



Sensitivity of Financial Factors to n and i

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Sensitivity of Financial Factors to n and i

Sensitivity analysis is the art of expressing how one variable responds to another. The calculus derivative is a first pass at sensitivity, but it is useful to divide the derivative by the function, getting the fractional rate of change. Multiplying that by 100 you get the percentage rate of change, a common and familiar concept.

The derivative divided by the function is also the derivative of the natural log of the function, called the "log derivative." You have the choice of taking the log first, and then the derivative, if that saves work.

A. The PIPO case

Take the basic PIPO compounding factor, with $P=1$ for simplicity.

1. *Method of log derivatives*

$$F = I^n \quad (1)$$

$$\ln F = n \ln I \approx ni \quad (2)$$

Differentiate w.r.t. n :

$$F'/F \approx i \quad (3)$$

For example, if n increases by one, $\Delta F/F = i$. Of course we already knew that, but it's reassuring when everything checks. In fact when anything doesn't check you have surely goofed.

Here is something we didn't know, at least not as well. Differentiate w.r.t. i this time, and:

$$F'/F \approx n \quad (4)$$

When i rises by .01 that doesn't look like much, but empires have fallen from such causes. Here's why. When i rises by .01, F rises by $.01n$, which is $n\%$. The Rule is the same when n rises from .01 to .02, or from .20 to .21. Thus if n is 25, every point rise in i makes F rise by about 25% (actually, we will see, it's a little bit more, but ni is good for a quick estimate.)

See the paper on "The $n\%$ Rule of Thumb." It brings out one of the vital facts of economics, that small differences of i result in large differences in future values, and *the differences*

increase with the futurity, as measured by $\underline{n}$. (It is a rule "of thumb" because it is less than 100% accurate; it understates F' . It needs serious adjusting when $\underline{n}$ gets above 35 or so.)

The $n\%$ Rule applies equally in discounting, when you divide by the compounding factor (or multiply by the discount factor, which amounts to the same thing). Higher interest rates depress present values in proportion to $\underline{n}$, the futurity of the values being discounted. Let's show this directly. This time we let $F=1$.

$$P = I^{-n} \quad (5)$$

$$\ln P = -n \ln I \approx -ni \quad (6)$$

Differentiating w.r.t. $\underline{n}$,

$$d \ln P / d n \approx -i \quad (7)$$

$$P' \approx -P i \quad (8)$$

(8) says that when you raise $\underline{n}$ by one, you lower P by iP . Sensitivity to changes in $\underline{n}$ rises with $\underline{i}$.

Differentiating w.r.t. $\underline{i}$,

$$d \ln P / d i \approx -n \quad (9)$$

$$P' / P \approx -n \quad (10)$$

(10) says that a 1-point rise of $\underline{i}$ reduces P by $n\%$. Sensitivity to changes in $\underline{i}$ rises with $\underline{n}$. See paper on "The $n\%$ Rule of Thumb."

2. Method of ratios

Another way to compare the growth of money at different rates, $\underline{i}$ and $\underline{j}$, is to take their ratio. We have already defined $I = 1+i$. Likewise, let $J=1+j$, where $\underline{j}$ is another interest rate, and $K=1+k$, where $\underline{k}$ is to be defined below as I/J .

$$I^n/J^n = [I/J]^n = K^n \quad (11)$$

$$k = I/J - 1 = \frac{1+i}{1+j} - 1 = \frac{1+i-j-1}{1+j} = \frac{i-j}{1+j} \quad (12)$$

(11) shows that the ratio of two amounts growing at different rates is not constant, but keeps growing with n , at the rate k . (12) shows that k is $[i-j]/[1+j]$.

For example, try $n=25$, $i=.09$, and $j=.07$.

$$1.09^{25} = 8.623$$

$$1.07^{25} = 5.427$$

$$8.623/5.427 = 1.59$$

$$[1+.02/1.07]^{25} = 1.59$$

For "quick and dirty" approximations, people often omit the refinement of dividing .02 by 1.07. Then they slightly overstate the divergence of values, as below.

$$1.02^{25} = 1.64$$

Work out some more numerical examples on your calculators, until you have a feel for the sensitivity.

A frequent application of this technique is in adjusting for inflation. Inflation is measured by price indexes, e.g. the CPI (Consumer Price Index). Say your money is growing at 6%, and the CPI at 3%, and you want to know the growth of your real wealth. A standard technique is just to subtract 3% from 6%, and say your real wealth grows at 3%. That is tolerably accurate for 3% and 6%.

However, in Mexico these days inflation is at much higher rates, and so are nominal interest rates. There, you had better get more careful, and divide by $[1+j]$. Say your money grows at 90%, and the inflation rate is 75%, are you really making 15%? Not by a far piece. $.15/1.75 = 8.6\%$.

3. Method of semi-log scales

An excellent way to visualize this is to plot the growth curves on a semi-log scale, with n on the abscissa and F on the ordinate. See the paper on "Using Semilog Scales."

B. The Discounted Cash Flow Factor (DCFF) Case

Starting with the DCF factor, let $a=1$ for generality. We know (from the derivation of the DCFF as the sum of a geometrical progression) that when n rises by one, P rises by I^{-n} . Now we solve for the fractional increase, $\Delta P/P$:

$$\Delta P/P = \frac{I^{-n}}{[1-I^{-n}]/i} = \frac{i}{I^n - 1} \quad (13)$$

(13) shows that $\Delta P/P$ is a sharply decreasing function of n . It starts at One when $n=1$, and approaches zero as an asymptote as n rises. This underscores the low returns to building durability into capital. For example, if $i=.09$, and $n=50$, (13) = .0012. Check it out on your calculator. That says that a 2% rise in life (from 50 to 51 years) yields a rise of DCF of .12%.

"Building durability" into capital as an abstract virtue is highly regarded, in semi-popular rhetoric. A lot of well-meaning reformist economists get caught up in this idea, too. Things that wear out are denigrated; producers are accused of skimping, deceiving, and catering to consumers' meretricious values. Their products are called tacky-tacky, jerry-built, flimsy, and such rude names. Buyers are called short-sighted. However, considering discounting, it does not make sense to spend a lot of money to tack on another 10-20 years to the life of something that already lasts 50 years.

A common form of "building durability" is building in excess capacity, to care for expected or possible higher future demand. This was common in the electric utility business, for example, until in the 1980s this "perpetually rising demand" failed to rise on cue, and sellers were left with large excess capacity. Utilities could take the attitude of building ahead of demand because, owing to quirks in regulation, they had access to captive capital and could conceal the heavy costs of carrying excess capacity.

Water utilities, both urban and rural, routinely carry great excess capacity for the same reason, but aggravated by a "rent-seeking" factor. Excess capacity is useful to help lay claim to natural water sources in advance of need - another quirk in regulation. The amount of capital wasted and sterilized this way is a caution.

C. The IPF case

Let $a = \text{IPF}$. Take the log of both sides.

$$\text{Lna} = \text{Lni} - \text{Ln}(1-I^n) \quad (14)$$

Now differentiate w.r.t. n :

$$a'/a = 0 - \frac{-iI^n}{1-I^n} = \frac{-i}{I^n-1} \quad (15)$$

(15) says that IPF is very sensitive to n when n is low, but the sensitivity $\rightarrow 0$ when n is high. This jibes with what we already know about IPF as a function of n . To repeat from p.15 of the paper "Basic Mathematics of Finance, Skeletal Outline,"

Think of IPF as a combination of dividing by n , as in a simple yearly average; and multiplying by the interest rate, i . IPF approaches $1/n$ when n and i are both small; and approaches i when n and i are large, but it always remains above both $1/n$ and i .

Remember that (15) is derived using calculus, and that means it is only valid strictly at a point. Often we see that, and something like a' is a good surrogate for $\Delta a/n$. In this case, though, beware, for small values of n . (15) seems to say that a'/a is -100% when $n=1$, but that cannot be, for then a (or IPF) would be zero when $n=2$ - and it is not. It is just the instantaneous rate of change that is at that level. If it kept up at that rate, clear from $n=1$ to $n=2$, the drop would be 100%. It doesn't, though. By the time n has risen partway there, say to 1.5, the rate of change has dropped sharply down. Doubling n still does cut IPF way down, though, almost by half.

D. Elasticities

Elasticity is another useful measure of sensitivity. It is the ratio of the log derivatives of the dependent and independent variables. We already have the first, and the second is a piece of cake, it is simply the reciprocal of the independent variable, in this case $\underline{n}$. Dividing by $1/\underline{n}$ is the same as multiplying by $\underline{n}$, and there you have elasticity, generally described as the ratio of the percentage changes of two variables.

We can translate what we already have into elasticities immediately. Let ϵ denote elasticity.

ϵ of F w.r.t. $n = n_i$
 ϵ of F w.r.t. $i = i_n$ (same thing)
 ϵ of P w.r.t. $n = -n_i$
 ϵ of P w.r.t. $i = -i_n$ "

Let DCF denote the present value of a flow, as before.

$$\epsilon \text{ of DCF w.r.t. } n = \frac{n_i}{I^{n-1}} \quad (15)$$

$$\epsilon \text{ of IPF w.r.t. } n = \frac{-n_i}{I^{n-1}} \quad (16)$$

The ratio seen in (15) and (16) pops up often in finance. You can know how it behaves by realizing that it is the ratio of simple interest to compound interest. That means the top grows nearly as fast as the bottom for a few years, but then the bottom races far ahead and the ratio falls swiftly.

The trouble with these elasticities (and the log derivatives, too) is that they are true only at one point. If the changes are large, e.g. if you double $\underline{n}$ from 15 to 30, these point elasticities misstate the response. They are accurate for small changes, however; and always useful to put you in the right ballpark.

~ In (7), P is quite sensitive to i . The most orderly way of expressing this particular sensitivity is in terms of "elasticity", which is the % change in P caused by a % change in i . (You may wonder why I choose elasticity here when I did not do so before, in Table 2 and Eqn. (6). The reason is that we found an orderly rule there without going to that extra trouble. You may convert (6) to an elasticity rule if you like, by dividing it by $1/i$).

~ (7) is so simple it is intuitively obvious that an $x\%$ increase of i causes a corresponding $x\%$ decrease in P , so long as x is small. This is just the same relationship that you should remember from price theory when you studied elasticity of demand and learned about elasticities equal to one, when price times quantity was constant. In the present subject case it is Pi which is a constant, being always equal to a .

~ I suggest you confirm this by calculus. Take the log derivative of P w.r.t. (with respect to) i . You should get $-1/i$. Take next the log derivative of i w.r.t. i : you should get $1/i$. To get elasticity divide, and get -1 .

~ What if the $x\%$ increase in i is not small? What if x is 100%, for instance? Remember in price theory how they distinguished between what they called "arc elasticity" and point elasticity? It is the same here, but there is a simple algebraic expression to give it exactitude. The percentage response of P to i is not exactly $-x$ when x is large. It is $-x/(1+x)$.

~ To prove that, let x be the fractional increase of i , and y be the corresponding fractional change in P . Then:

$$\sim\sim\sim\sim P(1+y) = a/i(1+x) \sim\sim\sim\sim (8)$$

Divide (8) by (7)

$$\sim\sim\sim\sim 1+y = 1/(1+x) \sim\sim\sim\sim (9)$$

$$\sim\sim\sim\sim y = 1/(1+x) - (1+x)/(1+x) = -x/(1+x) \sim\sim\sim\sim (10)$$

~ I suggest you remember this rule in the simple form, that elasticity here equals -1 . As for Eqn.(10), that is a general rule applicable to all cases of large changes in divisors, and you should remember that as a general algebraic rule, and not just in connection with this case.

~ The sensitivity of a/i to i is comparable to the sensitivity shown in Table 2 when raising i from .05 to .06, a 20% increase, raises the discount factor by about 20% also. Thus at a 5% rate of interest the sensitivity of a perpetuity is the same as that of a single or point payment due in 20 years. Going to Eqn. (6) you can infer that it is also the same as a 50% increase when $n = 50$; a 10% increase when $n = 10$; and so on.

~~ b. It follows from a) that doubling the life of capital from a small base, say from one year to two, nearly doubles the PV. Doubling life from 20 years to 40 does not nearly double the PV. And doubling from 50 years to 100 adds very little to PV.

~ Putting that more generally, the elasticity of PV w.r.t. n is a decreasing function of n . In Table 3, "Elasticity (ep) of P w.r.t. n ", n increases by doubling, i.e. in steps of 100%. dP/P is thus the ep, since $dn/n = 1$.

~~~~~ Table 3: Arc Elasticity (ep) of P w.r.t.  $n$

| $i = .05$ |      |        |       |        | $i = .10$ |      |        |       |        |
|-----------|------|--------|-------|--------|-----------|------|--------|-------|--------|
| $n$       | $P$  | $dP/P$ | $P$   | $dP/P$ | $n$       | $P$  | $dP/P$ | $P$   | $dP/P$ |
| 1         | .95  | ---    | .91   | ---    | 2         | 1.86 | .96    | 1.74  | .91    |
| .91       | 3.17 | .82    | 8     | 6.146  | .82       | 5.33 | .68    | 16    | 10.84  |
| 7.82      | .47  | 32     | 15.80 | .46    | 9.53      | .22  | 64     | 19.12 | .21    |
|           |      |        |       |        |           |      |        | 9.98  | .05    |

~ You might think about Table 3 when you're deciding how much it is worth spending today to make a product last for the second 50 years. And it is not just simple durability, it is also capacity. Many buildings and engineering works are built with excess capacity, in expectation of higher future demand. You have to balance the extra cost of that extra capacity against the fact that its extra value doesn't come into play until after several years.

~ Even more generally we can find an algebraic expression for elasticity (ep) at a point. It is clear from (12) that the increment to PV is  $I-n$  whenever  $n$  increases by one. The fractional increase in PV is therefore:

$$~~~~dP/P = I-n/(1-I-n)/i = i/(In-1)~~~~~ (15)$$

~The fractional increase in  $n$  w.r.t.  $n$  is always just  $1/n$ . To get  $ep$ , divide by that, and we have:

$$~~~~ ep = ni/(In-1)~~~~~(16)$$

~ You can also struggle with calculus and get (16) as the ratio of the log derivatives of PV and  $n$ , and it would be a revelatory experience which I commend to you. However the answer would be no different.

~ Now (16) probably looks to you like mush when you first eyeball it, but there are nice easy ways to grasp it. First, I trust you have gathered that "ep" is for epsilon, the conventional notation for elasticity, since I have no epsilon symbol on this machine which I am stroking at 1 A.M. at home. On the right side,  $ni$  is simple interest; and  $In-1$  is compound interest. Thus  $ep$  is just the ratio of simple to compound interest, which we may call the "simp/comp ratio". (If you are into baiting reds you may call it the reciprocal of the com/simp ratio.) This ratio pops up often in finance, so much so that I have it all programmed on my calculator and can give you a table of numbers in Table 4, "Point Elasticity of PV w.r.t.  $n$ ". Absent the table you can easily remember that which is not otherwise obvious, that the  $ep$  is always less than one, because simp is always less than comp.

Table 4: Point Elasticity of PV w.r.t.  $n$

|    |     |     |     |
|----|-----|-----|-----|
| i= | .02 | .05 | .10 |
| n  |     |     |     |
| 2  | .99 | .98 | .95 |
| 4  | .97 | .93 | .86 |
| 8  | .93 | .84 | .70 |
| 16 | .86 | .68 | .45 |
| 32 | .72 | .42 | .16 |
| 64 | .50 | .15 | .01 |

~~c. PV is also sensitive to  $i$ , of course. Here, however, the sensitivity is negative, that is higher  $i$  means lower PV. If you want the practise of taking the log derivative of PV w.r.t.  $i$ , I recommend it, you will gain a lot of insight. Remember that  $(1+i)^{-n}$  is also expressed as  $e^{-in}$ , which tells you that the effects of  $i$  and  $n$  on the discount factor are parallel. However the effects on PV are not parallel because  $i$  also appears alone in the basement. Taking the log derivative this causes the subtraction of  $1/i$  from an expression otherwise parallel to the

one in (15); dividing by  $1/i$  makes that become one. The upshot is that  $ep$  of PV w.r.t.  $i$  is  $\text{simp/comp} - 1$ :

$$\text{~~~~}ep = ni/(In-1) - 1 \text{ ~~~~~~(17)}$$

~ And the upshot of that is that you can construct your own table simply by subtracting 1 from the corresponding figures in Table 4. That operation of course makes the figures negative, as they should be, indicating that PV is a decreasing function of  $i$ . Perform this exercise, and note how  $ep$  changes with both  $n$  and  $i$ . Now you are really into sensitivity analysis.

## 6. Sensitivity

## B. The PVF case

## C. The IPF case

For generality, set  $P=1$ . Take the log of both sides, and differentiate.

$$\ln a = \ln i - \ln(1-I-n)$$

$$a'/a = d\ln a/dn = \text{-----} \quad (18)$$

## D. Elasticities

Let PV denote the present value of a flow, as in (3,A), to distinguish it from P, used for the PIP0 case.

$$E \text{ of PV wrt } n = \text{-----} \quad (19)$$

$$E \text{ of IPF wrt } n = \text{-----} \quad (20)$$

The trouble with these elasticities (and the log derivatives, too) is that they are true only at one point. If the changes are large, e.g. if you double  $n$  from 15 to 30, these point elasticities understate the response. They are accurate for small changes, however; and always useful to put you in the right ballpark.