

~~AD~~
Old Math

UNION No. 1012 FILE

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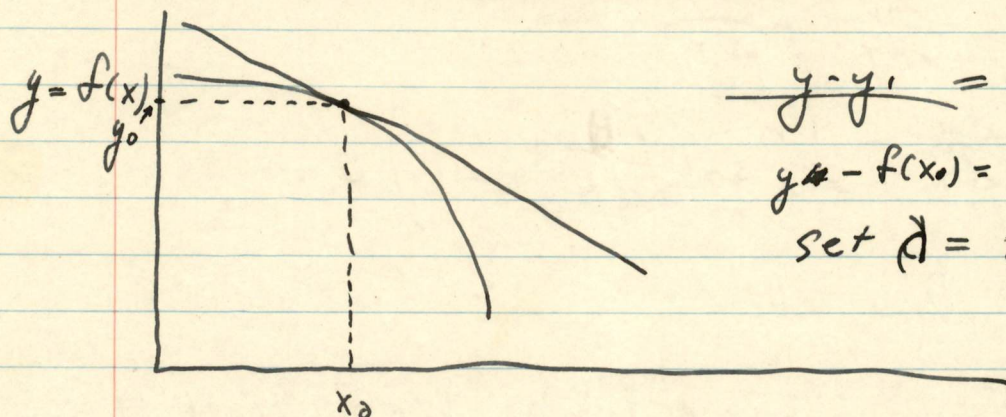
PURCHASE ADDITIONAL SETS OF FOLDERS
FROM YOUR DEALER OR DIRECT FROM—
UNION STEEL CHEST CORPORATION

LE ROY, N. Y.

**FOUND
AS
SCANNED**

Loomis - Nov. 19

Tangents & Normals to curves



$$y - y_0 = d(x - x_0)$$

$$y_0 - f(x_0) = d(x - x_0)$$

$$\text{Set } d = D_x f(x) \Big|_{x=x_0}$$

Ex. $f(x) = x^3$

find equation of tangent at point $(\frac{1}{2}, \frac{1}{8})$

We have now but to find d , $D_x x^3$ or $3x^2 = \frac{3}{4}$

$$\frac{y - y_0}{y_0 - \frac{1}{8}} = \frac{d(x - x_0)}{\frac{3}{4}(x - \frac{1}{2})} \quad \text{sub.}$$

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ an implicit function.

Tan at (x_0, y_0)

$$d = D_x y \Big|_{(x_0, y_0)}$$

$$\frac{2x}{a^2} + \frac{2y D_x y}{b^2} = 0$$

$$D_x y = \frac{-b^2 x}{a^2 y}$$

$$y - y_0 = \frac{-b^2 x_0}{a^2 y_0} (x - x_0)$$

$$a^2 y y_0 - a^2 y_0^2 = -b^2 x x_0 + b^2 x_0^2$$

$$a^2 y_0 y + b^2 x_0 x = a^2 y_0^2 + b^2 x_0^2 = a^2 / b^2$$

since (x_0, y_0) must satisfy equation of ellipse.

Ass.

p. 48 - 1, 2, 3, 4, 8

$$2.4 = 8.1$$

$$\frac{4}{6} = \frac{1}{2}$$

$$\frac{2}{14} = \frac{8}{4}$$

Limits

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right)$$

$$D_x y = D_x u + D_x v$$

Limit of sum = sum of limits. We ain't prove it.

We assumed all below.

$$1) \begin{matrix} X \rightarrow A \\ Y \rightarrow B \end{matrix} \text{ then } (X + Y) \rightarrow (A + B)$$

$$2) XY \rightarrow (A \cdot B)$$

$$3) \frac{X}{Y} \rightarrow \left(\frac{A}{B} \right) \quad B \neq 0$$

What do we mean by $f(x) \rightarrow B$ as $x \rightarrow A$ or $\lim_{x \rightarrow a} f(x) = b$ }
 or

Does not make sense to say a b/c $\rightarrow$ limit.

Loomis Nov.

Differentiation of Implicit Function

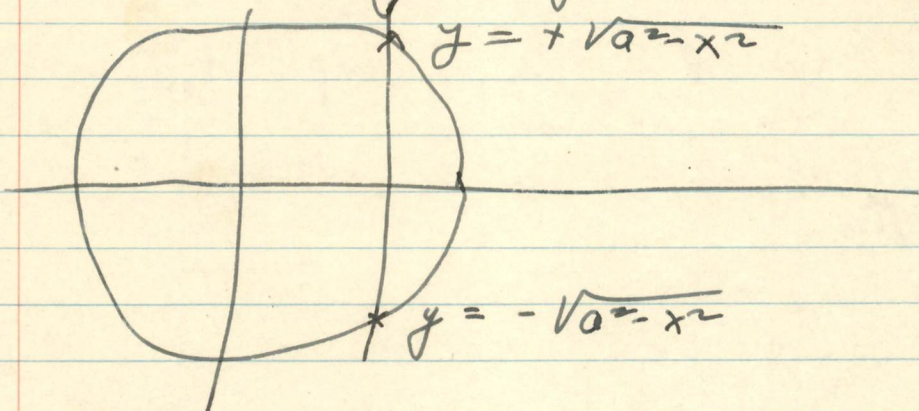
$$x^2 + y^2 = a^2$$

$$x^2 - 2xy + y^2 = 0$$

Can't be solved for y , so it defines y as a function of x implicitly.

But you get more than 1 value for y , so it's a multiple value function.

Over restricted ranges of x we can get unique values ~~for~~ of y for values of x .



These functions are special ones of a class Algebraic functions. (if $y = f(x)$) (irreducible)

$P(x, y)$ is a polynomial in x & y which cannot be factored. Then $P(x, y) = 0$ defines y as an algebraic function of x .

Ex. $y = x^2 \sqrt{a-x}$
 $y^2 = x^4(a-x)$
 $y^2 - x^4a + x^5 = 0$

implicit functions can always be turned into algebraic.

If it can be factored you get 2 sep. = tions, 2 separate curves.

Ass. p. 38 - 8, 12, 16
 p. 43 - 1, 2, 6, 8, 26, 27, 29, $n \neq 0$, neg. rec.

$P(x, y) = 0$ Def $y = f(x)$, algebraic function.
 To find $D_x y = D_x f(x)$, assume y is a $f(x)$ in
 $D(x, y) = 0$. & differentiate the equation as it stands.

Ex $x^2 + y^2 = a^2$
 $D_x(x^2) + D_x(y^2) = D_x(a^2)$

$$2x + (2y \cdot D_x y) = 0$$

y is for f - use the rule

$$D_x y = -\frac{x}{y}$$

Ex. $x^3 - 2xy + y^5 = 0$ implicit f.

again derive right across

$$3x^2 - 2 \underset{\text{prod. rule}}{x D_x y} - 2y + 5y^4 D_x y = 0$$

$$D_x y = \frac{2y - 3x}{5y^4 - 2x}$$

To prove general rule for fractional exponents.

$$D_x(x^n) = nx^{n-1} \quad \text{we already know}$$

$$y = x^{\frac{p}{2}}$$

$$D_x y$$

an algebraic function

$$D_x(y^2) = D_x(x^p)$$

$$2y^{2-1} \cdot D_x y = px^{p-1}$$

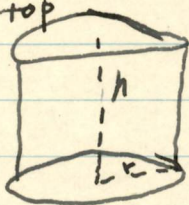
$$D_x y = \frac{p}{2} \frac{x^{p-1}}{y^{2-1}}$$

$$\text{Den. } (x^{\frac{p}{2}})^{2-1} = x^{\frac{p}{2}(2-1)} = x^{p-\frac{p}{2}}$$

$$D_x y = \frac{p}{2} \frac{x^{p-1}}{x^{p-\frac{p}{2}}} = \frac{p}{2} x^{p-1-p+\frac{p}{2}} = \frac{p}{2} x^{\frac{p}{2}-1}$$

Ex. $y = x^{\frac{1}{4}}$, $D_x y = \frac{1}{4} x^{-\frac{3}{4}}$

How can you find minimum tin at given vol.



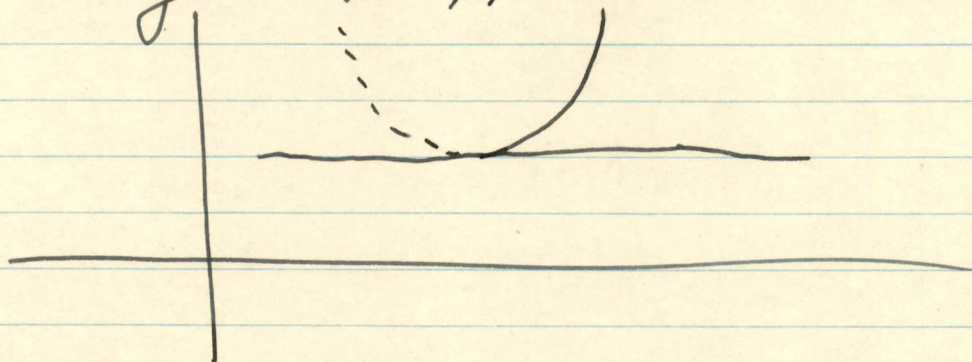
$$V = \pi r^2 h$$

$$A = \pi r^2 + 2\pi r h$$

Assume ~~that~~
 $h = f(r)$

Auxiliary vbles.

Plotting $A = f(r)$, to show 1st it swoops



$$A > \pi r^2$$

$$D_r A = 2\pi r + 2\pi(r D_r h + h)$$

$$0 = \pi(r^2(D_r h) + h(2r))$$

$$D_r h = -\frac{2h}{r}$$

$$A = 4xy \quad \begin{matrix} y = f(x) \\ [by \ x^2 + y^2 = r^2] \end{matrix}$$

Compute point where $A=0$

$$D_x A = 4(x D_x y + y)$$

Now compute $D_x y$ for (2), implicit function rule.

$$\begin{matrix} x^2 + y^2 = r^2 \\ 2x + 2y D_x y = 0 \end{matrix}$$

$$D_x y = -\frac{x}{y}$$

Now sub.

$$D_x A = 4\left(-\frac{x^2}{y} + y\right) = 0$$

$$-x^2 + y^2 = 0$$

$$x^2 = y^2$$

$x = \pm y$, but everything in this physical problem must be positive, so

$$x = y$$

Ass. p. 48 - 6, 10

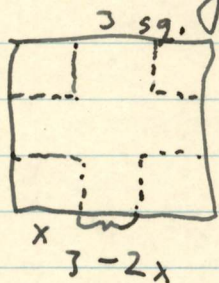
p. 51 - 2

p. 54 - 2, 1

1st show that it has a maximum in the interval.
use auxiliary vbles.

Loomis

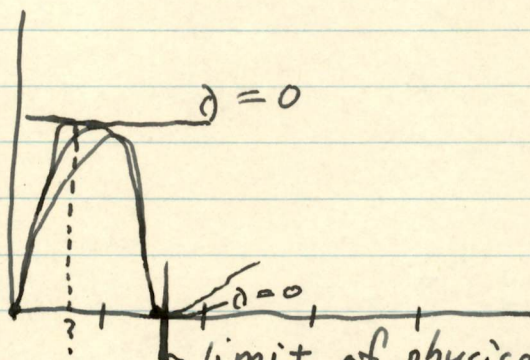
Define tangents & normals to curve.



fold it up to make a box. Problem: to get greatest volume.

$$V = x(3-2x)^2$$

1. we could plot V as $f(x)$



limit of physical meaning.

But this does not give precise value, and it is tough.

2. Do it by calculus.

$$\text{Take } D_x V = x(3-2x)(-2x+1)$$

$D_x V = 0 = 0$ at maximum so

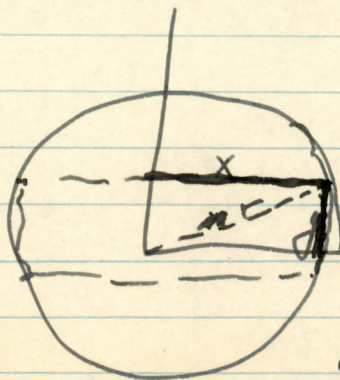
$$(3-2x)(-2x+1) = 0$$

$$x = \frac{3}{2}$$

$$x = \frac{1}{2}$$

Or it can be other way around. Wanted a box of given volume, with what minimum of tin can it be made.

Method of Auxiliary variable



Find dimensions of inscribed rectangle having the largest area.

$$x^2 + y^2 = r^2 \quad (1)$$

such that $4xy$ is maximum. (2)
Consider y as $f(x)$ in (1).

~~$\frac{y}{y}$~~

$$\frac{2}{4} = \frac{1}{2}$$

$$2 = \frac{4}{2}$$

$$\frac{4}{4} = 1$$

$$\frac{4}{2} = 2$$

$$\left(3\frac{5}{9}\right)^2 + \frac{64}{729} =$$

$$\frac{32^2}{9^2} = \frac{1024}{81}$$

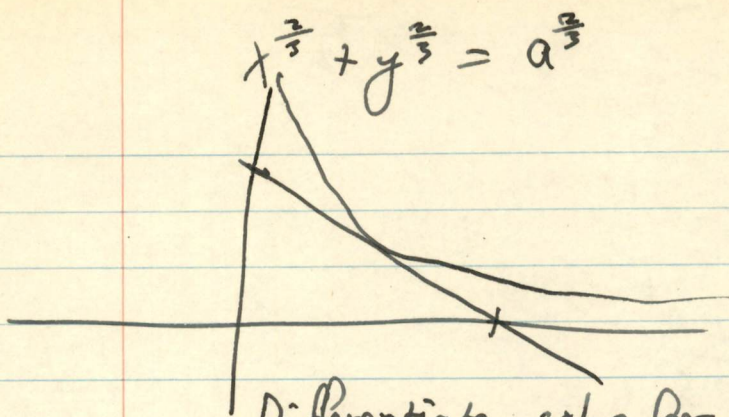
$$\sqrt{\frac{64}{729}} = \frac{8}{27}$$

$$\begin{array}{r} 32 \\ 32 \\ \hline 64 \\ 96 \\ \hline 1024 \end{array}$$

$$\begin{array}{r} 12 \\ 81 \overline{) 1024} \\ \underline{81} \\ 214 \end{array}$$

$$\frac{81}{729}$$

$$\frac{1}{3} h \pi r^2 = \text{vol. of cone}$$

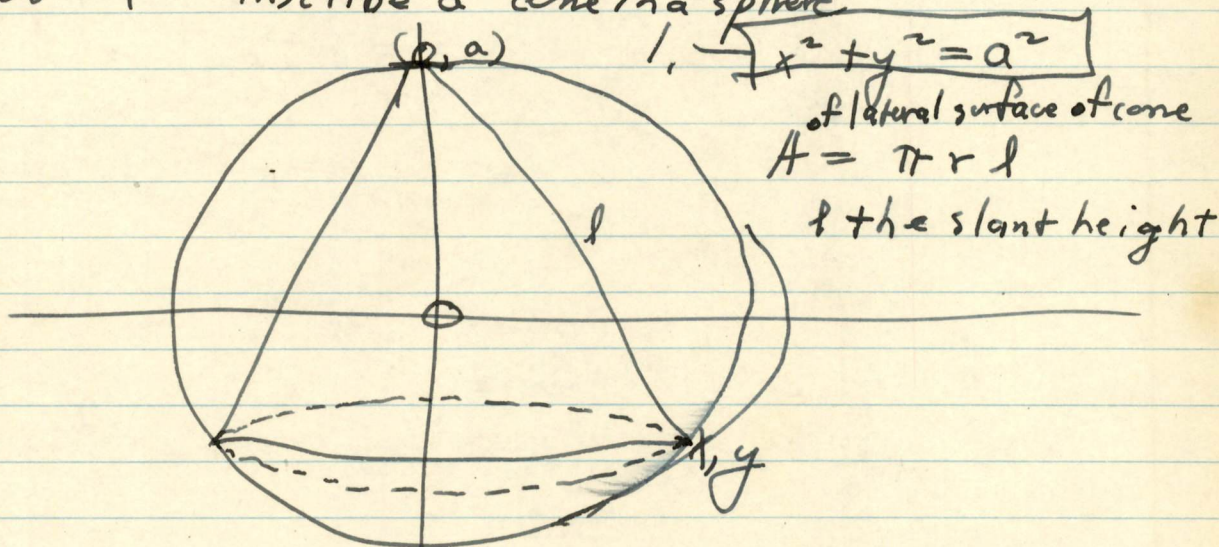


Differentiate, solve for $D_x y$, make = 0, solve for intercepts, use distance formula.

Use analytic geometry.

Using auxiliary variables helps in differentiation.

Ex. 58-4 - inscribe a cone in a sphere



$$l = \sqrt{x^2 + (a-y)^2}$$

$$A = \pi x \sqrt{x^2 + (a-y)^2}$$

Differentiate right across, sub $D_x y$ for one in other a terrific problem.

$$y = \frac{-a}{5}$$

$$h = \frac{4}{3} a$$

Ass. p. 58 - 2, 5

10 (discuss the answer - you'll find 2 answers, tell what it means & why, & which is right, I guess)

Maths, Dec. 1, (cont)

Ice Balls

Ice is stratified. Red. Green winter layers, dirty summer layers.

minimum for the d.

Pt. where d stops increasing & starts decreasing is a rel. max. for the d.

Some can use same tests.

Test for point of inflection

$[\text{If } D_x^2 f(x) = 0]_{x=x_0}$

$[D_x^3 f(x) \neq 0]_{x=x_0}$

} we have a pt. of inflection.

if $[D_x^3 f(x)]_{x=x_0} = 0$, the test fails.

Ex. $y = x^4$

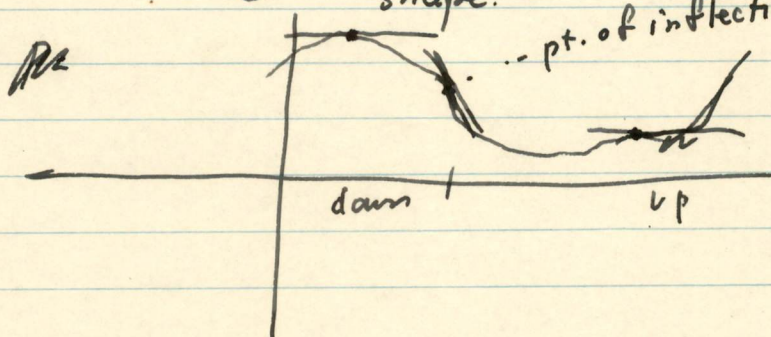
$$D_x = 4x^3$$

$$D_x^2 = 12x^2$$

$$D_x^3 = 24x$$

If $y = x^5$, they all fail.

Curve tracing - when you only want to know general shape.



D_x

D_x^2

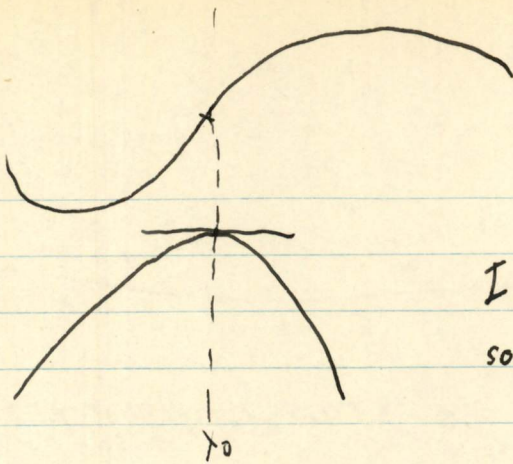
D_x^3

tests won't fail.

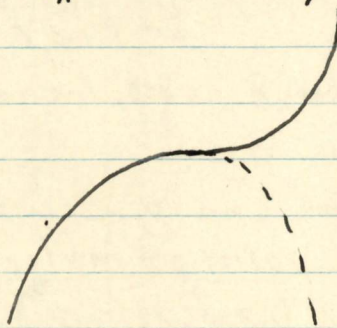
p. 67 - 1, 4, 12

p. 70 - 1, 4, 5

When $D_x^3 = 0$, D_x^2 has pt. of inflection



If $\partial_x^2 \tan \tau = 0$, it's a pt. of inflection
so



Loomis, Dec. 1

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

→ $\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$ = equation of tangent to ellipse at x_0, y_0

take intercepts.

$$\left(\frac{a^2}{x_0}, 0\right)$$

$$\left(0, \frac{b^2}{y_0}\right)$$

$$A = \frac{1}{2} \frac{a^2 b^2}{x_0 y_0}$$
$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$$

use auxiliary vbles.

$$D_x y = \frac{-b^2 x}{a^2 y} = \left(-\frac{1}{y^2}\right) D_x y$$
$$D_x A = \frac{a^2 b^2}{2} \left(\frac{1}{x} D_x \frac{1}{y} + \frac{1}{y} \left(-\frac{1}{x^2}\right) \right) = 0$$

$$\frac{a^2 b^2}{2} \left[x \left(\frac{-b^2 x}{a^2 y} \right) - y \right] = 0$$

$$x^2 b^2 = a^2 y^2;$$

rewrite first equation, & sub, & solve

$$x = \frac{a}{\sqrt{2}} \quad x^2 b^2 + a^2 y^2 = a^2 b^2$$
$$y = \frac{b}{\sqrt{2}}$$

Terminology - Intervals

Finite intervals - all the no.s between 2 fixed no.s.
all x such that $a < x < b$

Infinite intervals

1) All no.s.

2) all x such that $x < a$

3) " " " " $x > a$

The limiting a & b are the end-~~interval~~ points

Open Intervals

Endpts. not included. $a < x < b$

closed Intervals

End pts. included.

$$a \leq x \leq b$$

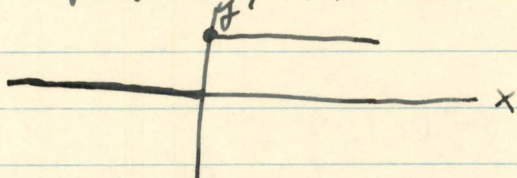
Continuous Functions

Def. $f(x)$ is cont. at x_0 if there is an open interval containing x_0 in which $f(x)$ is defined, and if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

(i.e., if graph $f(x)$ is "smooth" at x_0)

Ex. $f(x) = \frac{1}{x}$ not continuous at $x=0$
not defined at $x=0$

Ex. $f(x) = 1$, if $x \geq 0$, $f(x) = 0$, if $x < 0$



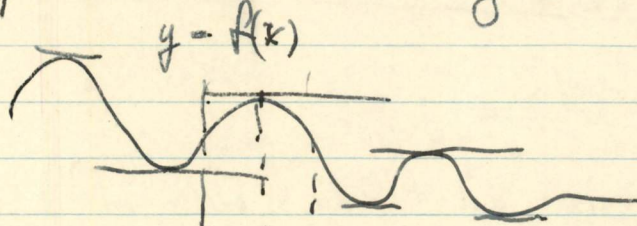
It's not true, then, that if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ where $x_0 = 0$.

Def. $f(x)$ is cont. at an interval if it is cont. at every pt. in the interval.

Thm. Polynomials are cont. everywhere

Thm. A rational function is continuous everywhere except where the denominator vanishes.

Thm. If $f(x)$ is cont. at pt. x_0 , & if $f(x_0) \neq 0$, then there is an open interval containing x_0 in which $f(x) \neq 0$



Relative maximum

Relative maximum

x_0 is a relative max. if $f(x_0) >$ any value of $f(x)$ in the neighborhood of x_0 .

Relative minimum

Loomis, Dec. 1, (cont.)

- 1) If x_0 is rel. max. for $f(x)$, then $D_x f(x) = 0$ at x_0 .
- 2) If $D_x f(x) > 0$ in an interval, then $f(x)$ is increasing in that interval.
 < 0 decreasing
- 3) If $D_x^2 f(x) > 0$ in an interval, then $f(x)$ is concave up in that interval.
 < 0 down
- 4) ~~If~~ If x_0 is a pt. of inflection, for graph of $f(x)$, then $D_x^2 f(x) = 0$ at $x = x_0$.

None of these statements have true converses.

p. 59 - 14

p. 63 - 6, 7

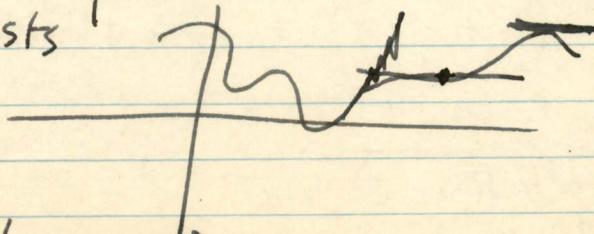
p. 64 - 13, 15

Loomis, Dec. 3.

1. If $f(x)$ has relative max. at $x = x_0$, then $D_x f(x) = 0$ at $x = x_0$.
min.
2. If $D_x f(x) > 0$ in an interval, then $f(x)$ is increasing in that interval.
 < 0 decreasing
3. If $D_x^2 f(x) > 0$ in an interval, then $f(x)$ is concave up in that interval.
 < 0 down
4. If $f(x)$ has a pt. of inflection at $x = x_0$, then $D_x^2 f(x) = 0$ at $x = x_0$.

Converses not true.

The problem - to make them work the other way.
Tests



1st set $D_x y = 0$

1. If curve in that interval is concave down. i.e., if

whenever $\left[\begin{array}{l} D_x f(x) = 0 \text{ at } x = x_0 \\ D_x^2 f(x) < 0 \text{ at } x = x_0 \end{array} \right]$ it is a rel. max. Converse not true.

$$D_x^2 f(x) < 0$$

if at that point $D_x^2 f(x) < 0$, then for small interval about that pt. curve is concave down, it is a rel. max.

Test for a relative maximum.

$$\left. \begin{array}{l} [D_x f(x)]_{x=x_0} = 0 \\ [D_x^2 f(x)]_{x=x_0} < 0 \end{array} \right\} \text{ then } x_0 \text{ is a relative maximum.}$$

Test for a relative minimum

$$\left. \begin{array}{l} [D_x f(x)]_{x=x_0} = 0 \\ [D_x^2 f(x)]_{x=x_0} > 0 \end{array} \right\} \text{ then } x_0 \text{ is a relative minimum.}$$

When neither of above is true,

$$\left. \begin{array}{l} [D_x f(x)]_{x=x_0} = 0 \\ [D_x^2 f(x)]_{x=x_0} = 0 \end{array} \right\}, \text{ the test fails. (it still may be rel. min. or max.)}$$

Ex. $x^2 - 4x + 3 = f(x)$

$$D_x f(x) = 2x - 4$$

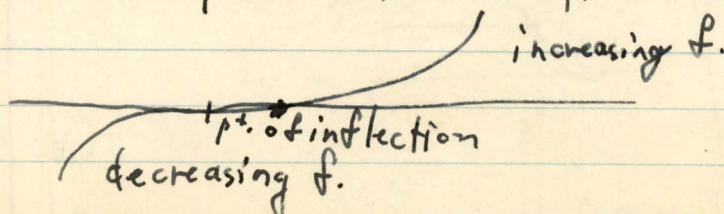
$$[D_x f(x) = 0]_{x=2}$$

$$[D_x^2 f(x)] = 2 \text{ everywhere, \& i; at } x=2$$

so $x=2$ is a rel. min.

- 1 - Calculate $D_x f(x)$ & find pts where it vanishes. ($D_x f(x) = 0$)
- 2 - " $D_x^2 f(x)$ & evaluate it for this same set of pts.

Now a test for pt. of inflection. that works the other way



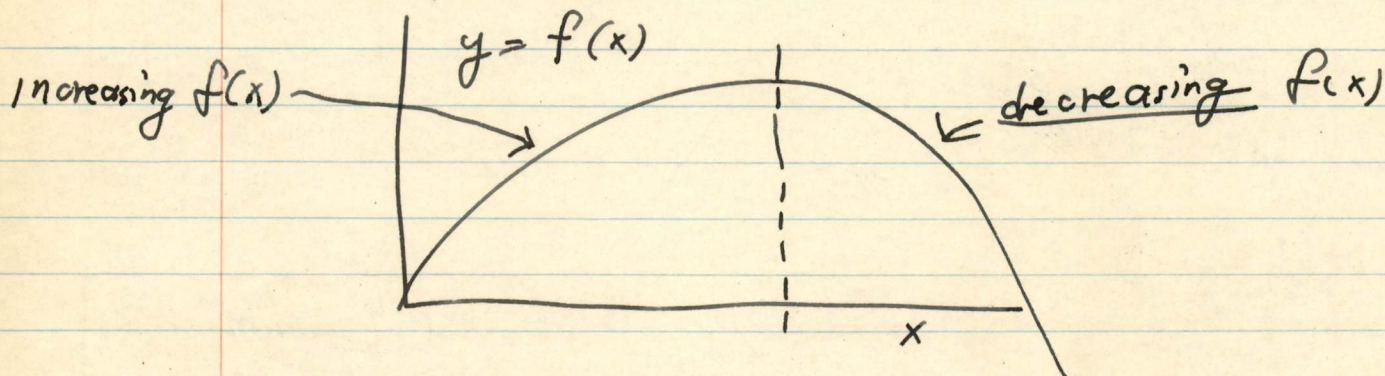
Pt. where f stops decreasing & starts increasing is a rel.

Ass. p. 59 - 11, 13
 p. 63 - 3, 5
 p. 64 - 9, 12

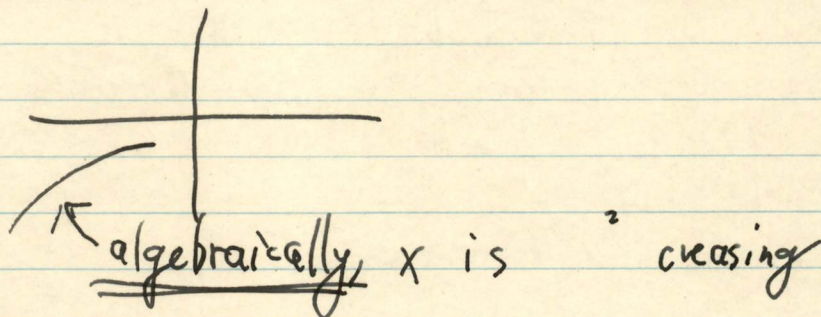
1. Find tangents 2. Find maxima & minima
 3. Find shape of curves.

$y = f(x)$

$f(x)$ is an increasing function if $f(x)$ increases as x increases.



These def.^s are for x in certain intervals



1. Restriction of x to some interval
2. Algebraic increase & decrease.

$f'(x) > 0$ when f is increasing $f'(x) < 0$ when f is decreasing.

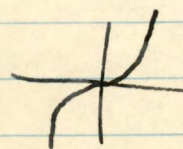
$y = f(x)$

$f(x)$ increasing when slope or Derivative positive
 $f(x)$ decreasing when " " " negative

$y = x^3$

$\frac{dy}{dx} = 3x^2$

> 0 save $x=0$.
 So it's an increasing f .



concave down because it's always turning that way. - neg. 2nd D.
 concave up - pos. 2nd D.

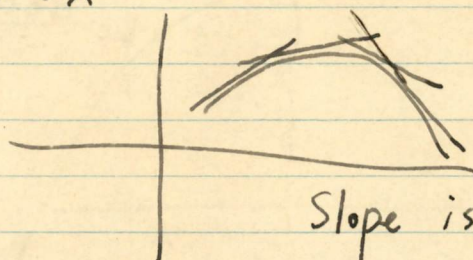
3rd
 second
 to go!

$$y = f(x)$$

$$D_x y = D_x f(x)$$

$$D_x (D_x f(x)) = D_x^2 f(x)$$

ex. $f(x) = x^3 - 3x$
 $D_x f(x) = 3x^2 - 3$
 $D_x^2 f(x) = 6x$



Slope is decreasing f .

D_x is negative

but $D_x^2 = D_x^2 f(x)$ negative

We can tell by just differentiating 2ice

Slope = $D_x y$
 Increase-decrease shown by D_x^2 .
 So use double f .

ex. $x^2 - 1$ / if $-1 < x < 1$ then $x^2 - 1 < 0$, first

P is decreasing

$x > 1$ } $f(x)$ decreasing
 $x < -1$ }

from $y = x^3 - 3x$

$$D_x y = 3x^2 - 3$$

$$D_x^2 y = 6x$$

When 2nd D is zero, we have pt. of inflection.
 Where $D_x^2 f(x) = 0$ at such a point.
 Converse not always true.

#4) work by new method

(Dec. 5, cont.)

$$y = (x-1)^3 (x+2)^2$$

$$0 \times y = (x-1)^3 \cdot 2(x+2) + (x+2)^2 \cdot 3(x-1)^2$$

$$= (x-1)^2 (x+2) \left[\frac{2x-2}{x+2} + \frac{3x+6}{x+2} \right]$$

a) $x=1$, no change of sign, neither

b) $x=-2$, it changes sign, from + to -; Rel. max.

c) $x=-\frac{4}{5}$, it changes sign, from - to +; rel. min.

Ass. By these methods. just rel. min. & max.

$$1) x^2 - 6x^2 + 12x + 4$$

$$2) (x-1)^4 (x+2)^3$$

By sign-change method, only for rel. min. & max.

$$p. 52, -6$$

$$p. 59, -7$$

$$x=1, x>1, f = + \cdot + \cdot +$$

$$x<1, f = + \cdot + \cdot +$$

$$x=2$$

$$x>1$$

Similarly for pt. of inflection.

A necessary & sufficient condition that $x=x_0$ be a pt. of inflection for $f(x)$ is that $D_x^2 f(x)$ change sign at $x=x_0$, (^{either} ~~any~~ way)

Ex $y=x^3$ $D_x y = 3x^2, = 0$ | at origin

$D_{x^2} y = 6x$, test fails

But apply other test; ~~take~~ take D_x

$$\left. \begin{array}{l} 3x^2 > 0 \text{ if } x < 0 \\ = 0 \text{ if } x = 0 \\ > 0 \text{ if } x > 0 \end{array} \right\}$$

x is neither a rel. min. nor rel. max.

$D_{x^3} y = 6 \neq 0$ at origin,

But apply other test any way.

$D_{x^2} y$ changes sign at $x=0$, it is a pt. of inflection

Ex $y=x^4$; $D_x y = 4x^3$, $D_{x^2} y = 12x^2$; $D_{x^3} y = 24x$; $D_{x^4} y = 24$; $D_{x^5} y = 0$
Everything to vanishes at $x=0$ $D_{x^6} y = 0$ etc.

Apply other test to ^{lower} ~~higher~~ D .

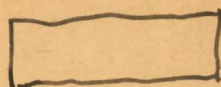
$D_x y = 4x^3$ x^3 goes from $-$ to $+$, so since $\begin{array}{l} 4x^3 < 0 \quad x < 0 \\ = 0 \quad x = 0 \\ > 0 \quad x > 0 \end{array}$
rel. min

pt. of inflection.

$D_{x^2} y$ cannot change sign.

Necessary & Sufficient conditions

Loomis, Dec. 5



1- A necessary condition that a quadrilateral be a rectangle is that opp. sides are $\parallel$.

Not sufficient. ~~with~~

2- Sufficient condition

is that it is a square

not necessary

3- Necessary & sufficient condition - 4 es be rt. $\angle$ s.

1- A necessary condition that $x = x_0$ be a rel. max. of $f(x)$ is that $D_x f(x) = 0$ _{$x=x_0$}

Not sufficient.

2- A necessary condition that $x = x_0$ be a pt. of inflection of $f(x)$ is that $D_x^2 f(x) = 0$ _{$x=x_0$}

Not sufficient

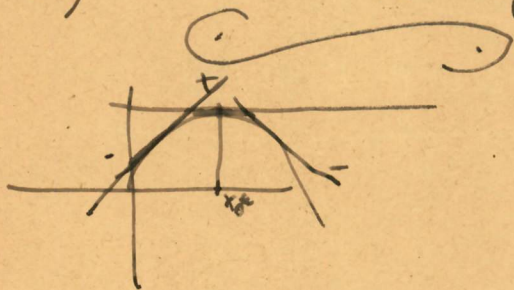
3a) A sufficient condition that $x = x_0$ be a rel. max. for $f(x)$ is that

$$1) D_x f(x) \Big|_{x=x_0} = 0$$

$$2) D_x^2 f(x) \Big|_{x=x_0} < 0$$

Not necessary

3b) - Rel. min. $D_x^2 f(x) \Big|_{x=x_0} > 0$



$$1. D_x f(x) \Big|_{x=x_0} = 0$$

2. Slope to left + pos.

3. " " right neg.

x_0 is a rel. max.

Necessary & sufficient

$D_x f(x)$ changes sign from + to - as x crosses point x_0 & min the other way.

We write it $\int_a^b f(x) dx = F(b) - F(a)$ where $F(x)$ is any integral of $f(x)$

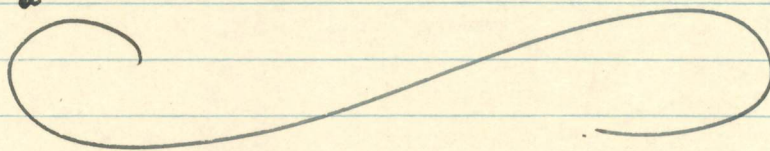
Therefore, the other integral is called the indefinite integral.

Indefinite integral is function of x
 definite " " a number

Now we can restate area theorem

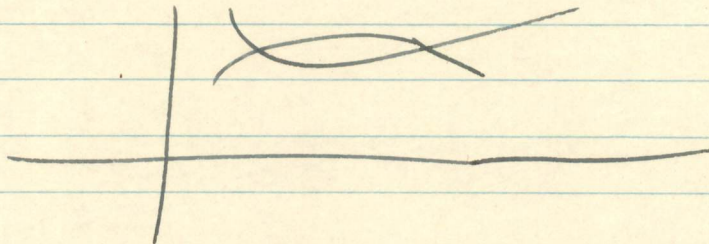
Area under $y = f(x)$ between lines $x = a$ & $x = b$ is the definite integral from a to b of $f(x)$

$$\int_a^b f(x) dx$$



If curve is under x -axis, call area negative.

Suppose we have 2 curves, & they cross.



1. Find pts. of intersection.
2. Find 2 areas.
3. Subtract.

Ass.

p. 228, - 5, 6 - miles instead of Acres
 p. 234, - 8, 11, 13, 14

Quenshapes of curves

If curve has max. in interval, take both sides. ●

1

$F(x)$ is an integral of ~~$f(x)$~~ $f(x)$
Most general " " is $F(x) + C$

Problem

1. ~~Find~~ Given $f(x)$, find $F(x)$
2. Then we want particular $F(x) = F(a) = 0$
So we Get $F_1(x)$, the desired particular integral,
choose C so that $F_1(a) + C = 0$

That is $F(x) = F_1(x) + C - F_1(a)$

$$\text{Desired Area} = F_2(b) - F_1(a) = F(b)$$

Suppose $F_1(x)$ & $F_2(x)$ are any 2 integrals of $f(x)$

$$\text{Then } F_1(b) - F_1(a) = F_2(b) - F_2(a)$$

Reason — a 2nd integral can differ at most by a constant.
 $F_2(x)$ has the form $F_1(x) + C$
Evaluate the difference.

$$F_2(b) - F_2(a) = [F_1(b) + C] - [F_1(a) + C] = F_1(b) - F_1(a)$$

Definite integral of $f(x)$ from $x=a$ to $x=b$

is $F(b) - F(a)$ where $F(x)$ is any integral of $f(x)$.

Loomis, Jan. 8

The integral of $f(x)$

Suppose $F(x)$ is an integral of $f(x)$
Every integral is of the form $F(x) + C$
The integral of $f(x)$ is $F(x) + C$

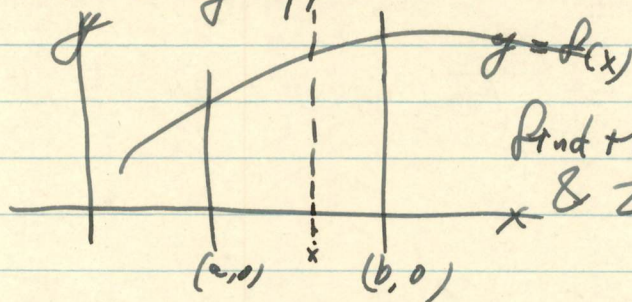
Analogy - The equation for a str. line is $Ax + By$

$+ C = 0$. Represents no particular line, just a str. one.

Same, with $F(x) + C = \underline{\text{The integral of } f(x)}$

Area under a curve -

~~most~~ only application of integration this term.



Find the area bounded by $y = f(x)$, $x = a$,
& 2 lines $\parallel$ to y -axis, $x = a$ & $x = b$

Have to use the calculus. We must have some function.

1. Make x ,
2. Consider area to left. Get F called $A(x)$
- Let $A(x)$ = area bounded by curve $y = f(x)$, the x -axis, & lines $\parallel$ to y -axis whose x -co-ordinates are a & x .
3. ~~Def~~ Derived area is $= F|_{x=a}^{x=b}$ or $F(b)$

This F is one whose derivative is $f(x)$
We discover $D_x \underset{F}{F}(x) = f(x)$ (remember squeeze play?)

Loomis - Jan. 7

Integration

1. Definition

If $f(x) = D_x F(x)$, then $F(x) \equiv$ an integral of $f(x)$

Then: if $F(x)$ is an integral of $f(x)$, then so is

$$F(x) + C$$

Then: if $F(x)$ is an integral of $f(x)$, then every ~~integral~~ of $f(x)$ has the form $F(x) + C$.

Then Theorem

$$a) \quad c \int f(x) dx = \int c f(x) dx$$

$$b) \quad c \int f(x) dx = \int c f(x) dx + C$$

proof - just differentiate
 c [any ~~integral~~ of $f(x)$] = [any ~~integral~~ of $f(x)$]
~~arbitrary~~ ~~arbitrary~~
 ~~$c f(x)$~~ $+ C$

Ex $3x^2$ is integral $3\left(\frac{x^3}{3}\right)$ - - - - - ?

$$\text{Take } D_x [c \int f(x) dx - \int c f(x) dx]$$

$$= D_x c \int f(x) dx - D_x \int c f(x) dx$$

$$= c D_x \int f(x) dx - D_x \int c f(x) dx$$

D_x cancels $\int dx$

$$= c f(x) - c f(x) = 0$$

So difference has $D_x = 0$, so it is C .

2nd Theorem

take 2
arbitrarily
of their depends

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

$$D_x \left[\int (f + g) dx - \int f dx - \int g dx \right] \quad \text{+C}$$

$$= D_x \int (f + g) dx - D_x \int f dx - D_x \int g dx$$

$$= (f + g) - f - g = 0$$

$\therefore$ the difference is constant.

3rd Theorem

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

With these 3 we can handle all polynomials

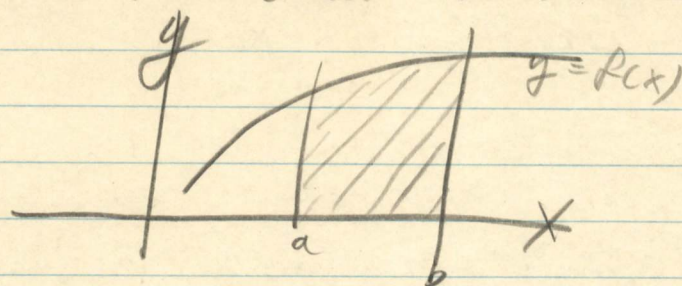
$$\text{Ex. } \int (4x^2 + 6x + 2) dx = \int 4x^2 dx + \int 6x dx + \int 2 dx$$

$$+ C_1 = 4 \int x^2 dx + 6 \int x dx + 2 \int 1 dx + C$$

$$= \frac{4}{3} x^3 + \frac{6}{2} x^2 + \frac{2}{1} x + C$$

Practical problem

Find area under a curve



1. Ask for area between a & moving pt. x

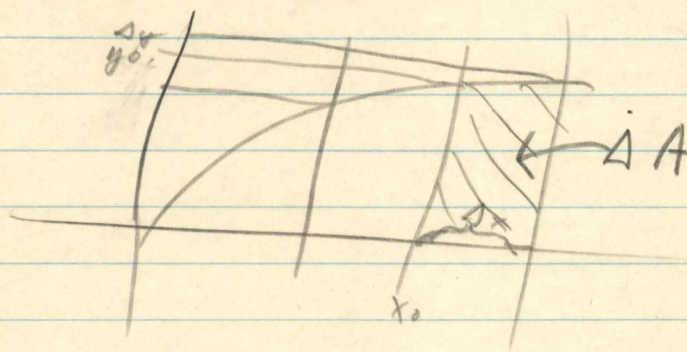
$A(x)$ = area under $y = f(x)$ from $x=a$ to $x=x$

Move x to right, area gets larger.

Area between $x=a$ & $x=b$, $= A(b)$

Solution

Take D_x of $A(x)$



$$y_0 \Delta x < \Delta A < \Delta x (y_0 + \Delta y) \quad \text{Now } \div \Delta x$$

$$y_0 < \frac{\Delta A}{\Delta x} < y_0 + \Delta y$$

Let $\Delta x \rightarrow 0$

a) $\frac{\Delta A}{\Delta x} \rightarrow D_x A(x)$ - def.

b) $\Delta y \rightarrow 0$

$y_0 = D_x A$ because

$$y_0 \leq D_x A(x) \leq y_0$$

i.e., $D_x A$ gets squeezed

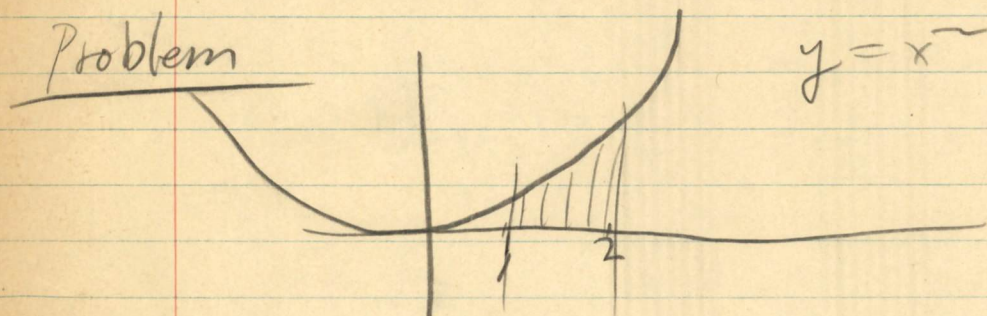
Ass. p. 234 - 3, 6
p. 228 - 2, 3

$$f(x) = D_x A(x)$$

$A(x)$ is an integral of $f(x)$

So to find area under curve, take ^{an} integral.

An integral of f chosen so that $A|_a = 0$



1. Integrate f i.e., ~~find~~ $\int x^2 dx = \frac{x^3}{3} + C$

2. Must ~~be~~ $= 0$ at 1

$$\frac{x^3}{3} + C = 0 \text{ when } x=1$$

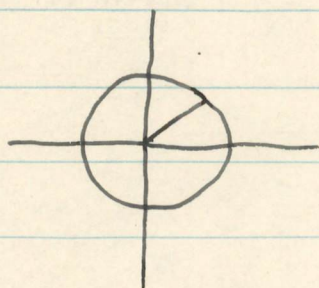
$$\frac{1}{3} + C = 0, \quad C = -\frac{1}{3}$$

$$A = \frac{x^3}{3} + -\frac{1}{3}, \text{ evaluate at } x=2$$

$$\frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Circle

②



$$x^2 + y^2 = r^2$$

General ~~Circle~~ $(x - \alpha)^2 + (y - \beta)^2 = r^2$

$$x^2 + y^2 + Ax + By + C = 0$$

$$\frac{A^2 + B^2 - 4C}{4} = r^2$$

$$A^2 + B^2 - 4C > 0, \rightarrow \text{circle}$$

$$A^2 + B^2 - 4C = 0, \rightarrow \text{pt.}$$

$$A^2 + B^2 - 4C < 0, \rightarrow \text{nothing}$$

$$\alpha = -\frac{A}{2}, \beta = -\frac{B}{2}$$

Tangents - $(x_1 - \alpha)(x - \alpha) + (y_1 - \beta)(y - \beta) = r^2$

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1 \quad - \text{tangent to ellipse.}$$

$x^2 + y^2 + Ax + By + C = 0$
find circle thru 3 points.
(forget orthogonality.)

Distance from pt. to line

$$D = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \quad \text{forget + proof}$$

Area of triangle (forget proof)

$$A = \frac{1}{2} |(y_1 - y_2)x_3 + (y_2 - y_3)x_1 + (y_3 - y_1)x_2|$$

3 lines on a pt.

3 pts on a line.

3 Medians of $\triangle$ meet in a pt.

" altitudes " " " " " "

" 1 bisectors " " " " " "

these 3 pts. lie on a line

Remember to choose axes skillfully.

Review - st. line & circle

Easiest

Slope of a st. line - $\frac{y_2 - y_1}{x_2 - x_1} = \tan \theta$

Distance between 2 pts. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Mid point formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Know derivations & proofs (easier ones)

Harder

Directed line segments (vectors)

Proportional division of line segment.

Straight line

a) 2pt. form $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ gives eqn of line
in terms of 2 pts. $(y_2 \neq y_1)(x_2 \neq x_1)$

b) pt. - slope form ~~$y - y_1 = m(x - x_1)$~~ $y - y_1 = d(x - x_1)$

c) slope-intercept form - $y = dx + b$

d) double intercept form $\frac{y}{b} + \frac{x}{a} = 1$

e) $x = c, y = c$ lines || to axes

General equation of 1st degree

$$Ax + By + C = 0 \quad (A^2 + B^2 > 0)$$

|| & $\perp$ lines

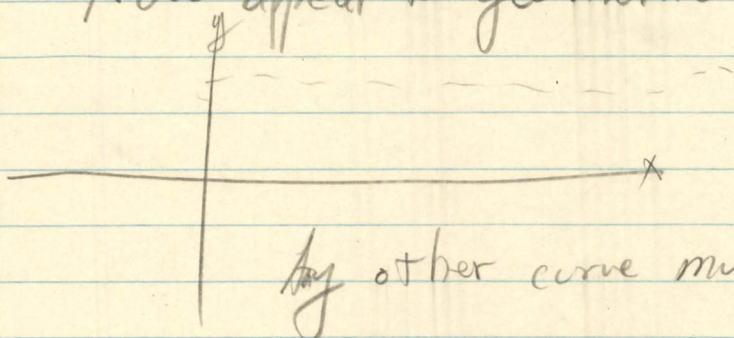
$$\perp \quad a_1 = -\frac{1}{a_2} \quad \text{or} \quad a_1 a_2 = -1 \quad (a_1^2 + a_2^2 > 0)$$

$$D_x \int f(x) dx = f(x)$$

$$\int [D_x f(x)] dx = f(x) + C$$

If a function $y = g(x)$ has derivative ^{always} $= 0$
 $\{D_x g(x) = 0 \text{ for all } x\}$, then $y = c$

Now appeal to geometric reasoning



Any other curve must have pos. or neg. d.

$$\text{So } F_1(x) - F_2(x) = c$$

$$x^n = D_x \left\{ \frac{x^{n+1}}{n+1} + c \right\}$$

Terminology If $F(x)$ is an integral of $f(x)$, then
 $F(x) + c = \int f(x) dx$

Thm. 3:

$$\int c f(x) dx = c \int f(x) dx$$

Thm. 4:

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

not the different

$\int dx$ is ambiguous - so what does $\int dx =$

$\int dx$ mean?

$\int = \int$
 If we choose an integral of one side it = that on other side.

$$= \frac{x^{\frac{4}{3}}(x-x^3)^{\frac{1}{3}}(1-3x^2) - (x-x^3)^{\frac{4}{3}}4}{x^5}$$

$$= \frac{\left[x^{\frac{4}{3}}(1-3x^2) - 4(x-x^3) \right] (x-x^3)^{\frac{1}{3}}}{x^5}$$

$$= \frac{\left(\frac{4}{3}x \cancel{4x^4} - 4x \cancel{4x^3} \right) (x-x^3)^{\frac{1}{3}}}{x^5}$$

$$= \frac{-2\frac{2}{3}x(x-x^3)^{\frac{1}{3}}}{x^5} = \frac{-2\frac{2}{3}(x-x^3)^{\frac{1}{3}}}{x^4}$$

Calculus

Chap. III - Applications.

1. Equation of tangent to a curve at (x_0, y_0) is

$$y - y_0 = (y')_0 (x - x_0)$$

Take 2 boats

20 mph

↓
40 N.

-(40 - 20t) + his leg

→
18 m.p.h.
(18t)

Then compute diagonal

$$d = \sqrt{(40 - 20t)^2 + (18t)^2}$$

~~Implicit~~ Implicit method

$$2 \frac{d}{dt} d = 2(40 - 20t)(-20) + 36t$$

$$= -600 + 800t + 36t$$

$$\frac{d}{dt} d = \frac{1448 - 1600}{\sqrt{(40 - 20t)^2 + (18t)^2}}$$

evaluate at any value for t.

$$t = 1, \quad \frac{-76}{\sqrt{724}}, \quad \text{sign means they're getting}$$

When does hyp. get smaller? at $d/dt = 0$, closer.
(also with a value for t)

Book calls D. Th "rate of increase". If you have a - rate of increase, it's a decreasing f.

1. Sign tells if it's getting larger or smaller
2. Abs. value tells how fast.

p. 78 — 1, 3, 5

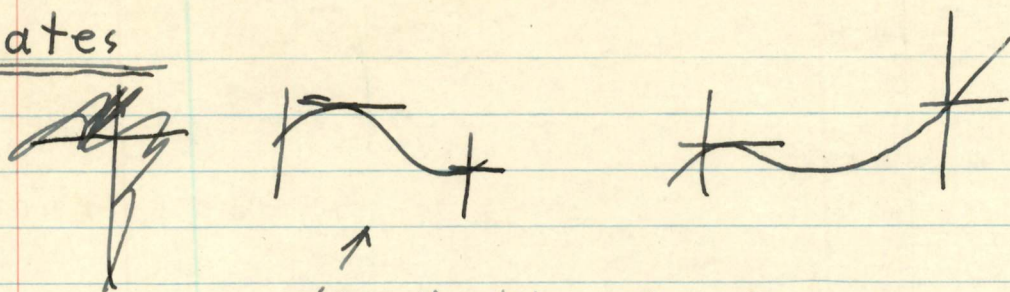
Scratch

$$ds = \frac{-a^2 d(atc)^2}{(a+tc)^4} = \frac{-a^2 2atc d(atc)}{(a+tc)^3} = \frac{-a^2 2dt}{(a+tc)^3}$$

$$\frac{dv}{dx} = \frac{x^{4/3} (x-x^3)^{1/3} (1-3x^2)}{x^8} \cdot \frac{1}{6} = \frac{1}{12}$$

EXAM. next MON.

Rates



Usually like This, but always test end points.
Even if it has several $Dx=0$ in the interval, we can tell which is greater

1. Calculus - no analyt. & rates •
Theory & application mostly
derivation of general procedure of differentiation

Rates:

If one vble. is $f(\text{time})$

Suppose $y = f(t)$

Average rate of change from t to $t + \Delta t$,
$$\frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Calculus will give instantaneous ~~time~~ rate.

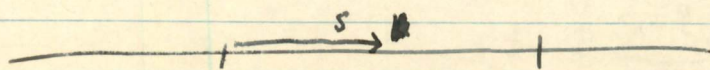
Rate of change of y at time t is the Derivative

Def $D_t f(t) = \text{rate of change at time } t$.

The limit as $\Delta t \rightarrow 0$, of the average rate.

Falling body, dropped from rest.

Drop at time $t=0$
 $s = 16t^2$



If object approaches point, it has decreasing or neg. f .
V after 1 second = 32 ft. per second.

Loomis, Jan. 5

Ass. p. 103 - 6, 7, 8, 12, 16,
19, 20, 21, 23,
24

Differentials, then integration.

~~Definition~~ $y = f(x)$
 $dy = df(x) \equiv D_x f(x) \cdot \Delta x$

In particular $f(x) = x$
~~then~~ $dx = \Delta x$

rewrite original def. by virtue of above

$$dy = df(x) = D_x f(x) \cdot dx$$

$$\frac{dy}{dx} = D_x f(x)$$

One main theorem - The equation $dy = D_x f(x) \cdot dx$ holds even when x is not the independent vble, but is for a 3rd vble, $x = g(t)$, $y = f(g(t)) = h(t)$

$$f(t) = (t^3 + t^2)^{\frac{3}{2}}$$

$$df(t) \approx \frac{3}{2} (t^3 + t^2)^{\frac{1}{2}} d(t^3 + t^2) \\ \{ 3t^2 dt + 2t dt \}$$

Forget about infinitesimals.

" " differentials of higher order

Integration

Find F of which given f is derivative

Problem - Given $f(x)$ to find $F(x)$ such that

$$D_x F(x) = f(x)$$

$$F \equiv \underline{\text{An integral of } f(x)}$$

Ex. ~~find~~ integrate $4x^3$

$$4x^3 = D_x x^4$$

x^4 is an integral of $4x^3$

ex. $x^3 = D_x \frac{x^4}{4}$

$$x^n = D_x \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

Why an integral?

$x^4 + \text{any constant}$ are all integrals of $4x^3$

Generally,

$$x^4 + c$$

Theorem I: (trivial)

If $F(x)$ is an integral of $f(x)$, then so is $F(x) + c$.

Theorem II: If $F(x)$ is an integral of $f(x)$, then every integral is of form $F(x) + c$

pseudo-proof: Suppose $F_1(x), F_2(x)$ are integrals of $f(x)$. To prove that $F_2(x) = F_1(x) + c$ or $F_2(x) - F_1(x) = c$

$$\text{hyp. } \begin{cases} D_x F_1(x) = f(x) \\ D_x F_2(x) = f(x) \end{cases}$$

of the difference

$$D_x [F_2(x) - F_1(x)] = D_x F_2 - D_x F_1 = f(x) - f(x) = 0$$

The catch - Now prove if f has $D=0$ it is c .

Ass. - p. 99; 1, 2, 6, 7
p. 103; 10, 11, 17, 18, 23, 25

Now use differential notation instead of derivative.

I. $y = u + v$ $D_x(u + v) = D_x u + D_x v$

multiply by dx

$$D_x(u + v) dx = D_x u \cdot dx + D_x v \cdot dx$$

$$d(u + v) = du + dv$$

Exactly the same thing happens to all other rules.

II ~~derivative~~ $d(cu) = c du$

III $d(uv) = u dv + v du$

IV ~~derivative~~ $d\frac{u}{v} = \frac{v du - u dv}{v^2}$

~~derivative~~ - - - - -

V ~~derivative~~ $d(c) = 0$

VI $d(x^a) = a x^{a-1} dx$

This makes differentiation easier. All trivial. Hold by definition of ~~derivative~~ "d"

Ex. $y = \frac{(x^2 - a^2)^{\frac{3}{2}}}{x^3}$ Find $d(y)$

$$d(y) = \frac{x^3 d(x^2 - a^2)^{\frac{3}{2}} - (x^2 - a^2)^{\frac{3}{2}} dx^3}{x^6} = 3x^2 dx$$

We have $\frac{[x^3 \cdot \frac{3}{2}(x^2 - a^2)^{\frac{1}{2}} \cdot 2x - (x^2 - a^2)^{\frac{3}{2}} 3x^2] dx}{x^6}$

Divide by dx $\frac{dy}{dx} = D_x y = \frac{[x^4 - 3x^4 + 3a^2 x^2](x^2 - a^2)^{\frac{1}{2}}}{x^6}$

$$= \frac{3a^2 - 2x}{x^4} (x^2 - a^2)^{\frac{1}{2}}$$

Keep going till you get dx , factor it out & divide by it.

$y = (1 + x + x^2)^{\frac{1}{3}}$; $dy = d\{(1 + x + x^2)^{\frac{1}{3}}\} = \frac{1}{3}(1 + x + x^2)^{-\frac{2}{3}} d(1 + x + x^2)$

Get dx out, & divide by dx $\frac{dy}{dx} = D_x y = \frac{1}{3}(1 + x + x^2)^{-\frac{2}{3}} (1 + 2x)$

$x^3 - 2xy + y^5 = 0 \rightarrow$ Implicit function, get $dy(\text{---}) + dx(\text{---}) = 0$

Loomis - Dec. 17

Differentials

Def if $y = f(x)$ then $\boxed{dy \equiv D_x y \cdot \Delta x}$

dy is a function of 2 variables.

$$y = x^3; dy = 3x^2 \cdot \Delta x$$

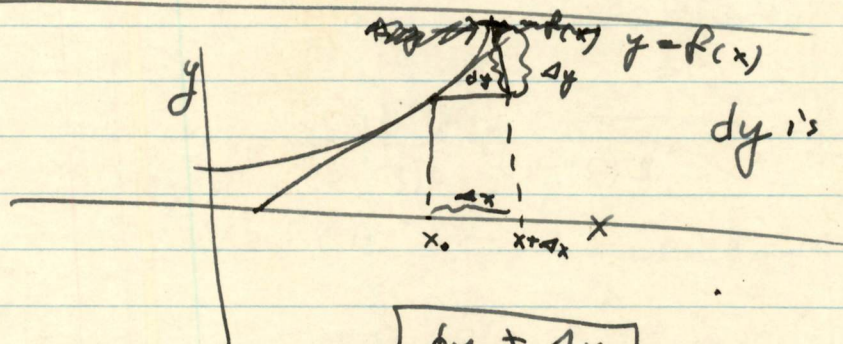
Theorem; $f(x) = x$ then
 $\boxed{df(x) = dx = 1 \cdot \Delta x}$

Corollary If $\Delta x = dx$

$$df(x) = dy = D_x y \cdot dx$$

$$dg(x) = dy = D_x y \cdot dx$$

holds in general, for any $f(x)$.



$$\boxed{dy \neq \Delta y}$$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$dy = D_x y \cdot \Delta x$$

If x 's independent vble, then
 $\Delta x = dx$

Cor. $D_x y = \frac{dy}{dx}$; by virtue of this definition

Function of function. $x = g(t); y = f(x) = f(g(t))$

$$D_t y = D_x y \cdot D_t x$$

Key Theorem The equation $dy = D_x y \cdot dx$ is true, even if y & x are both dependent variables, both $f(t)$

i.e.

$$x = g(t)$$

$$y = f(x) = f(g(t))$$

Def. $dy = D_t y \cdot dt = D_x y \cdot D_t x \cdot dt$, these 2 by def. = dx so $= D_x y \cdot dx$

~~the~~ Corollary: The derivative of the sum of any number of functions = the sum of their derivatives.

7. III. The derivative of a product is given by the formula:

$$D_x(uv) = u D_x v + v D_x u$$

Proof (typical of all these theorems)

1. let $y = uv$

2. then $y_0 = u_0 v_0$

3. $y_0 + \Delta y = (u_0 + \Delta u)(v_0 + \Delta v) = \cancel{u_0 v_0} + \Delta u v_0 + \Delta v u_0 + \Delta u \Delta v$

subtract (2)

$$\Delta y = \Delta u v_0 + \Delta v u_0 + \Delta u \Delta v$$

$$\frac{\Delta y}{\Delta x} = v_0 \frac{\Delta u}{\Delta x} + u_0 \frac{\Delta v}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

The last one drops out, because $\lim \Delta u = 0$

The first 2 are $v D_x u$ & $u D_x v$ q.e.d.

Corollary. $D_x(uvw) = (uvw) \left(\frac{D_x u}{u} + \frac{D_x v}{v} + \frac{D_x w}{w} \right)$

IV. The derivative of a quotient is given by the formula:

$$D_x \left(\frac{u}{v} \right) = \frac{v D_x u - u D_x v}{v^2}$$

By the way $D_x x^n = n x^{n-1}$ when n is a neg. integer.

8. V. If u is expressed as a function of y , & y in turn as a function of x , then:

$$D_x u = D_y u \cdot D_x y$$

9. Differentiation of implicit algebraic functions. Do it.

10. $D_x x^{\frac{p}{2}}$

$$y = x^{\frac{p}{2}}$$

$$y^2 = x^p$$

$$2y^{2-1} D_x y = p x^{p-1}$$

$$D_x y = \frac{p x^{p-1}}{2 y^{2-1}}; \quad y = x^{\frac{p}{2}}$$

$$D_x y = \frac{p x^{p-1}}{2 (x^{\frac{p}{2}})^{2-1}} = \frac{p}{2} \cdot \frac{x^{p-1}}{x^{p-\frac{p}{2}}} = \frac{p}{2} x^{p-1-p+\frac{p}{2}}$$

$$= \frac{p}{2} x^{\frac{p}{2}-1}$$

Calculus

Chap II - Differentiation of algebraic functions.

1. The limit which the ratio $\frac{\Delta y}{\Delta x}$ approaches when Δx approaches 0:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad \text{or} \quad \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called the derivative of y with respect to x and is denoted by $D_x y$.

2. Differentiation (determining derivative)

1. $y = x^2$

2. $y_0 = x_0^2$

3. $y_0 + \Delta y = (x_0 + \Delta x)^2$

4. $y_0 + \Delta y = x_0^2 + 2x_0\Delta x + \Delta x^2$

Subtract (2)

$$\Delta y = 2x_0\Delta x + \Delta x^2$$

Divide by Δx

$$\frac{\Delta y}{\Delta x} = 2x_0 + \Delta x$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 2x$$

$$D_x x^n = n x^{n-1}$$

3. Derivative of a constant = 0

4. " " $\sqrt{x} = \frac{1}{2\sqrt{x}}$

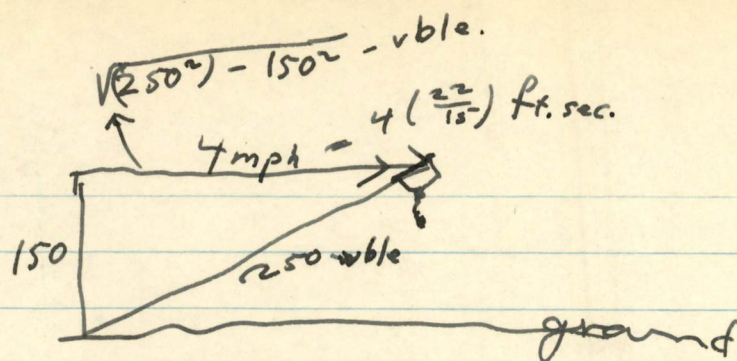
~~5. I. Limit of the sum of 2 vbles. = the sum of their limits.~~

~~The limit of the sum of any fixed number of vbles. = the sum of the limits of these vbles.~~

6. I. The derivative of the product of a constant & a function = the product of the constant & the derivative of the function; $D_x(cu) = c D_x u$

II. The derivative of the sum of 2 functions = the sum of their derivatives; $D_x(y + v) = D_x y + D_x v$

Another ex.

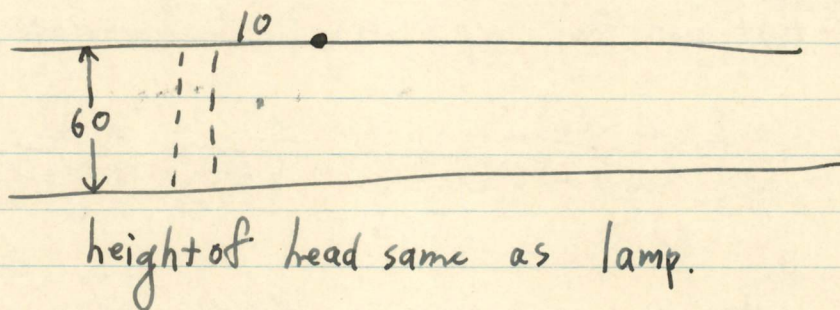


How fast is cord going at that instant.

Get derivative as value of diagonal.

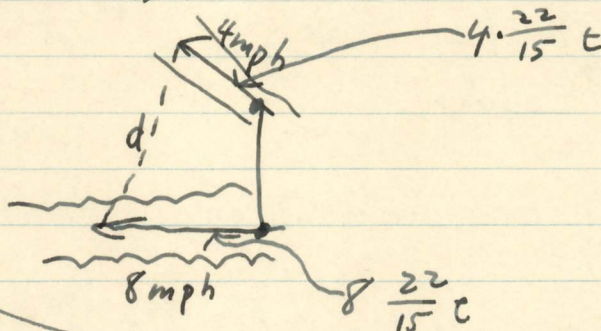
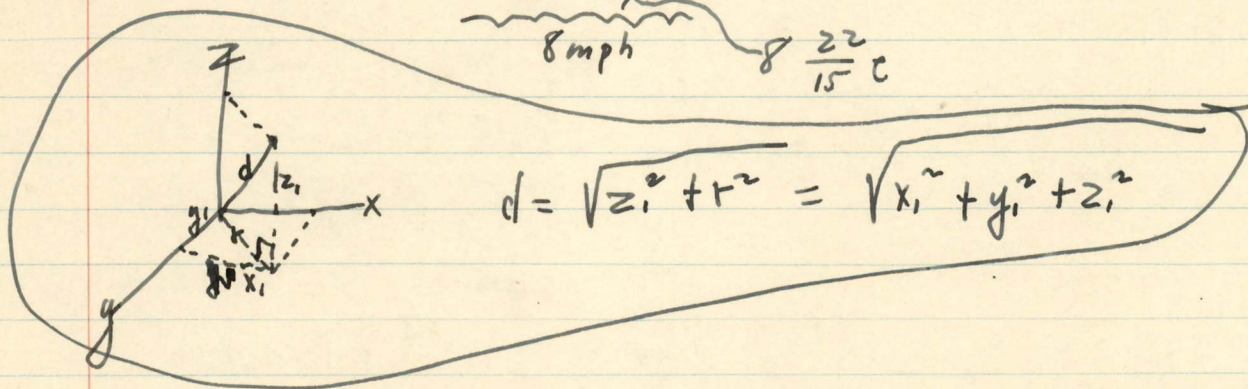
Ass. - p. 79, - ~~8~~, 10, 12

12.



Next hour review.

9. Three dimensions

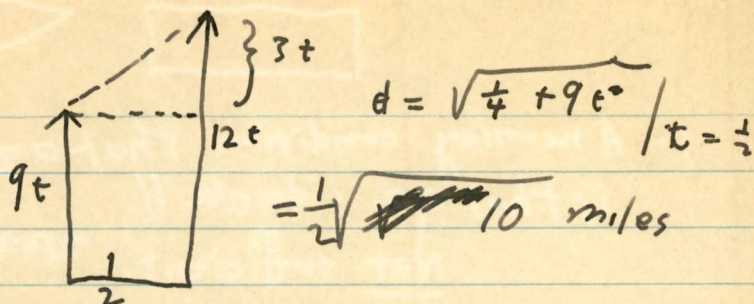


$$d = \sqrt{400 + 64 \left(\frac{22}{15} t\right)^2 + 16 \left(\frac{22}{15} t\right)^2} = \sqrt{400 + 80 \left(\frac{22}{15} t\right)^2}$$

$$P \text{ of } d = \frac{1}{2} \frac{160 + \left(\frac{22}{15}\right)^2 t}{\sqrt{400 + 80 \left(\frac{22}{15} t\right)^2}} \quad \left| \frac{d}{dt} = \right.$$

$t = 180 \text{ seconds.}$

p. 79, no. 3



How fast are they separating? Differentiate

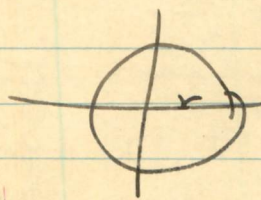
$$D_t d = \frac{1}{2} \frac{9 \cdot 2t}{\sqrt{\frac{1}{4} + 9t^2}} \Big|_{t=\frac{1}{2}} = \frac{\frac{9}{2}}{\sqrt{\frac{10}{4}}} = \frac{9}{\sqrt{10}} \text{ m.p.h.}$$

sign is +, so d is increasing, ~~distance~~ distance gets greater.

To find rate of change, differentiate d with respect to t .

$$\begin{aligned} \text{velocity} &= D_t s \\ \text{speed} &= |D_t s| \end{aligned}$$

rate is time-derivative of any function of time.
p. 79, no. 7.



$$\begin{aligned} r &= 6t \\ A &= \pi r^2 = \pi 36t^2 \\ \boxed{D_t A \Big|_{t=2}} & \text{ we want} \end{aligned}$$

$$D_t A = 72\pi t = 144\pi$$

$$r = f(t)$$

$$D_t r = 6$$

$$r = 4$$

Now

$$r = f(t)$$

$A = \pi r^2$ How fast does A change when $r = 4$
Function of function situation.

$$D_t A = D_r A \cdot D_t r = 2\pi r \cdot \boxed{D_t r} \quad \& \text{ sub.}$$

$$D_t A \Big|_{r=4} = 2\pi \cdot 4 \cdot 6 = 48\pi$$

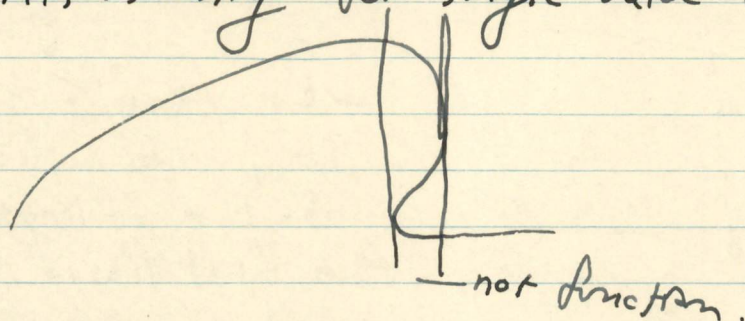
Restrict $|x| < 4$ arbitrarily

~~Then $|x+2| < 6$~~

Then if $|x-2| < \delta = \frac{\epsilon}{6}$

Then $|x^2-4| = |x-2||x+2| < \frac{\epsilon}{6} \cdot 6 = \epsilon$

This is only for single-value functions.



$f(x)$
 $g(x)$
 $h(x)$

$g(x) \rightarrow A, \text{ as } x \rightarrow a$ } hypotheses
 $h(x) \rightarrow B, \text{ as } x \rightarrow a$

I take $f(x)$ as sum of those 2 functions $[f(x) = g(x) + h(x)]$

Then $f(x) \rightarrow (A+B) \text{ as } x \rightarrow a$

take a value of ϵ . To find $\delta: |f(x) - (A+B)| < \epsilon$ whenever $|x-a| < \delta$

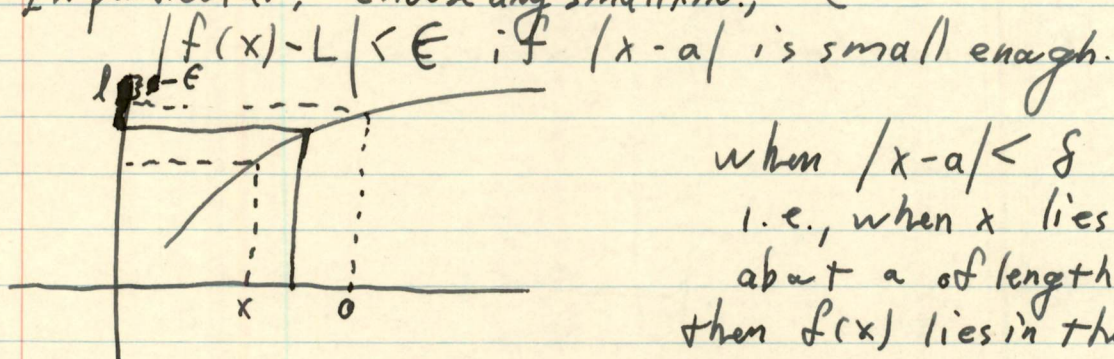
$$|a+b| \leq |a| + |b|$$

Loomis Nov.

Limits

$$f(x) \rightarrow L \text{ as } x \rightarrow a \quad \text{or} \\ \lim_{x \rightarrow a} f(x) = L$$

$|f(x) - l|$ is as small as we wish, if $|x - a|$ is small enough.
In particular, choose any small ^{pos.} ϵ



when $|x - a| < \delta$
i.e., when x lies in the interval
about a of length 2δ
then $f(x)$ lies in the given
interval about L
i.e., then $|f(x) - L| < \epsilon$

True for picture:

given ϵ : there is a δ such that
whenever $|x - a| < \delta$, then $|f(x) - L| < \epsilon$

Def Mathematical Def. of a limit
 $f(x) \rightarrow l$ as $x \rightarrow a$ or
 $\lim_{x \rightarrow a} f(x) = l$

is to mean for every ^{at least one} pos. ϵ , there is a pos. δ
such that $|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$

Alternate Def. there is a ^{pos.} function, $g(\epsilon)$, such that $|f(x) - l| < \epsilon$ whenever $|x - a| < g(\epsilon)$

Ex. $x^2 \rightarrow 4$ as $x \rightarrow 2$?

given ϵ , to find a δ such that $|x^2 - 4| < \epsilon$
whenever $|x - 2| < \delta$

$$|x^2 - 4| = |x + 2| |x - 2| < \epsilon$$

Find a δ such that $|x - 2| < \delta$ will give above.

$$D_y = \frac{1}{\sqrt{y}} \cdot \frac{1}{2}$$

multiply by $-2x! - D_y(x)$

$$\frac{\frac{1}{2} - 2x}{\sqrt{y}} = \frac{\frac{1}{2} - 2x}{\sqrt{a^2 - x^2}}$$

Ass. p. 35 - 8, 12 4 steps
 p. 37 - 1, 2, 5, 7, 9, 15, 20 f of f process
 fix prob. 2

$$\left[x^2 + 2x\Delta x + \Delta x^2 + px + p\Delta x + q \right]$$

$$\left[x^2 + px + q \right] \left[x^2 + px + q \right]$$

$$2(5+3) \quad \begin{array}{r} 10+6 \\ 16 \end{array}$$

$$(2+4)(5+3) = \begin{array}{r} 15+9 \\ 9 \\ 20 \\ 12 \\ \hline 56 \end{array}$$

$$x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3$$

$$\begin{array}{cccc} & & 1 & \\ & 1 & 1 & \\ & 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{array}$$

$$x^2 + 2x\Delta x + \Delta x^2$$

Find rule.

1. Δu but u refers either to y or x . f shouldn't occur at all, but we must differentiate w.r.t respect to y

$$\Delta u = f(y + \Delta y) - f(y)$$

$$\frac{\Delta u}{\Delta x} = \frac{\Delta u}{\Delta y} \cdot \frac{\Delta y}{\Delta x} = \frac{f(y + \Delta y) - f(y)}{\Delta y} \cdot \frac{\Delta y}{\Delta x}$$

Let $\Delta x \rightarrow 0$ $g(x + \Delta x) - g(x)$
 Δy will also $\rightarrow 0$
 $\Delta y \rightarrow 0$ as $\Delta x \rightarrow 0$

$D_x u$
 or $F'(x)$
 or $f'(g(x))$

$=$

$D_y(f(y))$

$D_x g$ or
 $(D_x g(x))$

$D_x F(x)$

$$\boxed{D_x u = D_y u (D_x y)}$$

$$D_x(f(g(x))) = D_y(f(y)) D_x g(x)$$

where $y = g(x)$

write without ambiguity if we use f'

$$\boxed{D_x f(g(x)) = F'(x) = f'(g(x)) g'(x)}$$

Example - $u = \sqrt{a^2 - x^2}$

Let $y = a^2 - x^2$

$u = \sqrt{y}$
 $y = a^2 - x^2$ } we have a double sequence, can apply rule.

We ought to have D of f , regardless of vble.

f' or $f'()$ indicates derivative without vble.

How To evaluate Derivative of $f()$ at $3x+4$?

$D_{(3x+4)} f(3x+4)$, clumsy & confusing

or
 $D_y f(y) \big|_{y=3x+4}$ - lousy

instead of these, write

$f'(3x+4)$

$f' = D_x f()$

evaluate the derivative function of f at $3x+4$

Example

$f() = ()^3$, then D_x

$f'() = 3()^2$

we just put $3x+4$ in the blank

$$f'(3x+4) = 3(3x+4)^2$$

f is a function.

Reason - suppose you had 2 functions - $u = f(y)$; $y = g(x)$
related in this way - u is a function of x .

$$u = F(x) = f(g(x))$$

ex. let $u = y^3$; $y = 4x$

$$y^3 = f(y); \quad 4x = g(x)$$

$$y^3 = f(4x)$$

$$u = 4x^3$$

replace y by $4x$ - - - - -

How to differentiate it?

$$D_x f(g(x)) = f'(g(x)) g'(x)$$

Loomis

Nov. 14

$$\frac{\sin x (1-x^3)}{\sqrt{1-x^2} \sqrt[3]{x}} - \text{easier to use product rule}$$

$$D_x(x^n) = nx^{n-1} \quad \left\{ \begin{array}{l} \text{when } n \geq 0 \text{ (an integer)} \\ \text{now let's verify it for } - \text{ integers} \end{array} \right.$$

let n be $-$; $n = -m$ (m pos.)

$$x^n = x^{-m} = \frac{1}{x^m}$$

use quotient rule

$$D_x(x^n) = D_x\left(\frac{1}{x^m}\right) = \frac{x^m D_x(1) - 1 D_x x^m}{x^{2m}} = \frac{-m x^{m-1}}{x^{2m}}$$

$$= -m x^{-m-1} \quad \text{replace } -m \text{ by } n = n x^{n-1} \quad \text{QED}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$D_x(x^{\frac{1}{2}}) = \frac{1}{2} x^{-\frac{1}{2}}$$

$$D_x \frac{1}{\sqrt{x}} = -\frac{1}{2} \frac{1}{\sqrt{x^3}}$$

$$D_x(x^{-\frac{1}{2}}) = -\frac{1}{2} x^{-\frac{3}{2}}$$

$$D_x(x^{\frac{p}{2}}) = \frac{p}{2} x^{\frac{p}{2}-1}$$

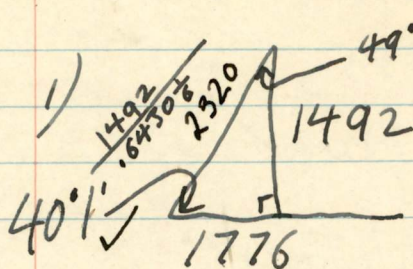
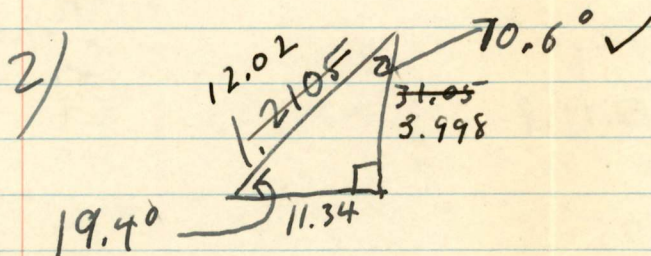
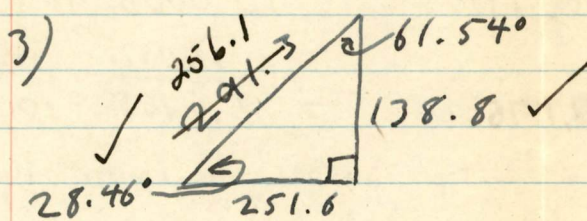
Rules yet to go. Easiest to write down, hardest to write down.
Notation for functions:

$f(x)$ same function, different independent variables.
 $f(y)$

$$D_x f(x), D_x f(y)$$

again the same function

For Oct. 21



FOR Oct 23

$$\begin{array}{r} 1710 \\ 1694 \\ \hline 16 \end{array}$$

$$\begin{array}{r} 1707 \\ 4579 \\ 4564 \\ \hline 12 \\ \hline 4576 \end{array}$$

$$\begin{array}{r} 516 \\ 517 \end{array}$$

$$\begin{array}{r} 90.00 \\ 22.77 \\ \hline 67.23 \end{array}$$

$$1) \tan 55.98^\circ = \frac{BQ}{286.8} \quad \log BQ = \log \tan 55.98^\circ - \log 286.8$$

$$3.1707 - 3$$

$$- 2.4576$$

$$.7131 - 3 = \log 5160.25$$

$$\tan 67.23 = \frac{AQ}{286.8} \quad \log AQ = \log \tan 67.23^\circ - \log 286.8$$

$$3.3770 - 3$$

$$- 2.4576$$

$$.9194 - 3 = \log 8,305.$$

$$AB = AQ - BQ$$

$$\begin{array}{r} 8,305. \\ - 5,160. \\ \hline 3,145. = AB \end{array} \quad X$$

$$AB = 258.3$$

$$2) \frac{x}{386} = \cos 27.4^\circ; \log x = \log 386 + \log \cos 27.4^\circ$$

$$2.5866$$

$$\frac{0.9483 - 1}{\cancel{3.5349} - 1} = \log \cancel{77.63} \quad 3423 \checkmark$$

$$b) \frac{x}{386} = \cos 56^\circ; \log x = \log 386 + \log \cos 56^\circ$$

$$2.5866$$

$$\frac{.7476 - 1}{3.3342 - 1} = \log 215.9 \checkmark$$

$$3) \tan 47.86^\circ = 1.105$$

$$\frac{2.105}{\log 1 - \log 2.105} \quad 10.0000 - 10$$

$$- .3232$$

$$= 9.6768 - 10$$

~~47.6%~~ $\log 1.4750$ velocity pushes horizontally
52.4% " " " vertically

$$.476 \times 1167 = 555.05$$

$$783.1$$

$$.524 \times 1167 = 611.95$$

$$865.3$$

FOR OCTOBER 28

$$1) \frac{1.4687}{0.2732}$$

$$.8742 - 2$$

$$\frac{2.6161 - 2}{0.6161}$$

$$0.6161$$

$$0.8524$$

$$0.6193 - 4$$

$$4.9746$$

$$6.4463 - 4$$

$$2.4463$$

$$10.6161 - 10$$

$$- 2.4463$$

$$8.1698 - 10 = \log \boxed{0.14783} \quad X$$

$$.01860$$

October 28 homework, (cont.)

2.) $2 \times \log 3.142 + 3 \times \log 18.74 - 2 \log 0.2617 = \overset{\log}{IT}$

$$\begin{array}{r} 0.4972 \\ \hline 2 \\ \hline .9944 \end{array} \quad \begin{array}{r} 1.2728 \\ \hline 3 \\ \hline +3.8184 \end{array} \quad \begin{array}{r} 0.4178 - 1 \\ \hline 2 \\ \hline -0.8356 - 2 \end{array}$$

$$\begin{array}{r} .9944 \\ 3.8184 \\ \hline 4.8128 \\ -0.8356 - 2 \\ \hline 3.9772 + 2 = \log \boxed{94,880.0} \end{array} \quad \begin{array}{l} X \\ 51.26 \end{array}$$

3) a) $\frac{\log .7183}{3} = \log IT; \quad \boxed{\text{[scribbled out]}}$

~~$\frac{\log .8408}{3} = \log IT; \quad \boxed{\text{[scribbled out]}}$~~

$$\frac{2.8563 - 3}{3} = 0.9521 - 1 = \log \boxed{.8957} \quad \checkmark$$

b) $\frac{\log .007183}{3} = \log IT; \quad \frac{0.8563 - 3}{3} = 0.2834 - 1$
 $= \log \boxed{.19205} \quad \checkmark$

c) $\frac{\log .07183}{3} = \log IT; \quad \frac{0.8563 - 2}{3} = \frac{1.8563 - 3}{3} =$
 $0.6188 - 1 = \log \boxed{.4158} \quad \checkmark$

d) $\frac{\log .0007183}{3} = \log it; \quad \frac{7.8563 - 3}{3} =$
 $2.6188 - 1 = \log \boxed{41.58} \quad X \quad .08957$

For Oct. 30

$$1) \frac{a}{\sin A} = \frac{b}{\sin B}, \frac{218.7}{\sin 36^\circ 52'} = \frac{b}{\sin 73^\circ 9'}$$

$$\log b = \log 218.7 - \log \sin 36^\circ 52' + \log \sin 73^\circ 9'$$

$$\begin{array}{r} 2.3399 \\ - 0.7780 - 1 \\ \hline 1.5619 + 1 \\ + .9809 - 1 \\ \hline 2.5428 \end{array} = \log \boxed{349. = b}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}, \frac{218.7}{\sin 36^\circ 52'} = \frac{c}{\sin 69^\circ 59'} = c$$

$$\log c = \log 218.7 - \log \sin 36^\circ 52' + \log \sin 69^\circ 59'$$

$$\begin{array}{r} 2.5619 \\ 0.9701 - 1 \\ \hline 3.5320 - 1 \end{array} = \log \boxed{342.4} \quad \boxed{340.4 = c}$$

$$2) \frac{a}{\sin A} = \frac{b}{\sin(180-B)}, \frac{129.6}{\sin 53.41^\circ} = \frac{b}{\sin 75.23^\circ}$$

$$\log b = \log 129.6 - \log \sin 53.41^\circ + \log \sin 75.23^\circ$$

$$\begin{array}{r} 2.1126 \\ - 0.5341 - 1 \\ \hline 1.5785 + 1 \\ 0.9854 - 1 \\ \hline 2.5639 \end{array} = \log \boxed{366.7 \text{ feet.}} \times 156.1$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}, \frac{129.6}{\sin 53.41^\circ} = \frac{c}{\sin 21.82^\circ}, \text{ angle } C \checkmark$$

$$\log c = \log 129.6 - \log \sin 53.41^\circ + \log \sin 21.82^\circ$$

$$\begin{array}{r} 180.00 \\ 104.77 \\ \hline 75.23 \end{array}$$

$$\begin{array}{r} 53.41 \\ 104.77 \\ \hline 158.18 \\ 180.00 \\ \hline 21.82 \end{array}$$

For Oct. 30, (cont.)

$$\begin{array}{r} 2.5785- \\ + 0.5702-1 \\ \hline 3.1487-1 \end{array} = \log \boxed{140.8} = C \quad 59.99$$

3) a) Find $\angle B$

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \frac{29.38}{\sin 32^\circ 47'} = \frac{24.91}{\sin B}$$

$$\begin{array}{r} \log \sin B = -\log 29.38 + \log \sin 32^\circ 47' + \log 24.91 \\ + 2.3964-1 \\ - 1.4680 \\ \hline .9284-1 \\ + 0.7336-1 \\ \hline 1.6620-2 = \log \sin \boxed{27.33^\circ} = \angle B \end{array}$$

$$b) \frac{a}{\sin A} = \frac{c}{\sin(180-C)}, \quad \frac{29.38}{\sin 32^\circ 47'} = \frac{c}{\sin 60.11^\circ}$$

$$\log c = \log 29.38 + \log \sin 60.11^\circ - \log \sin 32^\circ 47'$$

$$\begin{array}{r} 27.33 \\ 32.78 \\ \hline 60.11 \end{array}$$

$$\begin{array}{r} 1.4680 \\ 0.9380-1 \\ 2.4060-1 \\ - 0.7336-1 \\ \hline 1.6724 = \log \boxed{47.04} = c \end{array}$$

$$\angle C = 119.89^\circ$$

I forgot to take down the assignment for Nov. 4.
or something. Anyway I don't have it.

For Nov. 6

1. $a^2 = b^2 + c^2 - 2bc \cos A$

$$a^2 = 43^2 + 46^2 - 2 \cdot 46 \cdot 43 \cos 29^\circ$$

$$a^2 = 505$$

$$a = 22.47 \quad \checkmark$$

2. $b^2 = a^2 + c^2 + 2ca \cos(180 - B)$

$$b^2 = 13^2 + 16^2 + 2 \cdot 13 \cdot 16 \cos 63^\circ$$

$$b^2 = 613.85$$

$$b = 24.77 \quad \checkmark$$

3. $b^2 = a^2 + c^2 - 2ac \cos B$

$$9 = 4 + 16 - 16 \cos B$$

$$\frac{11}{16} = \cos B = .6875; \quad \boxed{B = 43.43^\circ} \quad 46.6^\circ$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{4 + 9 - 16}{12} = \frac{-3}{12} = -\frac{1}{4}$$

It's negative, so the angle is obtuse, and

$$\cos(180 - C) = \frac{1}{4} = \sqrt{\frac{1}{4}}$$

$$\angle(180 - C) = 75.5^\circ$$

$$\angle C = 104.5^\circ \quad \checkmark$$

$$\angle A = 32.07$$

$$28.99$$

For Nov. 11

$$1.) \frac{a-b}{a+b} = \frac{\tan \frac{1}{2} (A-B)}{\tan \frac{1}{2} (A+B)}$$

$$\frac{a-a}{c+a} = \frac{\tan \frac{1}{2} (C-B)}{\tan \frac{1}{2} (C+B)}$$

$$\frac{21.4}{166.2} = \frac{\tan \frac{1}{2} C - \tan \frac{1}{2} 71.87^\circ}{\tan \frac{1}{2} C + \tan \frac{1}{2} 71.87^\circ}$$

$$.1289 \tan \frac{1}{2} C + .09 = \tan \frac{1}{2} C - .7247$$

$$.8147 = .8711 \tan \frac{1}{2} C$$

$$\tan \frac{1}{2} C = 1.069$$

$$\frac{1}{2} C = \text{was } 6.10$$

$$C = 12.2^\circ$$

$$A = 95.93^\circ$$

$$60.40^\circ$$

$$47.74^\circ$$

$$2) \begin{array}{r} 180.00000^\circ \\ - 54.61166^\circ \\ \hline \end{array}$$

$$125.38834^\circ - L C$$

$$\text{given } 27.305^\circ - L B$$

$$b = 9.087$$

$$27.305^\circ - L A$$

$$\begin{array}{l} \frac{b-c}{b+c} = \frac{\tan \frac{1}{2} (B-A)}{\tan \frac{1}{2} (B+C)} \\ \frac{9.087-c}{9.087+c} = \frac{\tan \frac{1}{2} (98.083^\circ - 27.305^\circ)}{\tan \frac{1}{2} (98.083^\circ + 27.305^\circ)} \end{array}$$

$$\frac{c-b}{c+b} = \frac{\tan \frac{1}{2} (C-B)}{\tan \frac{1}{2} (C+B)}$$

$$\frac{c-9.087}{c+9.087} = \frac{\tan \frac{1}{2} (98.083^\circ)}{\tan \frac{1}{2} 152.693^\circ} = \frac{\tan 49.0415^\circ}{\tan 76.346^\circ}$$

$$= .2802$$

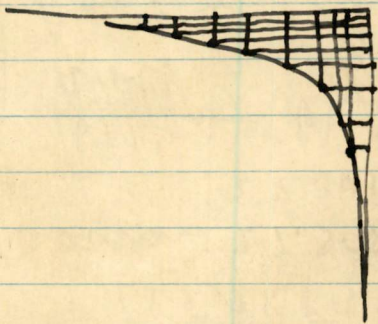
$$c - 9.08 = .28 c + 1.382$$

$$.72 c = 9.462$$

$$c = 13.14$$

$$15.12$$

	11	12	2	3
Mon		Philips	Caywood	Rickart
Tues.	Alaoglu			
Wed.	Grandos	Hewitt	Norris	
Thurs.	Whiteman	Lorenz	Putnam	



Mon. - Thurs.
2-4
Holyoke 43

M. Mason
Gaffney

~~Write Frank, summer class.~~

~~29, 31, 31-2, 32, 33-35, 36-37, 42-43.~~
~~\$ Kings, T. P. Penells, Const. + - 324-328~~
~~E.C.~~
~~Will a price increase be followed by a decrease in demand?~~

$$\frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b}$$

$$\frac{a}{b} = \tan \theta$$

$$\frac{c}{b} = \cos \theta$$

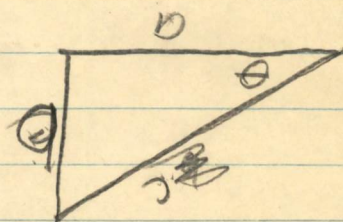
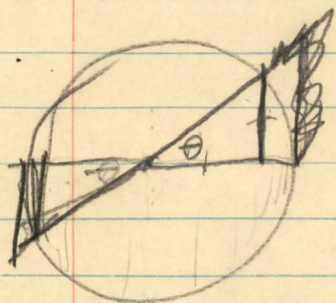
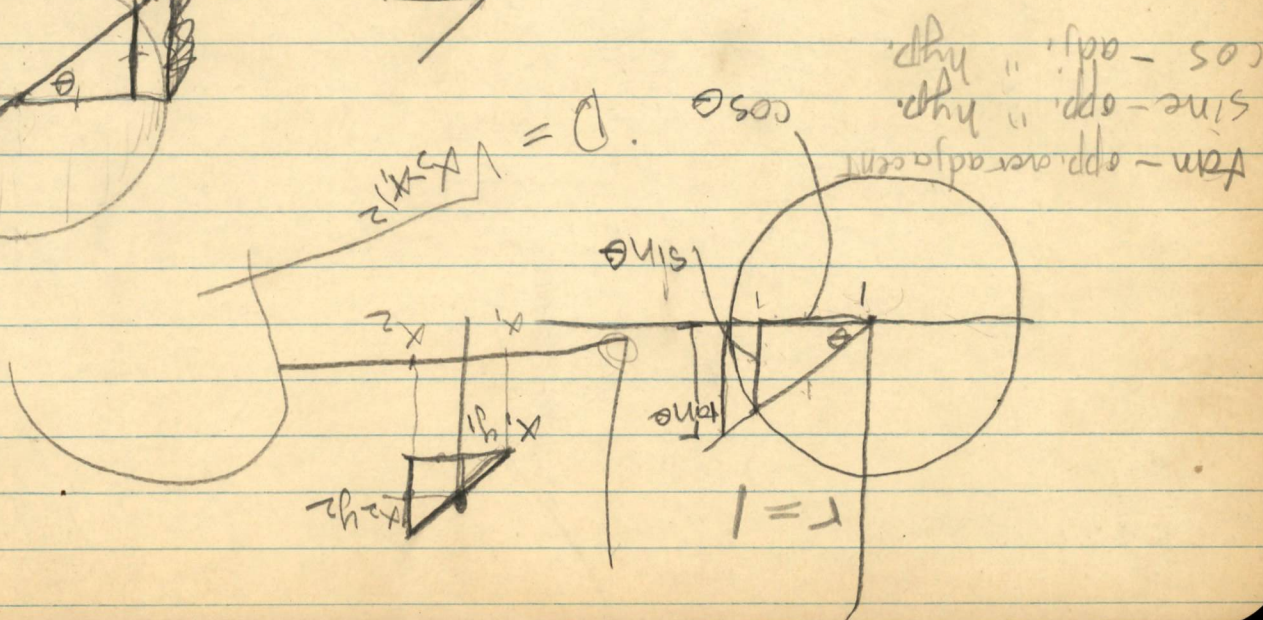
$$\frac{p}{q} = 0.118$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan x + y = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad \text{I think}$$

$$\frac{\cos x}{\sin x} = \cot x$$



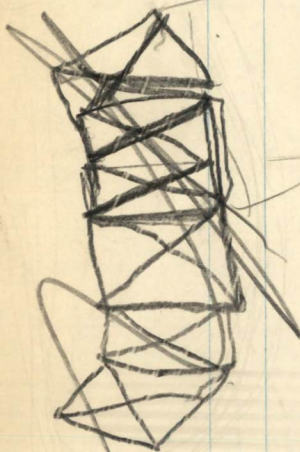
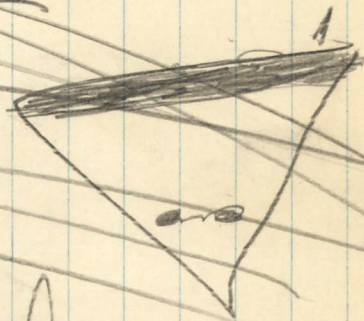
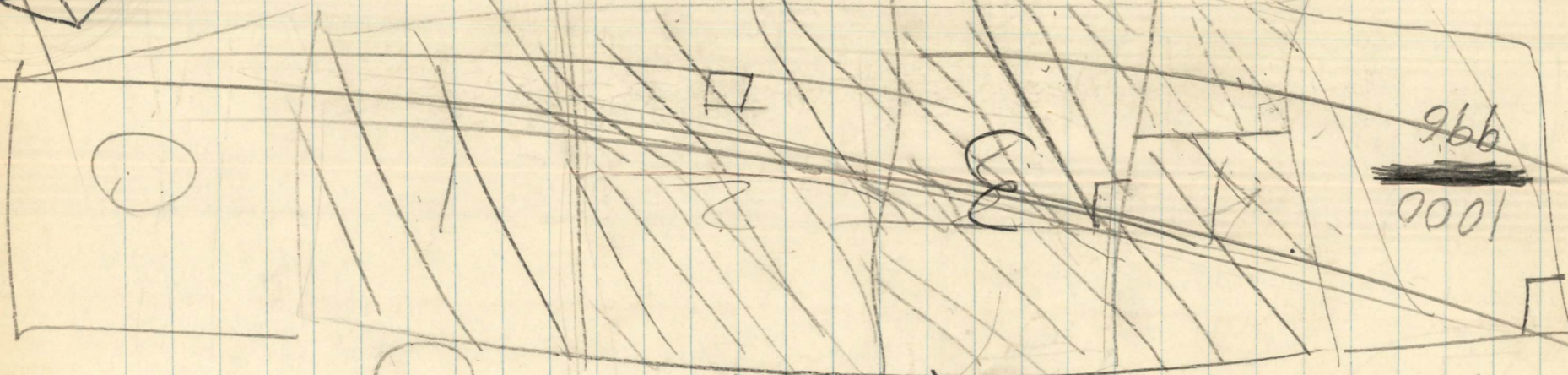
10/21/91

end to a minimum

Wages

~~996~~
1000

Rent



8

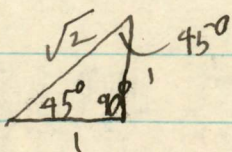
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8

8

$$\Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y$$

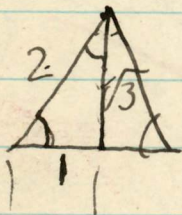
0° ~~45°~~ 30° 45° 60° 90°



$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\tan = 1$$



$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\tan 60^\circ = \sqrt{3}$$

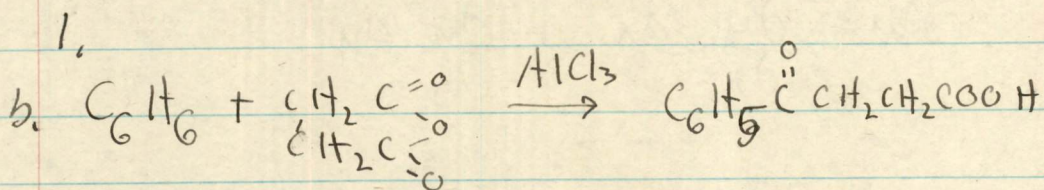
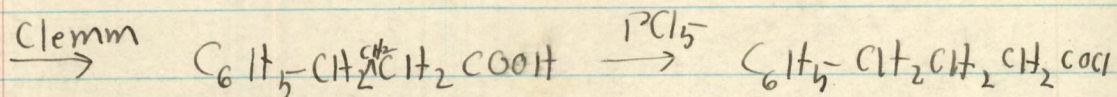
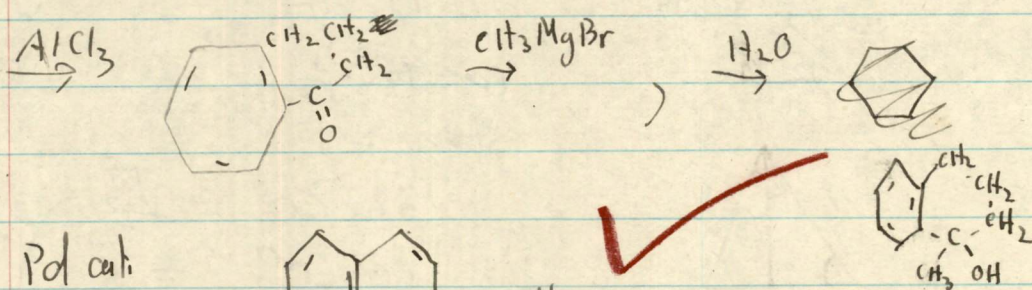
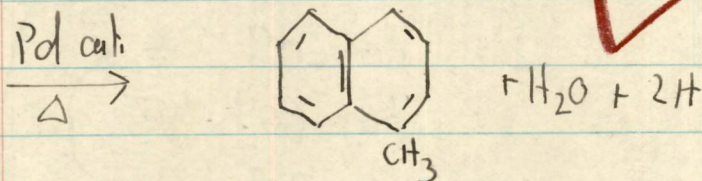
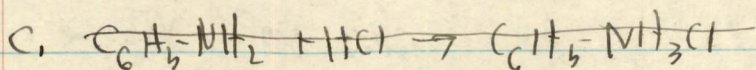
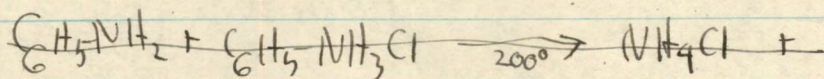
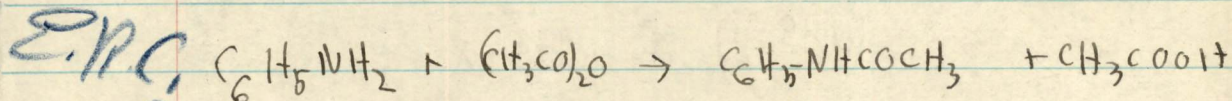
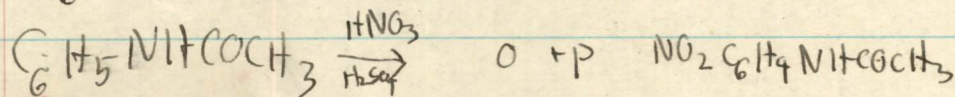
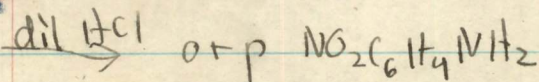
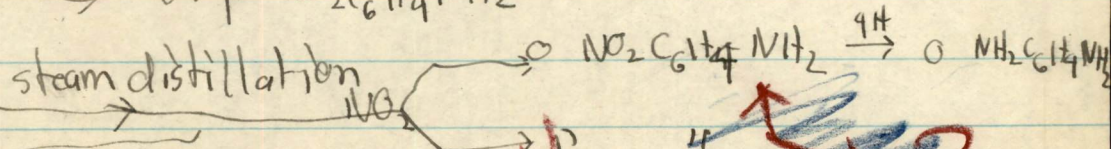
$$\sin 30^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\sin(90^\circ - x) = \cos x$$

Thurs. - Mike & Kate



because of chelate ring. *acid*

Given

~~$\frac{f(x)}{g(x)}$~~ , ~~$x \rightarrow a$~~ , ~~$y \rightarrow 0$~~

~~$\frac{f(x)}{g(x)}$~~ $\frac{f(z)}{g(w)}$

To find the limit of $\frac{f(z)}{g(w)}$ as $z \rightarrow a$, $w \rightarrow b$

If $g(b) \neq 0$, $\lim_{z \rightarrow a, w \rightarrow b} \frac{f(z)}{g(w)} = \frac{f(a)}{g(b)}$, if $f(a) = 0$
 $= 0$

If $g(b) = 0$, and $f(a) \neq 0$, $\lim$ does not exist.

If $g(b) = 0$ and $f(a) = 0$, $\lim$ may or may not exist.

Try to find some common factor which approaches 0, and divide thru by it.

2. $\lim_{z \rightarrow a, w \rightarrow b} \frac{f(z)}{g(w)} = \frac{f(a)}{g(b)}$
even if both 0.

II b. 1. A transitional element in
Group 3a, 4a, 5a, ~~6a~~ or 7a.

3

middle of EMF.

6

2. A trans element
in Group 1b
bottom of EMF.

$$f(x) = \log(1+x)$$

(x-a)

$$f(0) = 0$$

$$f(h) = \log(1+h)$$

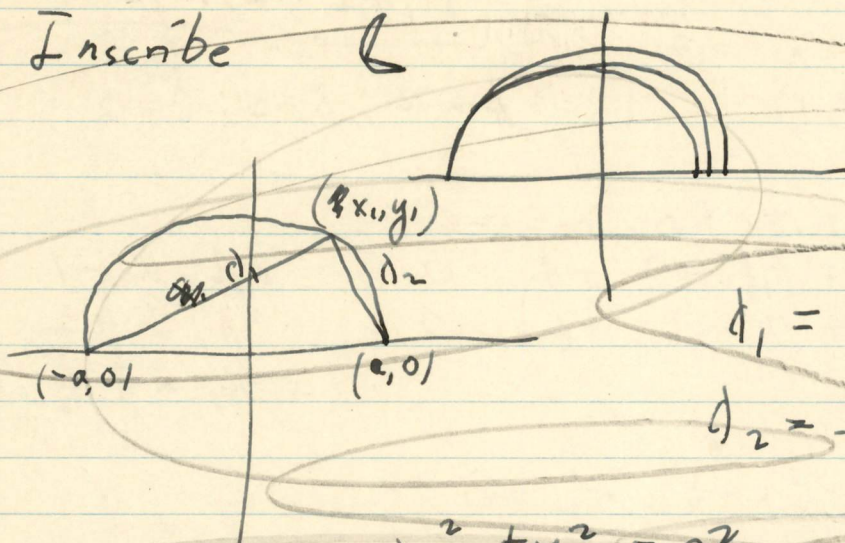
$$\frac{f(h) - f(a)}{b-a}$$

$$\frac{f(h) - f(0)}{h} = \frac{1}{1+h}$$

$$\frac{\log(1+h)}{h} = \frac{1}{1+h}$$

$$\frac{1}{1+h} < \frac{1}{1+h} < 1$$

17 Inscribe



$$d_1 = \frac{y_1}{x_1 - a}$$

$$d_2 = \frac{y_1}{x_1 - a} = \frac{y_1^2}{-y_1^2} = -1$$

$$x_1^2 + y_1^2 = a^2$$

$$x_1^2 - a^2 = -y_1^2$$

Conic sections

Parabola - locus of points in plane equidistant from point & line.

Ellipse - locus of points, sum etc. foci

Hyperbola

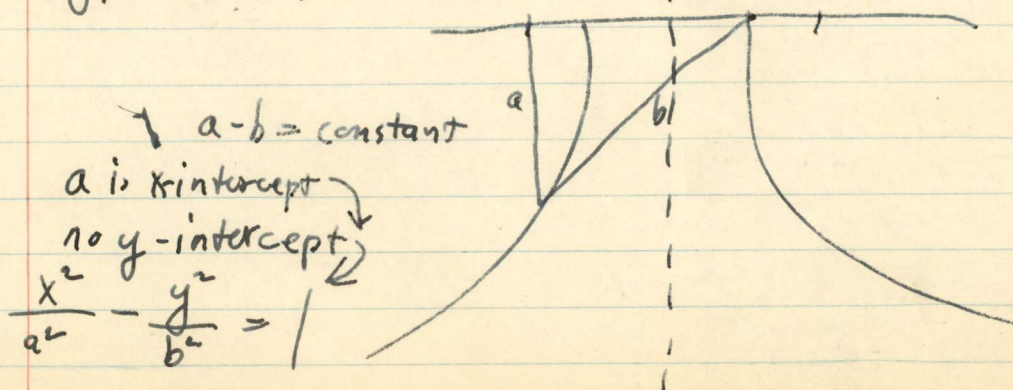
equations - $y^2 = ax$ ^{constant}

Parab. $= 2mx$ - positive x-axis
 $= -2mx$ - neg. x-axis

Ellipse - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ a, b are intercepts of ellipse

or define ellipse like parabola with $\frac{\text{distance}}{\text{distance}} = \text{constant} < 1$

Hyperbola - point moves so that difference between 2 distances constant.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Loomis, Oct. 29

sub, get $A_1 = C_1$

$$A_1 A_2 + B_1 B_2 = 2C_1 + 2C_2$$

24. $x^2 + y^2 + 2x = 0$

$(1,0) (0,1)$

$A = 2; B = 0; C = 0$

$x^2 + y^2 + A_1 x + B_1 y + C = 0$

$1 + 0 + A_1 + 0 + A_1 = 0$

$0 + 1 + 0 + B_1 + A_1 = 0$

$2A_1 = -1$

$A_1 = -\frac{1}{2}$

$B_1 = -\frac{1}{2} \text{ (sub)}$

$x^2 + y^2 - \frac{1}{2}x - \frac{1}{2}y - \frac{1}{2} = 0$

$2x^2 + 2y^2 - x - y - 1 = 0$

Purely a problem exam.

23. Given 2 circles

$x^2 + y^2 + A_1 x + B_1 y + C_1 = 0$

$x^2 + y^2 + A_2 x + B_2 y + C_2 = 0$

$(d_1, B_1) = \left(-\frac{A_1}{2}, -\frac{B_1}{2}\right) \quad \left(d_2, B_2\right) = \left(-\frac{A_2}{2}, -\frac{B_2}{2}\right)$

$r_1^2 = \frac{A_1^2 + B_1^2 - 4C_1}{4}$

$r_2^2 = \frac{A_2^2 + B_2^2 - 4C_2}{4}$

Square of
distance between 2
centers = ~~max~~
sum of radii squares.

$\frac{1}{4} [(A_1 - A_2)^2 + (B_1 - B_2)^2] = [A_1^2 + B_1^2 + B_2^2 - 4C_1 - 4C_2]$

$A_1 A_2 + B_1 B_2 = 2C_1 + 2C_2$

if 2 circles cut orthogonally

Loomis

Nothing about orthogonal circles.

Forget about general theory of Hs & Ls.

1. Slope $A = \frac{y_2 - y_1}{x_2 - x_1}$

2. Midpoint $(x_1, y_1)(x_2, y_2) \Rightarrow \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}$

Calculus - Chap. II

Def. - A function is a rule which assigns to every object from a certain set of objects, a unique object from a second set or the same.

Def. - Given ϵ arbitrarily, there is a corresponding small number, δ , such that whenever $|x-2| < \delta$ then $|x^2-4| < \epsilon$

Proof

(2)

Step I $y + \Delta y = (u + \Delta u) \cdot (v + \Delta v)$
Expand $uv + u\Delta v + v\Delta u + \Delta u \Delta v$

" II $\Delta y = u\Delta v + v\Delta u + \Delta u \Delta v$

" III $\frac{\Delta y}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$
↓ ↓ ↓ ↓

$D_x y = u D_x v + v D_x u + 0 \cdot D_x v$

IV Rule for differentiating a quotient
Def $f(x) = \frac{g(x)}{h(x)}$ or $y = \frac{u}{v}$

$D_x(f) = \frac{h(x) D_x(g) - g D_x(h)}{[h(x)]^2}$

or $D_x y = \frac{v D_x u - u D_x v}{v^2}$

Proof

Step I - $y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$

Manipulate $\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{uv + v\Delta u - u\cancel{v} - u\Delta v}{v(v + \Delta v)}$

" III $\frac{\Delta y}{\Delta x} = \frac{v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x}}{v(v + \Delta v)}$ let $\Delta x \rightarrow 0$

" IV $D_x y = \frac{v D_x u - u D_x v}{v^2}$, QED

Ass. p. 21 - 1, 4 by 4 step rule

p. 34 - 1, 4, 5, 7, 11, 13 apply general rules.

(I)
$$\left. \begin{aligned} 1) f(x) &= c g(x) \\ 2) D_x f(x) &= c D_x g(x) \end{aligned} \right\} \text{rewrite in terms of } u \text{ or } v$$

1) $y = cu$
 2) $D_x(y) = c D_x u$

Step I - $y + \Delta y = f(x + \Delta x) = c g(x + \Delta x) = c(u + \Delta u)$

" II - $\Delta y = c \Delta u$ - just subtract y or cu from $\uparrow$

" III - $\frac{\Delta y}{\Delta x} = c \frac{\Delta u}{\Delta x}$

" IV Let $\Delta x \rightarrow 0$, $\frac{\Delta y}{\Delta x} \rightarrow D_x y$ by def.

$c \frac{\Delta u}{\Delta x} \rightarrow c D_x u$

$D_x y = c D_x u$

(II) Function = sum of 2 functions

If $f(x) = g(x) + h(x)$ or $y = u + v$
 Then $D_x f(x) = D_x g(x) + D_x h(x)$ or $D_x y = D_x u + D_x v$

Step I - $y + \Delta y = u + \Delta u + v + \Delta v$

" II $\Delta y = \Delta u + \Delta v$

" III $\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$

" IV Let $\Delta x \rightarrow 0$
 $\downarrow \quad \quad \downarrow$
 $D_x y = D_x u + D_x v$

(III) Let $f(x) = g(x) \cdot h(x)$

$y = uv$
 $D_x f(x) = g(x) D_x h(x) + D_x g(x) h(x)$
 or $D_x y = u D_x v + v D_x u$

Proof

Loomis, Nov. 12

①

$$y = f(x)$$

Def. $\Delta y = f(x + \Delta x) - f(x)$

function of 2 variables, x & Δx , but we keep x fixed.

Then $y + \Delta y = f(x + \Delta x)$

In terms of above, write down 4 step rule of differentiation

- I) given $y = f(x)$
Calculate $y + \Delta y = f(x + \Delta x)$
II) Subtract y , find Δy : $\Delta y = f(x + \Delta x) - f(x)$ } Manipulation
III) Divide thru by Δx : $\frac{\Delta y}{\Delta x}$

IV) Let $\Delta x \rightarrow 0$
 $D_x f(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$

Example: $f(x) = \sqrt{x}$

I.) $y + \Delta y = \sqrt{x + \Delta x}$

II) $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

~~Manipulate~~ - multiply by $\frac{(\sqrt{x + \Delta x} + \sqrt{x})}{(\sqrt{x + \Delta x} + \sqrt{x})} = \frac{x + \Delta x - x}{\sqrt{x + \Delta x} + \sqrt{x}} = \Delta y$

III $\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$

IV Let $\Delta x \rightarrow 0$

$D_x(y) = \frac{1}{2\sqrt{x}}$ - ok to leave irrational.

Apply these rules generally. General Rules

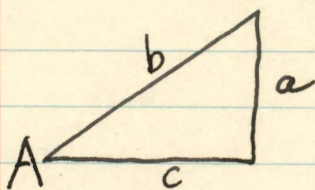
3 functions - $y = f(x)$; $u = g(x)$; $v = h(x)$

$$y + \Delta y = f(x + \Delta x)$$

$$u + \Delta u = g(x + \Delta x)$$

$$v + \Delta v = h(x + \Delta x)$$

Trigonometry



$$\begin{array}{l}
 \left[\begin{array}{l}
 \left\{ \begin{array}{l} \sin A = \cos(90-A) \\ \cos A = \sin(90-A) \end{array} \right. \\
 \left\{ \begin{array}{l} \tan A = \cot(90-A) \\ \cot A = \tan(90-A) \end{array} \right. \\
 \left\{ \begin{array}{l} \sec A = \csc(90-A) \\ \csc A = \sec(90-A) \end{array} \right.
 \end{array} \right.
 \end{array}
 \quad
 \begin{array}{l}
 \sin A = \frac{a}{b} \\
 \cos A = \frac{c}{b} \\
 \tan A = \frac{a}{c} \\
 \cot A = \frac{c}{a} \\
 \sec A = \frac{b}{c} \\
 \csc A = \frac{b}{a}
 \end{array}$$

I Law of sines (AAS, SSA)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or, if B is obtuse

$$\frac{a}{\sin A} = \frac{b}{180-B} = \frac{c}{\sin C}$$

II Law of cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

We define $\cos C \text{ obtuse} = \cos(180-C)(-1)$

When $a < b$, & angles aren't fixed, watch for ambiguity.

With SSS, you don't know if $\angle$ is obtuse.
but

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}; \text{ if it is positive, ok}$$

if it is negative, it is the absolute value $\cos(180-C)$

III If you have large numbers & SAS, alternative to law of cosines is Law of Tangents.

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

or

$$\frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}$$

IV

$$\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

where s means semiperimeter.

Best for SSS

Chap. IV - The circle

1. $(x-a)^2 + (y-b)^2 = r^2$; - = equation of a circle.
2. $x^2 + y^2 + Ax + By + C = 0$; - = equation of a circle.
3. co-ordinates are $(-\frac{1}{2}A, -\frac{1}{2}B)$, radius = $\frac{1}{2}\sqrt{A^2 + B^2 - 4C}$
4. Tangent to a circle - $(x-a)(x_1-a) + (y-b)(y_1-b) = r^2$
5. Circle thru 3 points - Solve 2 equations simultaneously, sub in 3rd.

Things which must be known

Chap. II

(I) 1) P_1, P_2 $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$; or $\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}}$ The equation of a line thru

- a) If line passes thru Origin, $P_1(0,0)$ any 2 pts.
or $x_1y_2 - x_2y_1 = 0$ $\frac{x}{x_2} = \frac{y}{y_2}$
b) If line || x-axis, $y_2 - y_1 = 0$, so (I) is out.
But then $y = y_1$ or y_2 .

~~Point slope given~~

To plot line from equation, use (3)(a), or (4).

2) Point-slope form $\frac{y-y_1}{x-x_1} = m$; or $y - y_1 = m(x - x_1)$

a) Slope intercept form $(P_1(0, b)) (P_2(x, y))$
 $\frac{y-y_1}{x-x_1} = m$; $y = mx + b$ (y_1)

3) The equation of every straight line is of the 1st degree in x & y .

Conversely, the gen. equation of the first degree ($Ax + By + C = 0$) always represents a straight line (A, B) $\neq (0, 0)$

a) the case $B \neq 0$, divide thru by it

$\frac{A}{B}x + y + \frac{C}{B} = 0$; $y = -\frac{A}{B}x - \frac{C}{B}$, which is slope-intercept form where $m = -\frac{A}{B}$ & $b = -\frac{C}{B}$

Therefore the general equation of the 1st degree represents a line with $m = -\frac{A}{B}$ & y-intercept $-\frac{C}{B}$

b) the case $B = 0$

$Ax + C = 0$; $x = -\frac{C}{A}$, the equation of a straight line || to y-axis. If $C = 0$, it is the y-axis.

4) In any equation, determine x & y intercepts by setting x & $y = 0$ alternately. Then draw line. (Intercepts)

5) Intercept form of equation of line. $P_1(0, b)$, $P_2(a, 0)$

$\frac{x-a}{0-a} = \frac{y-0}{b-0}$; or $\boxed{\frac{x}{a} + \frac{y}{b} = 1}$ $a = x$ intercept
 $b = y$ intercept

6. To be $\parallel$, 2 lines must have $=$ slopes.

7. " " $\perp$ " " " " negative reciprocal slopes.

8. Distance of a pt. from a line.

$$D = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

x_1 & y_1 are coordinates of pt.
 A, B & C are constants of line's eqn.

9. Δ area

$$A = \pm \frac{1}{2} [(y_1 - y_2)x_3 + (y_2 - y_3)x_1 + (y_3 - y_1)x_2] \text{ or sub } y\text{'s for } x\text{'s.}$$

10. a) 2 lines $\parallel$ if can be written with $A=A, B=B$

b) " " $\perp$ " " " " " " $\frac{A}{B} = -\frac{B}{A}$

c) " " $\parallel$ " " " " " " $\frac{A_1}{A_2} = \frac{B_1}{B_2}$

d) " " $\perp$ " " " " " " $\frac{A_1}{B_1} = -\frac{B_2}{A_2}$

e) " " identical " " " " " " $A_1 : B_1 : C_1 = A_2 : B_2 : C_2$

But remember exceptions

Chap. III 11. 3 ^{lines} ~~pts.~~ ^{point} on a line if results of solving 2 simultaneously will fit the 3rd.

12. 3 pts. lie on a straight line if eqn of line thru 2 is fitted by co-ordinates of 3rd.

13. Medians, altitudes, & perp. bisectors of sides of Δ go thru a pt.

14. 3 pts. determined above lie on str. line.

Things which must be known.

Chapter I (& introduction)

1. The sum of the directed projections of the broken line $\overline{AB}, \overline{BC}, \dots, \overline{YZ}$ = the projection of the directed line $\overline{AZ}$.
2. ~~The~~ The projections of 2 broken directed lines with same extremities are =.
3. Projections of directed line segments on the axes
Proj. of $P_1 P_2$ on x -axis = $x_2 - x_1$
" " " " y -axis = $y_2 - y_1$
4. Distance b/w 2 points
 $D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ (rt. Δ setup)
if $P_1 P_2$ is ~~is~~ $\parallel$ x -axis, $y_2 - y_1 = 0$, & $\therefore$
 $D^2 = (x_2 - x_1)^2$, $D = x_2 - x_1$
5. $\therefore = \frac{y_2 - y_1}{x_2 - x_1}$
6. To determine the midpoint of a line segment
 $x - x_1 = x_2 - x$
 $x = \frac{x_2 + x_1}{2}$
7. Division of line segment in any ratio $\left(\frac{m_1}{m_2}\right)$
 $\frac{P_1 P}{P P_2} = \frac{m_1}{m_2}$, $\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2}$, $x = \frac{m_2 x_1 + m_1 x_2}{m_2 + m_1}$
- This formula derived with m_1 representing & determining $P_1 P$, so in ~~making~~ assigning values to m_1 & m_2 act accordingly.
8. To find the pt. of intersection of 2 curves, solve their equations simultaneously

End Chapter. I

Math. / A Ex. graph (III)

Read Quiller Cech on Jargon

Plane & Solid Analytical Geom. - Osgood & Graustein
Introduction to the Calculus - Osgood

1. Quiller-Cech Next time find section lists.

2. Hist.

3. Oxford Dict. Osgood & Graustein, p. 18, 11, 7

Curve plotting = $\frac{y}{x}$ of a curve - (simple)

Points of intersection of 2 curves.

To find the co-ordinates of the pt. of intersection of 2 lines given by their equations, solve the latter as simultaneous $\frac{y}{x}$ in the unknown quantities x & y by the methods of elementary algebra.

Intercepts

4. Instead of using slope-intercept form of linear $\frac{y}{x}$, take intercepts of x & y axes. Set x at 0, solve for y & vice versa, draw straight line.

5. Intercept form of line's $\frac{y}{x}$. When both intercepts are known, have coordinates $(a, 0)$ & $(0, b)$ so

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \quad \frac{x - a}{0 - a} = \frac{y - 0}{b - 0} \quad \text{or} \quad \frac{a - x}{a} = \frac{y}{b}$$

$$\frac{a - x}{a} = \frac{y}{b} \quad \frac{a - x}{a} = \frac{y}{b} \Rightarrow 0$$

$$\frac{ay}{b} = a - x \quad ay = ba - bx$$

$$\frac{ay}{b} = -\frac{bx}{y - b}$$

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EXAMINATION BOOK

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Subject Math. A

Section Loomis

**Do not tear out leaves
from this book.**

I. a) single-valued function — the function, of a constant, I suppose, tho I never heard of it.

13 b) Derivative of the function $y = f(x)$ at $x = x_0$, is the limit that Δy approaches as x approaches x_0
 $\frac{\Delta y}{\Delta x} \rightarrow \text{as } x$

c) Relative maximum is the greatest value of a function in a certain interval.

d) P.t. of inflection is where a curve changes from being concave up to being concave down, or vice versa.

II a) $y = ax^2 + bx + c$

✓ $D_x y = a D_x x^2 + b D_x x = 2ax + b$

19 b) $D_x y = (m-x) D_x \sqrt{m+x} + \sqrt{m+x} D_x (m-x)$

✓ i. $D_x \sqrt{m+x}$ let $y = m+x$
then $u = \sqrt{y}$

$$D_x u = D_y \sqrt{y} \cdot D_x (m+x) = \frac{1}{2\sqrt{y}} \cdot 1 = \frac{1}{2\sqrt{m+x}}$$

$$D_x y = \frac{(m-x)}{2\sqrt{m+x}} - \frac{2\sqrt{m+x}}{2\sqrt{m+x}} = \frac{-m-3x}{2\sqrt{m+x}}$$

c) $x^3 - 2xy + y^5 = 0$

✓ $3x^2 - 2(x D_x y + y) + 5y^4 D_x y = 0$

$$3x^2 - 2x D_x y - 2y + 5y^4 D_x y = 0$$

$$\begin{aligned}
 3x^2 - D_{xy}(2x - 5y^4) - 2y &= 0 \\
 -D_{xy}(2x - 5y^4) &= 2y - 3x^2 \\
 -D_{xy} &= \frac{2y - 3x^2}{2x - 5y^4}
 \end{aligned}$$

$$D_{xy} = \frac{3x^2 - 2y}{2x - 5y^4}$$

$$d) \quad y = \frac{(x^2 - a^2)^2}{x^3}$$

$$D_x y = \frac{x^3 D_x(x^2 - a^2) - (x^2 - a^2) D_x x^3}{x^6}$$

$$= \frac{2x^4 - 3x^4 + 3x^2 a^2}{x^6} = \frac{2x^2 - 3x^2 + 3a^2}{x^4}$$

$$= \frac{3a^2 - x^2}{x^4}$$

$$\text{III} \quad f(x) = (x+3)^2(x-2)^3$$

$$\begin{aligned}
 25 \quad a) \quad D_x f(x) &= (x+3)^2 D_x(x-2)^3 + (x-2)^3 D_x(x+3)^2 \\
 &= (x+3)^2 3(x-2)^2 + (x-2)^3 2(x+3) \\
 &= (x+3)(x-2)^2 [3(x+3) + 2(x-2)] \\
 &= (x+3)(x-2)^2 [5x + 5]
 \end{aligned}$$

When this > 0 , $f(x)$ is increasing.

" " < 0 , " " decreasing.

When $x > 2$, $+, +, + = +$ f is increasing
~~When $x < 2$, $+, +, + = +$ " " "~~
 $x < 2$, $+, +, + = +$ " " "

function is positive, increasing

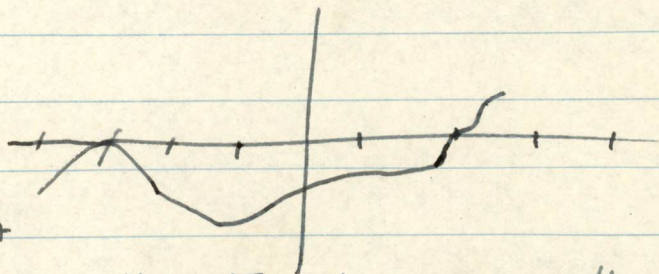
when $x > -3$ $+, +, - = -$ function is decreasing

$x < -3$ $-, +, - = +$ function is increasing

when $x > -1$, $+, +, + = +$ f is increasing

$x < -1$, $+, +, - = -$ f is decreasing

So.



$f(x)$ from the left increases till -3 , decreases till -1 , then increases. The zero slope at $x=2$ is only a pt. of inflection.

b) Rel. maximum at -3 , rel. minimum at -1 , neither at 2

c) substitute at $x = -3$ & at $x = 0$, see which value is greater.

$f(x)|_{x=-3} = 0$; $f(x)|_{x=0} = -72$, so max is at $x = -3$

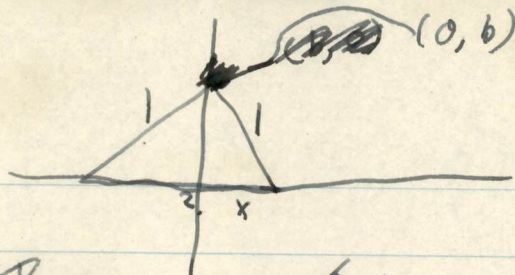
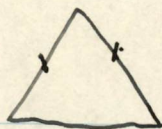
b) $(x+3)(x-2)^2[5x+5] = 0$
 therefore $f(x)$ has relative maxima or minima
 $x = -3, 2, -1$

we know that y at $x = -3$ is greater than
 at any pt. except those right of $x = -1$. Of
 these $|x = 0$ is the greatest, & if $|x = -3$ is
 greater, $|x = -3$ is the max.

Same is true of $|x = 4$

$f(x) = 7^2 \cdot 2^3$ which is 70, so
 $|x = 4$

It is the max.



~~Area~~

$$A = bx$$

$$x^2 = 1 - b^2$$

~~Area~~

The ~~max~~ Area, when x is very small will be very small, and when x is very large will be very small. Somewhere in between is a max.

Set $D_x A = 0$ to find this max.

$$1) D_x A = b D_x x + x D_x b = 0$$

$$2) D_x x^2 = -D_x b^2 = 2b \cdot D_x b = 2x$$

$$D_x b = \frac{x}{b}$$

$$D_x A = b + \frac{x^2}{b} = 0 \quad \text{at the max.}$$

$$b = -\frac{x^2}{b}$$

$b^2 = -x^2$, which ain't possible, so I've done something wrong.

$b = -1x$, the -1 drops out 'cause it's the neg. side of the x -axis

$$b = x, \quad b^2 + x^2 = 1$$

$$b^2 = \frac{1}{2}$$

$$b = \frac{1}{\sqrt{2}}, \quad x = \frac{1}{\sqrt{2}}$$

