



John Napier and Continuous Interest

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John Napier and Continuous Interest

John Napier, around 1590 A.D., noticed that money grows faster when you compound semi-annually, instead of once a year. He got to wondering what would happen if interest were compounded monthly, and found money grows even faster. If monthly, why not weekly? If weekly, why not daily? If daily, why not hourly? Finally, why not -- continuously! Actually, the advantage of frequent compounding is mostly exhausted by going from one year to one month, but by now his lust for gain gave way to an intellectual curiosity that obsessed him. He finally figured out how to do it. In the process he paved the way for calculus, which came along 100 years later.

Let:

$$(1+i) = (1+\rho/f)^f \quad (1)$$

where f is frequency of conversion per year, and ρ is rho, called the "force" of interest. ρ is that number, slightly lower than i , that yields the same result as i compounded annually, when compounding takes no time at all - i.e., $f \rightarrow \infty$, and $\rho/f \rightarrow 0$. It would be more fun, of course, to see how much more money you'd have if you stuck with i and raised f . You can do that if you like, but mathematicians have a way of taking the fun out of things, to keep us sober. This is how they did it.

The trick here is to get ρ up into the exponent, from where it started down in the base. Not to sweat, this is going to make things a lot simpler, when we're done. We do this by defining an artificial term, $x=f/\rho$. Then:

$$(1+\rho/f)^f = (1+1/x)^{x\rho} \quad (2)$$

Now, let $x \rightarrow \infty$. Then $(1+1/x)^x \rightarrow e$, the base of natural logs. (This is how e is defined. You'll see in a moment how to find its value, using just your calculator.) Now you can substitute e for all that flute-music, $(1+1/x)^x$. Therefore, the limit of $(1+\rho/f)^f$, as $f \rightarrow \infty$, is simply e^ρ . Pretty cute, eh? We did a complex thing not to be obscure, but to boil matters down into a simple thing.

Then, substituting in (1):

$$(1+i) = e^\rho \quad (3)$$

Take the Ln of both sides, reverse sides, and:

$$\rho = \ln(1+i) \quad (4)$$

(Recall that $\ln(1+i) = \ln I$, by definition of I .)

(4) is how you can find ρ on your calculator, anytime. ρ is the $\ln$ (natural log) of One plus the Interest Rate. Just push a button, the one marked LN. Try a few; get the feel of it.

Now to show how you can derive the constant, e . First, check the value of e on your calculator. It's there because your calculator uses it all the time. Next, raise 1.0001 to the 10,000 power: you get 2.71814593 -- getting pretty close to e , which is 2.718281828.... Try 1.000001 to the 1,000,000 power. You get 2.718280469, closer yet.

Next, pull a switcheroo. Nothing says that x must be positive, so try .999999 to the -1,000,000 power. You get 2.718283188, which is barely more than e . Take an average and you get 2.718281829. That should be close enough to convince you.

CONGRATULATIONS! You have just done in a few moments what must have taken John Napier many years, back in 1590 or so, when he first determined the value of e . He did it all by hand, poor chap. Can you imagine the labor?! What did his wife and children think? Calculators are marvelous: appreciate yours. That was towards the end of the long reign of Queen Elizabeth I, as she was mulling over whether to make James Stuart her successor (she did, just before she died in 1603). There were no calculators easily available at that time. It was the year they settled Jamestown, 17 years before the Pilgrims landed at Plymouth Rock, if that helps you place it. Napier knew Wm. Shakespeare: in *A Winter's Tale*, Shakespeare refers to logarithmic (exponential) growth, I've been told (I haven't found it yet). Certainly the concept enthralled Thomas Malthus, 1798.

ρ is called the continuous rate of interest. Marvelous as that seems, it is not much different from what you get if you let $f=12$ (monthly compounding). Let's call that rate $\underline{m}$, for monthly (although the monthly rate is $m/12$). If $i=.12$, and you compound monthly, resist the temptation to say the monthly rate is .01 ($.12/12$). $1.01^{12} = 1.1268$, which is closer to a 13% annual rate than a 12% annual rate. You find the monthly rate by letting $f=12$ in (1). Put in $\underline{m}$ where ρ is, and solve for $\underline{m}$. Work that out and you get $m = .1139$. The rate to charge each month is $m/12$, or .00949. Check it out: $1.00949^{12} = 1.12$.

$\rho = \ln 1.12 = .1133$. That is just a shade lower than the value of $\underline{m}$ you just found, $m = .1139$. Thus, ρ is a good approximation to $\underline{m}$. If you're in a hurry to find the monthly rate, just take $(\ln I)/12$.

Here's the bottom line, as a practical business matter. Interest rates are expressed annually, but payments are usually monthly, and compounding periods are months, not years. To find the monthly interest rate, $\underline{m}$, that corresponds to an annual rate, $\underline{i}$, substitute $\underline{m}$ for ρ in Equation (1), set $f=12$, and solve for $m/12$.

$$1 + m/12 = (1+i)^{1/12}$$

$$m/12 = (1+i)^{1/12} - 1$$

The result will be just a tad greater than $\ln(1+i)$, which may be used for a close estimate. For example:

i	$\ln(1+i)$	m	$\ln(1+i)/12$	m/12
.05	.0488	.0489	.0041	.0041
.10	.0953	.0957	.0079	.0080
.15	.1398	.1406	.0116	.0117
.20	.1823	.1837	.0152	.0153

When interest rates are high, it is worth your while to sharpen your pencil and figure this out right. There was a period in the early 1980s when rates soared. I knew a man who needed money so urgently he offered a lender 24%. He then figured the monthly rate at 2% - just divided 24 by 12. However, $1.02^{12} = 1.2682$. By that careless act, he was really paying nearly 27%! Since he borrowed \$200,000, it cost him several k each year. I am not sure if he ever even knew it, but I am sure he'd be better off today if he had known.

What monthly rate should he have paid? Over to you.