



Mathematics of Finance, Compressed

By Dr. Mason Gaffney

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MATHEMATICS OF FINANCE, COMPRESSED.

Corrected, Jan. 20, 1977.

1. P = present value, lump sum
F = future value, lump sum
a = level annual payment
i = interest rate, annual
n = number of years

2. "Money makes money, and the money money makes makes more money."

- B. Franklin

$$F = P(1 + i)^n$$

3. Shorthand

$$(1 + i)^n \equiv e^{in} \quad \text{where } e \text{ is the base of natural logarithms}$$

4. "Capitalization" in perpetuity.

$$P = \frac{a}{i}$$

5. Capitalization over n years. The key operation.

$$P = \frac{a}{i} - \frac{a}{i} e^{-in} = \frac{a}{i} (1 - e^{-in}) = a \frac{1 - e^{-in}}{i}$$

(Note that e^{-in} is a short way of writing $\frac{1}{e^{+in}}$)

$$P = a \frac{1 - e^{-in}}{i} \quad \text{is the "present value formula" } na$$

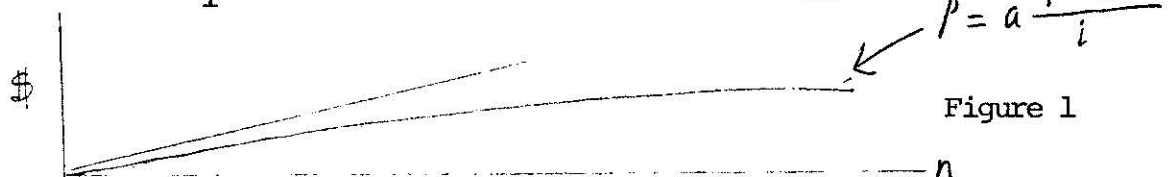


Figure 1

6. The instalment plan formula = annualizing a present lump sum

Solve (5) for a:

$$a = P \frac{i}{1 - e^{-in}}$$

(Figure 2)

7. Growth of an annuity at compound interest.

$$F = Pe^{in} = a \frac{1 - e^{-in}}{i} \times e^{in} = a \frac{e^{in} - 1}{i}$$

8. Sinking Fund instalment. Solve (7) for a

$$a = F \frac{i}{e^{in} - 1}$$

9. Forest Valuation

(8) is also the way to value forest land which yields regular periodic harvests. Say it yields \$F every n years.

Costs (C_i) are zero.

Annuity equivalent of one harvest is:

$$a = F \frac{i}{e^{in} - 1}$$

Capitalizing in perpetuity:

$$\frac{a}{i} = \frac{F}{e^{in} - 1}$$

10. Now let costs be incurred in year 0. Their future equivalent is found from

$$F = Pe^{in}$$

Here, P is C_0 , and future equivalent is $C_0 e^{in}$.

Forest land value, or "Soil Expectation Value" $\equiv S_e$ is:

$$S_e = \frac{F - Ce^{in}}{e^{in} - 1}$$

(10)

11. Solving (10) for $(F + S_e)$:

$$F + S_e = (C + S_e) e^{in} \quad (11)$$

(11) says forestry is an investment whose input is land plus capital and whose output is land plus timber.

12. Growth of investment in timber. Solve (11) for F:

$$F = (C + S_e) e^{in} - S_e \quad (12)$$

(12) is a modified compound interest curve.

13. Rate of growth of timber investment

Differentiate (12) w.r.t. n .

$$\frac{dF}{dn} = i(C + S_e) e^{in} \quad (13)$$

Substituting (12)

$$\frac{dF}{dn} = i(F + S_e) \quad (14)$$

(14) says the investment grows each year by interest on timber plus land value.

14. Financial maturity of timber

As long as the current stumpage value of timber is growing faster than (14), the timber is unripe, and vice versa.

15. Depreciation of forest roads.

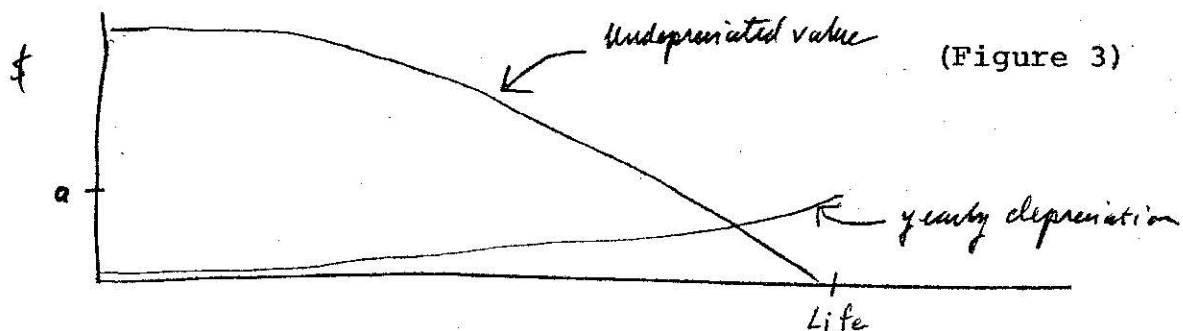
Life of forest roads is determined largely by "locational obsolescence". They are useful so long as there is wood to cut and haul. Under our appraisal system they earn a guaranteed (sort of) income, not decaying over time. This residual value is approximated $\therefore$ from (5), the Present Value Formula, where n is years of life remaining

$$P = a \frac{1 - e^{-in}}{i} \quad (5)$$

For depreciation, differentiate (5) w.r.t. n , and multiply $\times (-1)$.

$$-\frac{dP}{dn} = \frac{a}{i} (1 - ie^{-in}) = -ae^{-in} \quad (15)$$

This is called "sinking fund depreciation", [because the amount of allowable depreciation each year is equal to the growth of a sinking fund whose accumulated value after n years would equal the capital cost of the road.] It is a much slower write-off than normally allowed.



16. Valuation of Old Growth

Value depends on rate of cut.

Let A = total stumpage.

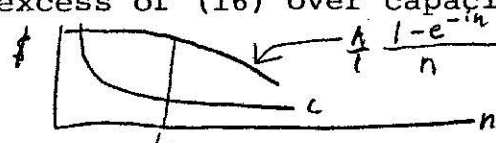
Annual gross income stream = $\frac{A}{n}$

$$PV = \frac{A}{n} \frac{1 - e^{-in}}{i} = \frac{A}{i} \frac{1 - e^{-in}}{n} \quad (16)$$

(16) is a decreasing function of n . To maximize (16) $\therefore$ we always want to cut everything right away. However:

- for small n , (16) decreases very slowly
- capacity costs also decrease with n . For small n , they decrease rapidly.

To maximize net value of old growth removal, maximize the excess of (16) over capacity costs:



(Figure 4)

L is optimal life

17. Valuation of old growth followed by new growth.

Modify (16) by adding S_e to A. This says that when you cut you get value of timber plus release of site for future growth. This makes the curve steeper and pushes L to the left, i.e. speeds the cut.

18. Valuation of peak load capacity, flood protection culvert.

Let p = period between floods of given level.

ΔC is added cost of insuring against p - year flood.

Expected frequency (or probability of p - year flood)

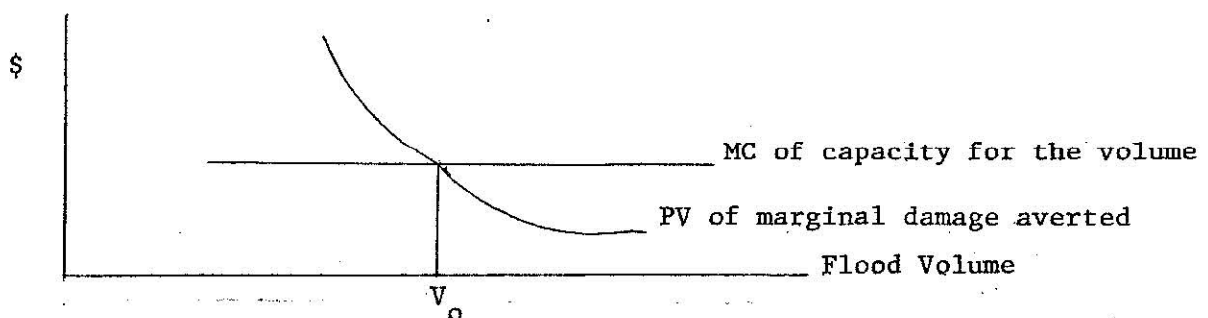
is $\frac{1}{p}$ each year. Full cost of washout is D (operating losses plus repairs)

$$PV = \frac{D}{p} \frac{1 - e^{-in}}{i} \quad \text{This is the PV of the extra capacity for the } p\text{-year flood}$$

As you add peaking capacity, marginal cost is below average, and probably declining by the 6/10 rule.

$$C = k x^{(.6)}$$

where k = a constant and x = capacity. $MC = 60\%$ of AC .



Is V_0 the optimal capacity? Two adjustments needed:

- Damage increases with capacity, because the culvert itself is part of what is damaged.
- Extra capacity adds to life of culvert, n .

Can you work out a solution? Carry on!