of what you paid on your loan was interest (which is deductible from taxable income) and how much was repayment of principle (which is not).

In the early years of a long loan, repayment is shockingly small. E.g. at 11.5%, with a loan amortized over 40 years, what % of the debt is repaid in year one?

$$-\frac{\Delta P}{P} = \frac{(1+i)^{-n}}{\frac{1 - (1+i)^{-n}}{i}} = \frac{i}{(1+i)^n - 1} = \frac{.115}{76.8} = .0015 \quad (8)$$

On a debt of \$100,000, you've worked it down by \$150. Pretty slow going! You've paid them \$11,650; all but \$150 was interest.

C. Capital Recovery for Recycling

But look at it from the banker's viewpoint. He has only recovered \$150 of his loan. The rest is committed to you--you're using his money. Assuming that the interest rate is fair, and the bank has to pay to attract money, and interest in general is either just or inevitable, then he cannot treat the interest portion as capital recovery. It is income, and in law, custom and logic it has a prior claim on cash flow. Recovery is just what's left over after paying interest.

Why should you as a manager worry about the banker's viewpoint? You might be managing a bank, but otherwise, every manager supervises investments. That's what it's all about. You're throwing down nickels and picking up dimes--later, and you hope not much later. Throwing down those nickels is the motive force that drives our economy, and any economy, so it's important socially, too.

Your investment in real capital is like the banker's investment in a loan to you. He must recover the principal before he can relend it; you must recover your capital before you can recycle it by reinvesting it. That recycled capital is most of what managers have to work with. It is "the ready." It is what lets you "get there firstest with the mostest" (in Stonewall Jackson's egregious English) and seize prizes before equilibrium sets in. It is what keeps your organization going without infusion of outside capital, and without any new saving. It's what lets you achieve high volume without having a lot of assets. It is your working capital. Treasure it, nurture it, recycle it, it's what keeps you alive.

7. C. Capital Recovery for Recycling (continued)

But if you bury it in assets with 40-year recovery periods, you won't remobilize most of it for a long time; half your life, perhaps.

You might think you recover $1/40$ of it, or $2\frac{1}{2}\%$, each year, which would be slow enough. That is "straight-line" depreciation. But we saw in (8) you recover only $.15\%$, which is 6% of $2\frac{1}{2}\%$.

8. DCF and marginal analysis.

DCF is the bottom line after forecasting revenues and then:

- a) Revenue - Expenses = Cash Flow
- b) Cash Flow - Interest = DCF

DCF is the most you can pay a contractor to build a thing, turnkey basis.

Marginal DCF is the discounted marginal product from adding inputs at time 0.