Finance. Misc.

Sec. Ch. 18 Cap. Budgeting

I. Rate of Return.

1. Add in salvage value.

Redeemable bond.

a. Canti's method is better than ignoring.

$$i = \frac{a}{p} + \left(\frac{R}{P}\right)^{\frac{1}{n}} - 1$$

b. Correct method

$$i = \frac{a}{P} \frac{1 - I^{-n}}{1 - \frac{R}{P} I^{-n}}$$

Rt. side is programmable; ∴ can be solved by trial + error, ∴ by machine.

PIPO

$$i = \left(\frac{R}{P}\right)^{\frac{1}{n}} - 1$$

But avoid common error of o'stating i by omitting L

$$\text{True } i = \left(\frac{R+L}{P+L}\right)^{\frac{1}{n}} - 1$$

10/20/83

Topics in finance, not covered, or just barely scratched

Replacement timing
Turnover, payback, capital recovery, mean recovery time
Solving for YTM & IRR
Defining taxes, advancing deductions
Normalizing
Borrowing & leverage
Valuation
Log scales, use of
Imputation
Continuous interest
Annualization as a "time-average"
Depreciation & depletion
Economizing on K -- ~~getting~~ intensifying its use through payback control
Risk
Cash Flow Analysis

Annuity formulas as averages through time.

1. Value crf. when $i \to 0$

$$\lim_{i \to 0} \frac{i}{1-(1+i)^{-n}} = \frac{1}{n}$$ Just recover P, no int.

Test on calcs.

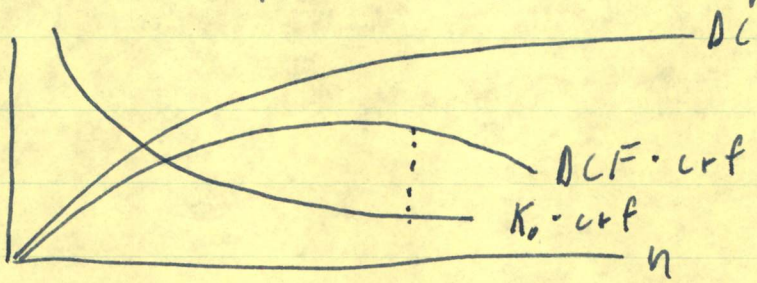
2. When $i > 0$, $crf > \frac{1}{n}$ Recover P + int.

E.g. $i = .05$

n	crf	1/n	Skg. Fund
1	1.05	1	1
14	.10	.07...	.05
∞	.05	0	

(Skg. fund form. $< \frac{1}{n}$)

3. This solves problem of opt. life. DCF rises forever.

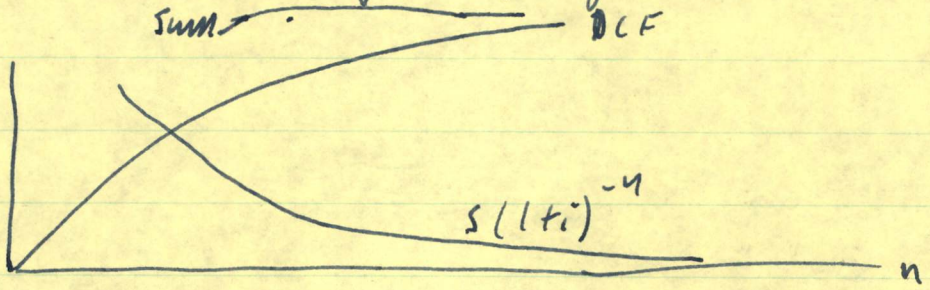


Thus max. a and S

Puts limit on gains from:
- spreading K cost over time
- durability
- long production runs

4. Alt. method:

~~Include~~ add salvage value of S ~~to~~ to DCF, use sum



But then, must add S_0 to K_0 ?

"Time-average"

~~Homework~~

Joe's ~~fat~~ aged mother has a nest egg of 100,000 for retirement. ~~At 15k/yr.~~ In 10 yrs. how much per yr. ~~How many yrs~~

1) $P \cdot \frac{1}{n}$, if no int.
2) $P \cdot i$, if no invasion of princip.

3) $P \cdot \frac{i}{1-I^{-n}}$ if both

$$\frac{100{,}000}{10} = 10k$$

$$100k \cdot .09 = 9k$$

$$100k \frac{.09}{1-1.09^{-10}} = 100k \frac{.09}{1-.422} = 100k \cdot .156 = 15{,}600$$

Thurs., 10-11-84
Ec. 162

1. Itemize Tues. points; at same time, recap. and illustrate

a. Growing annuity

$$P = \frac{a}{i - g}$$

Corrollary:

i) $P_i = a + P_g$

$P_g = \Delta P$ bcs. each passing year you begin from g higher, and each term is raised by g.

a is cash

$P_i > a$, hence neg. cash flow

ii) a is taxable income. P_g is silent, but large. Dominates many mkts. + decisions.

iii) Income basis of prop. assmt., ignores g.

iv) i and g leverage each other. "Amplify" ea. other. Unless they change by same ~~fraction~~ amount.

v) g may be real; or inflationary.

Cf. Reagan v. Mondale debate.
Reagan said i.r. ~~low~~ has fallen.
Mondale " real i.r. has risen.

How to find real i.r.? If P falls, is a sign real i.r. has risen.

b. Periodic payments, or cycles.

i) Periodic pmt. or yield.

$$\frac{a}{i} = \frac{P}{1 - I^{-n}} \qquad (1)$$

or, $\frac{a}{i} = \frac{F_n}{I^n - 1}$ Feenstmann (2)

If $n = 1$ ~~then~~

$$\frac{P}{1 - \frac{1}{I}} = \frac{P}{\frac{I-1}{I}} = \frac{PI}{i} \cong \frac{P}{i} \qquad (1)$$

$$\frac{F}{I - 1} = \frac{F}{i} \qquad (2)$$

L.V. Mult., sample values

u	i' .04	.08
1	26	13.5
5	5.6	3.1
10	3.1	1.9
20	1.8	1.3
40	1.3	1.05

ii. The pmt. need not be strictly PIPO. Any repeating cycle will do. You gather all values at beginnings of cycles (for (1)) or ends (for (2)).

d.) Finding i.

Depreciated calc.

Distrib. sheet

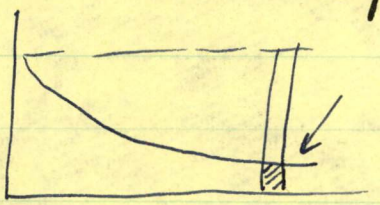
e) Deprecn & Amortn.

1. $\frac{1 - I^{-n}}{i} - \frac{1 - I^{-(n-1)}}{i} = \frac{I^{1-n} - I^{-n}}{i} = \frac{I^{-n}(I-1)}{i} = I^{-n}$

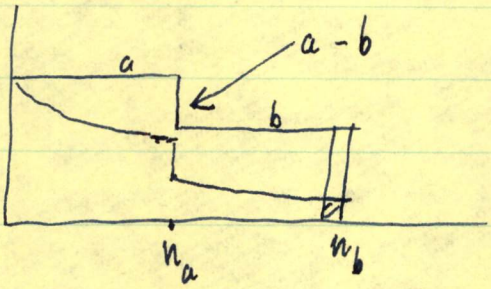
2. Deprecn = $1 - Pi = 1 - 1 + I^{-n} = I^{-n}$

It's nice how all the pieces fit together.

3.



4. If c.f. decays, deprecn. is greater.



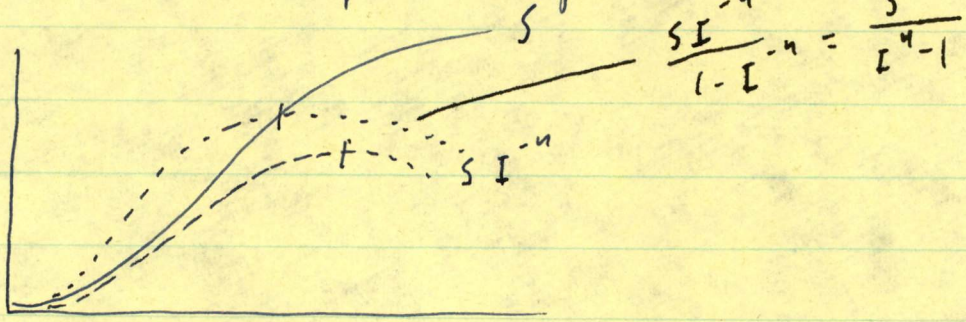
Net loss is $\left[b I^{-n_b} + a I^{-n_a} - b I^{-n_a}\right]$

$= I^{-n_b}\left(b + I^{-(n_a - n_b)}\right)$

~~$= a I^{-n_a}$~~

$= (a-b) I^{-n_a} + b I^{-n_b}$

g. Maturity + replacement



alt. solutions.

1) Max S.
2) Max S/n
3) Max SI^{-n} ~~Max IRR~~
4) Max $\frac{SI^{-n}}{1-I^{-n}}$

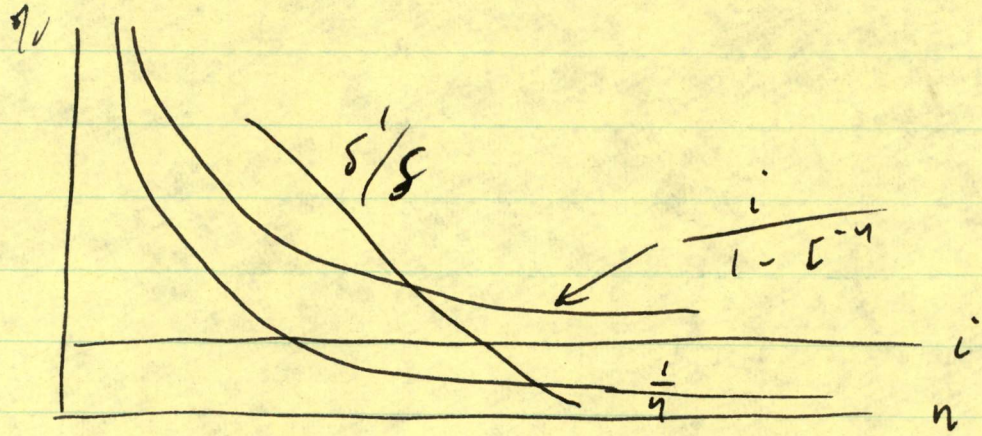
as mgnl. solutions

1) $s' = 0$
2) $s'/s = \frac{1}{n}$
3) $s'/s = i$
4) s'/s = log deriv. of $I^n - 1$

$= \frac{i I^n}{I^n - 1} = \frac{i}{1 - I^{-n}} > i$

$> \frac{1}{n}$ also

Tues Go over Sec
Thurs. Exam



$$u = 1, \quad \frac{i}{1 - I^{-u}} = \frac{i}{\frac{1+i-1}{1+i}} = 1 + i > u$$

Fig. — shows how the true, ~~unqualified~~ cost of time relates to $1/u$ and to i, ~~it is~~.

- it is always higher than either
- for low u, $TMCT \to u$
- for hi u, $TMCT \to i$

Warning for mgrs. "Product Champions" in a govt. agency (or a firm) are always looking for free capl.

USFS. uses it. "Good forestry". Hence move for privatizn. "Mal".

But also in pvt. forest indy. Mac & Blo. foresters, e.g. Mgr. has to hold such types in check.

The secret of Donaldson Brown.

Otherwise, product champions love to become free K. Once secured, stops appearing in budget

$$\frac{s'}{s} = \frac{i}{1 - I^{-n}}$$

$$s' = \frac{si}{1 - I^{-n}} = \frac{si\, I^n}{I^n - 1}$$

Purpose: to factor out si, to see what is the other element on which we are earning interest.

Technique: Subtract and add one to num., thus factor out one.

$$si\left[\frac{I^n}{I^n - 1}\right] = si\left[\frac{I^n - 1 + 1}{I^n - 1}\right] = si\left[1 + \frac{1}{I^n - 1}\right]$$

$$= si + i\frac{s}{I^n - 1}$$

But $\frac{s}{I^n - 1}$ is land value

$\therefore$

$$s' = i[s + L]$$

Installment #3 for Ec. 162A. F84, by M.G.

9. Practical influences, (cont.)

e. Depreciation and Amortization, level annuity

How much of annual payment is depreciation of asset? Same formula shows amortization of principal of debt.

Three approaches.

1) $D = \frac{1 - I^{-n}}{i} - \frac{1 - I^{-(n-1)}}{i}$

$= \frac{I^{1-n} - I^{-n}}{i} = \frac{I^{-n}(I-1)}{i} = I^{-n}$

(19)

2) Payment is divided between interest and amortization. (Amortization to the debtor means repayment of principal. He is unhappy because it is so slow. But the creditor is also unhappy it is slow: to the creditor it means recovery of his capital. Be ready to look at this from both sides.)

Interest is due at each passing year on the unrecovered principal P_n, where n is the number of years of life remaining (that is neither the original n nor the number of years that have passed.)

$$\text{Interest} = Pi = \frac{1 - I^{-n}}{i} \times i = 1 - I^{-n}$$

$$\text{Deprec.} = 1 - Pi = I^{-n} \qquad (20)$$

3) Graphically:

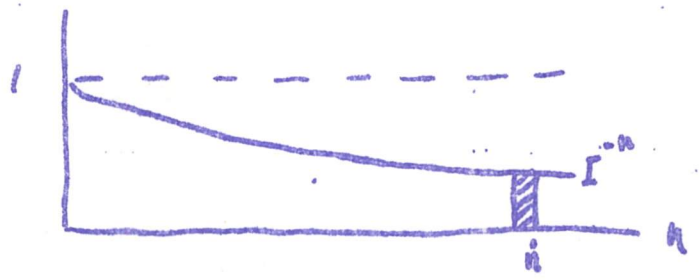


Fig. 6: Depreciation

Shaded column is Deprec. where n is $\dot{n}$. It is $I^{-\dot{n}}$, where $\dot{n}$ is the number of years of life remaining.

10. Appreciation: PV of a rising cash flow

Let C.F. $= 1 + (1+g) + (1+g)^2 + \dots + (1+g)^{\infty}$

Then $PV = I^{-1} + I^{-2}(1+g) + I^{-3}(1+g)^2 + \dots + \left(\frac{1+g}{I}\right)^{\infty}$

Sum the series.

$$PV = I^{-1}\frac{1}{1-I^{-1}(1+g)} = \frac{1}{I-1-g} = \frac{1}{i-g} \qquad (21)$$

(Hereafter, use P alone for PV)

Solving (21) for P_i:

$$P_i = 1 + P_g \qquad (22)$$

P_g is ΔP, (by obvious influence from the top line on this page, the definition of cash flow.)

1 stands for $\underline{a}$, any initial C.F. (22) may be rewritten:

$$P_i = a + \Delta P \qquad (22A)$$

Thus P_i equals the sum of current cf. plus current appreciation. Note the symmetry and consistency of this with (20).

11. Periodic payments and repeating cycles.

Solve (11) for a/i

$$\frac{a}{i} = \frac{P}{1 - I^{-n}} \qquad (23)$$

$\underline{a}$, as we have discussed in class, is the n-year annuity corresponding to P.

Dividing $\underline{a}$ by $\underline{i}$ "capitalizes" it in perpetuity. It gives the present value of a perpetual flow of $\underline{a}$. (See (7), p. 11.)

Thus the right side of (23) is the present value of a perpetual succession of P, received every $\underline{n}$ years. Label it Σ.

[If the P repeats, but not forever, we adjust Σ by subtracting the discounted value of the portion of Σ contributed by the later

years, the ones that are cut off. This is the same technique we used on p. 13 to find (11).

Let X = no. of reps of P. Then:

$$\Sigma = \frac{P}{1 - I^{-n}} - \frac{P I^{-Xn}}{1 - I^{-n}} = P \frac{1 - I^{-Xn}}{1 - I^{-n}} \qquad (24)$$

You need not "memorize" (24) for any exam. I just want you to see that and how it can be done.]

12. Positive NPV and Land Values

If investments normally yield positive net present values, NPV, after deducting the initial capital cost, that sounds too good to be true for long, and it is. It creates demand for the land on which these happy results can be achieved. Hence Land Values, which rise to soak up the surplus.

But Land lasts forever, so its value looks

up the value of not just the first cycle of investment and return, but all succeeding cycles as well. These are called "Generations."

The NPV of one Generation (G) or cycle of investment and recovery is summarized as P at the time the initial outlay is made. We have called this the "present" when dealing with one cycle, alone. In dealing with a succession of Generations, each P has its own future date.

A string of future P may be summed up as a string of periodic payments, as in (23). If a cycle repeats (in perpetuity):

$$L = \frac{P}{1 - I^{-n}} \qquad (23A)$$

where L = Land Value

Take out $\frac{1}{1 - I^{-n}}$ from that and

we have the "Land Value Multiplier". See Table 6.

Table 6: Land Value Multipliers

n \ i =	.04	.08	.12
1	26.00	13.50	9.33
5	5.62	3.13	2.31
10	3.08	1.86	1.47
20	1.84	1.27	1.12
40	1.26	1.05	1.01

To find Land Value, multiply P from Generation #1 by the appropriate figure.

To adjust for anticipated inflation, use a <u>real</u> interest rate — i.e. subtract the inflation rate from the nominal i.r. Thus, .04 or .08 is more realistic than .12.

13. Maturity and Replacement

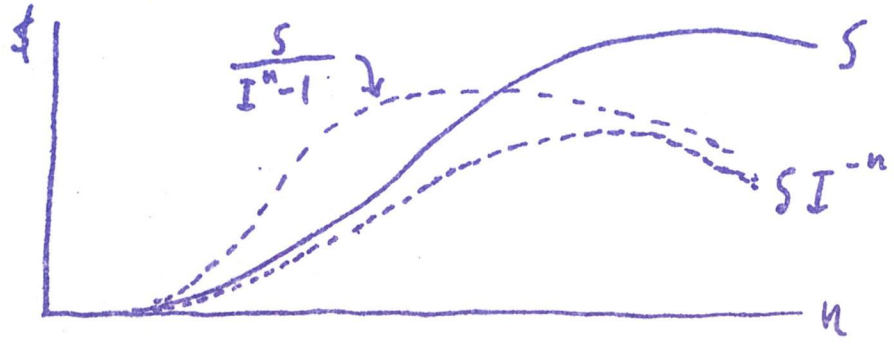


Fig. 7: Growth of Stumpage, Discounted Stumpage & Land Value

$S \equiv$ Stumpage, the market value of a tree "on the stump", before cutting.

alternative maximizing solutions.[‡]

1) Max S
2) Max S/n, "Mean Annual Increment"
3) Max SI^{-n}; PV of G_1
4) Max $\frac{SI^{-n}}{1-I^{-n}}$, PV of all G_j same as

Land Value, L.

Clearing the fraction:

$$L = \frac{S}{I^n - 1} \qquad (25)$$

as shown on Fig. 7

The economic solution is to max L, using (25), known as Faustmann's Formula. This advances the date of maturity over the other 3 criteria of maturity.

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‡ Solutions to the problem "When to cut the tree?"

[In (25) we have left unspoken the implicit assumption that all costs and revenues in the cycle are made commensurable with S by compounding them forward to year n. See p. 24, first full para., first sentence.]

Every maximizing criterion is also a marginal criterion. Let us see what marginal conditions correspond to the 4 maximizing solutions.

1) $S' = 0$ (S' is shorthand for dS/dn, or annual growth.)

2) $S'/S = \frac{1}{n}$

3) $S'/S = i$

4) $\frac{S'}{S}$ = log derivative of the denominator, $(I^n - 1)$. The log derivative is:

$$\frac{i I^n}{I^n - 1} = \frac{i}{1 - I^{-n}} > i$$

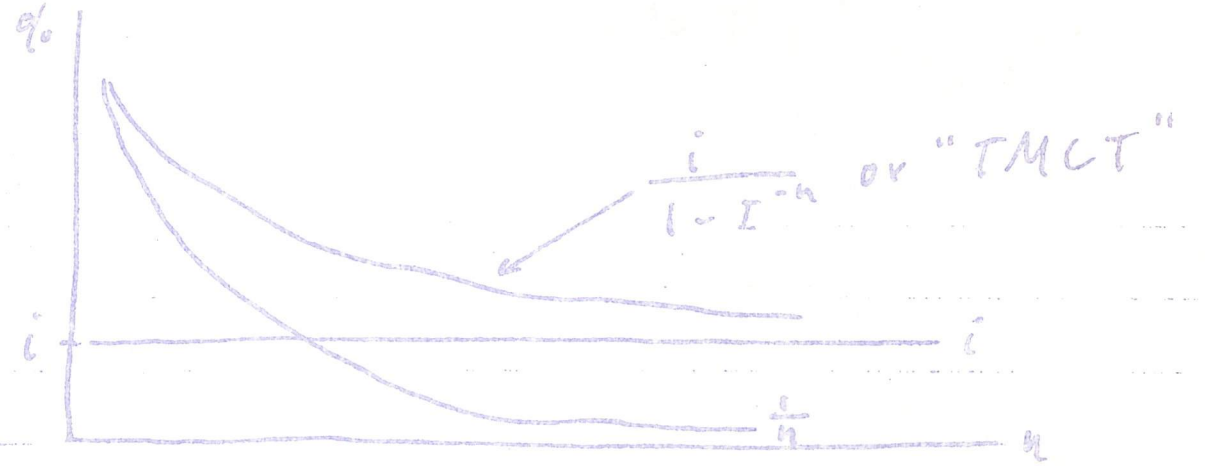


Fig. 8: Marginal Costs of Time Expressed as %

Fig. 8 shows how the true marginal cost of time, TMCT, relates to i and to $\frac{i}{n}$.

- TMCT is always higher than either i or $\frac{i}{n}$.

- For low n, TMCT $\rightarrow \frac{i}{n}$

- For high n, TMCT $\rightarrow i$

To use Fig. 8, simply use whatever curve is showing the lowest % TMC on a given day. (Use whichever curve is lowest.)

Leverage

1. % growth rate of I^n is $\frac{i I^n}{I^n} = i$

% growth rate of $I^n - 1$ is $\frac{i I^n}{I^n - 1} = \frac{i}{1 - I^{-n}} > i$

---

2. a) Costs = 1

Sales = $1.1 \pm .2$ or .9 to 1.3

Net Gain = $-.1$ to .3

Range / Sales = $\frac{.4}{1.1} = .364$

Range / Net Gain = $\frac{.4}{.1} = 4$

b) $\frac{\text{Sales}}{\text{Costs}} = \frac{1}{.90 \pm .05}$ or $\frac{1}{.90 \div (1.05)}$

3. Gain = $I^n - 1$ e.g. $2 - 1 = 1$

vs

$1.1(I^n) - 1$ $2.2 - 1 = 1.2$

a .1 rise of sales price becomes a .2 rise of gain.

Before rejoicing, though, what is this in

Leverage - 2

IRR? Depends on n.

n	$\left(\frac{2}{1}\right)^{\frac{1}{n}}$	$\left(\frac{2.2}{1}\right]^{\frac{1}{n}}$
2	.414	.483
5	.149	.171
10	.072	.082
50	.014	.016

Midterm, Ec. 162A, October 30, 1984

Answer any 5

1. Mom lends Sonny $10,000 at 8% amortized over 7 years.

a. In year one (the first year) how much is the principal reduced?

b. How much is it reduced in year 6?

c. After 7 years, what is Mom's capital recovery as a share of all the payments made?

2. a. Beginning from the PV formula, show how to find the PV of a perpetual succession of repeating cycles of investment and recovery.

b. In the above, what is the land value multiplier? What does it do -- explain.

c. In (a), where does the initial capital cost appear -- i.e. the investment that begins each cycle?

3. In the case of tree growth:

a. Show why maximizing SI^{-n}, discounted present value of stumpage, corresponds to the marginal condition that $S'/S=i$.

b. FOR DOUBLE CREDIT, THIS PART IS OPTIONAL. Show why maximizing $S/(I^n-1)$ corresponds to the marginal condition that $S'/S= i/(1-I^{-n})$.

4. What is the value of land yielding cash flow of $1:

a. At 8%

b. At 8% if cash flow is to rise at 3% per year

c. If you borrow to buy the land at its correct PV, and pay 8% on the loan, how much will you owe the bank after one year? After 10 years?

5. Find the PV of a 25 year cash flow that begins with $23.53 and declines at 4.658%/year. Discount at 7.452%. Make your answer accurate to 3 places.

6. You have to choose between two processes in your plant. The Red Process costs $4m and lasts 5 years; its GPV over that time is $6m, discounting at 13%. The Yellow Process costs $7m and lasts 11 years. Its GPV over that time is $8m. Which process makes better use of the plant? Give a numerical answer and show your work.

7. It costs $125/acre to reforest cutover timberland to yield stumpage of $14k after 55 years, in constant $. What monthly rent from grazing would be of equal value, using a nominal interest rate of 15%. You forecast inflation over the period at 11%/year.

$$\frac{P}{a} = \frac{a}{a}\,\frac{1 - I^{-n}}{i}$$

$$P = a\,\frac{1 - I^{-n}}{i}$$

$$\frac{Pi}{a} = 1 - I^{-n}$$

$$1 - \frac{Pi}{a} = I^{-n}$$

$$\ln I^{-n} = -n \ln I$$

Answers Midterm #1

1. P = 10k

i = .08

n = 7

a. $a = \frac{.08}{1 - 1.08^{-7}} \cdot 10k = 10k \cdot .19 = \underline{1920.72}$

$1.08^{-7} = .58$

$a \cdot 1.08^{-7} = \underline{1120.72}$

~~or~~

or $a\, I^{-7} = P \frac{i}{I^7 - 1} = 10{,}000 \cdot .1121 = 1120.72$

or

~~tot~~ $a\left[\frac{1.08^{-6} - 1.08^{-7}}{.08}\right] = 10k \frac{1.08^{-6} - 1.08^{-7}}{1 - 1.08^{-7}} = 1120.72$

b. $a\, I^{-2} = 1920.72 \cdot .8573 = 1646.70$

c. $\frac{10{,}000}{7 \cdot a} = \frac{10{,}000}{13{,}445} = .7438$

or $\frac{PV}{n} = \frac{1 - 1.08^{-7}}{7 \cdot .08} = \frac{5.206}{7} = .7438$

2. a. $\frac{a}{i} = \frac{P}{1 - I^{-n}}$

b. $\frac{1}{1 - I^{-n}}$

c. Deducted from P

3. a. Ratio of log derivatives. Same as E

b. Same. $\frac{\Delta(I^n - 1)}{I^n - 1} = \frac{i I^n}{I^n - 1} = \frac{i}{1 - I^{-n}}$

4. a. $\frac{1}{.08} = 12.5$

b. $\frac{1}{.08 - .03} = 20$

c. $.08 \cdot 20 - 1 = .6$

You owe $20 + .6$

Year	a	~~Net~~ New loan	Cum. loan	Int.	c.f.
1	1	.6	20.6	1.6	1
	1.03				
	1.03^2				
	1.03^3				

End of Yr.	c.f.	Int	Cum. debt	New debt.
-1	0		20	20

Begin Yr.	Int.	c.f.	Int - c.f.	New Debt	Cum. Debt
0	—	-20		20	20
1	1.6	1	.6	.6	20.6
2	1.648	1.03	.618		

$P_1 i = a + P_g = 1 + .03 \cdot 20 = 1.6$

$P_1 i = P_0 (1+g) i$

P_2 etc.

In each yr. you must borrow $P_n i - a_n = P_n g$

∴ in 10 yrs. your debt will = the new value.

$$P_0 (1+g)^{10} = 20 \cdot 1.03^{10} = 20 \times 1.344 = 26.88$$

$a = \$23,53$

5. $P = a \dfrac{1 - 1.12^{-25}}{.12} = a \times 7.78367 = \$183.1498$ $\quad \dfrac{1 - .05740}{.12}$

or —

~~or~~ $P = a \dfrac{1 - \left(\frac{1-d}{1+i}\right)^{25}}{i+d} = a \dfrac{1 - .05032}{.12} = a\ 7.84211$

$= 184.5249$

6. Red $\quad L = \dfrac{-4m + 6m}{1 - 1.13^{-5}} = 2 \cdot 2.181 = 4.374\ m$

Yellow $\quad L = \dfrac{-7 + 8}{1 - 1.13^{-11}} = 1 \cdot 1.353 = 1.353\ m$

7. real $i = .15 - .11 = .04$

$F = 14k - 125 \cdot 1.04^{55} = 14k - 125 \cdot 8.65$

$= 14k - 1,081 = 12,919$

$F = \dfrac{.04}{1.04^{55} - 1} = 12,919 \cdot .00523 = \$67.58$

in const. $.

Some elements of cap. thy for planning.

Topics. Cap. Thy Summary

(used for whole life)

1. Comp. int. . Doubling time. Semi-log scale.
2. Disc.
3. Capn.
4. PV of annuity
   Sum of geom. progn.
   Apply to S
   Confirm by a/i ... Capse for n years.
   Graphically 
   Depleem.
5. Subst. $e^{n \cdot}$
6. Crf.

   Instalment payment. $\rightarrow i$. Debt service. Fixed cap. cost.

   Int. $= a(1 - e^{-in})$

   Recov. $= a e^{-in}$

   ✓ Morals: a) Limited gains from spreading o'head over time.

   ✓ b) Is cost of using cap. to repl. labor for n yrs.

   ~~Acc Study gen.~~

7. Growth of annu. at comp. int.
   Sheep vs. trees

✓ 8. S.T.
   a Time-ave.

3. See notes from Hauke seminar
" " " UCLA course

4 Topics

a. Opt. life
Appre.
Depre.

b. Normalizing (TM/can)
App.
Dep.
w L w/o land

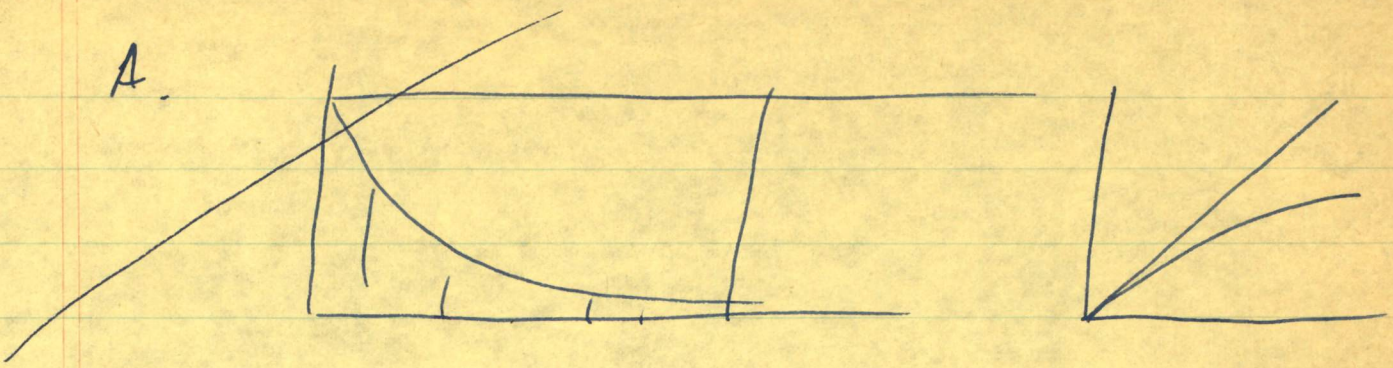
c. Factor propns.

d. Treat C as vble., too
(Use my App. on Intensity)(?)

e. Depreen. & depln. (& appreen?)
i) Effect of shorter future life - s.f.
Deriv. of p.v.; chop off terminal year; see RIMS

ii) Depre. for tax; ~~H.~~ depr. for mgmt., to know how much you have to replace, & what your assets are.

A.



B. $a = P \dfrac{i}{1 - e^{-in}}$

Cap. rec. → Pi as $n > 40$ or so.

Annuals:

C. $F = a \dfrac{e^{in} - 1}{i}$ cumulative amt.

~~Annuity found.~~

~~Sheep vs. trees.~~

D. $a = F \dfrac{i}{e^{in} - 1}$ sinking fund

Sheep vs. trees.

7. Replacement rules

A. Appreig Capl.

$$I = (C+s)e^{in} - S$$

$$\frac{dI}{dn} = i(C+s)e^{in}$$

$$= i(I + S)$$

Inv. grows each year by i on $I+S$

As long as wood grows faster, i is unripe.

This unpd. rule also means. DCF, in perp

B. Depreig K.

Challenger & Defender

I falls to 0

Max S.

$$S = \frac{R_m - C_0 e^{im} + R_n e^{i(m-n)}}{e^{im} - 1}$$

As a regul. solution

$R_m < Si$ means renew.

Better, 1st term is $(1+i)^{-1}$

$$S = (1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-n}$$

$$= \frac{1-(1+i)^{-1\cdot n}}{1-\frac{1}{1+i}}(1+i)^{-1} = \frac{1-(1+i)^{-n}}{1+i-1} = \frac{1-(1+i)^{-n}}{i}$$

Note: formula presupposes 1st payment is due at end of one year.

Can I handle e^{-i} as r?

$$S = e^{-i} + e^{-2i} + \dots + e^{-ni}$$

$$= e^{-i}\,\frac{1-e^{-in}}{1-e^{-i}} = \frac{1-e^{-in}}{e^{i}-1}$$

No, not this way.
Wd. have to use integral.

$$S = \int_{n=1}^{n} e^{-ni}\,dn = \frac{e^{-ni}}{-i} + C = C - \frac{e^{-ni}}{i}$$

~~Look up integral, give this as 2d option.~~

To evaluate C, let $n = \infty$, then $C = \frac{1}{i}$

Cap. Thy. 1 summary of math.

1. Sum of geom. progression

$$S = 1 + r + r^2 + \dots + r^{n-1}$$
$$rS = \quad r + r^2 + \dots + r^{n-1} + r^n$$
$$S - rS = 1 - r^n$$
$$S = \frac{1 - r^n}{1 - r}$$

2. PV of annuity

$$\Sigma = 1 + (1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-(n-1)}$$
$$r = (1+i)^{-1}$$
$$S = \frac{1 - \left(\frac{1}{1+i}\right)^n}{1 - \frac{1}{1+i}} = \frac{1 - (1+i)^{-n}}{\frac{1+i-1}{1+i}} = (1+i)\,\frac{1 - (1+i)^{-n}}{i}$$

$$S - 1 = \frac{(1+i)(1-(1+i)^{-n}) - i}{i} = \frac{1 - (1+i)^{-n} + i - i(1+i)^{-n} - i}{i}$$

$$= \frac{1 - (1+i)^{-n} - i(1+i)^{-n}}{i} = \frac{1 - (1+i)^{-n}(1+i)}{i}$$

$$= \frac{1 - (1+i)^{-(n-1)}}{i}$$

Annuity

Deprecn

Purposes:

a. To monitor changes in your wealth
b. To evaluate tax depn. rules
c. To allocate c.f. between int. & depn.

Numerical e.g. — ~~farm Alt. Study~~.

Taxn.

Mineral rents, insensitive to i.r.

Replmt. pol. & opt. life.

begin w. early years of neg. c.f., -C

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
n	$(R-C)_n$	$\frac{(4)+(2)}{(n+1)}$	(3)/1.1	Deprn.	% int. on DCF	% int. + dprn			
	$	$_n$	DCF_n	$(4)_n-(4)_{n+1}$		$=(2)$			
1	9								
2	10								
3	10								
4	10								
5	8								
6	6								
7	4								
8	3								
9	2	2.91							
10	1	1.0	.91	.91					
			.00						

Note: if you begin same process from the beginning, is called c.f. analysis. Figure int. on unrecovered balance.

CHAMPION LINE NO. 636

Ideas on Cap. Thy.

$\frac{1}{1.1} = 11\overline{)10.0}$ .909, 99, 100

"Time-average" — something to work.

$$a = \frac{\Sigma R}{m} \quad \text{when } i > 0$$

$$a = \frac{\Sigma \left[R_n e^{i(m-n)} \right]}{e^{im} - 1} \, i$$

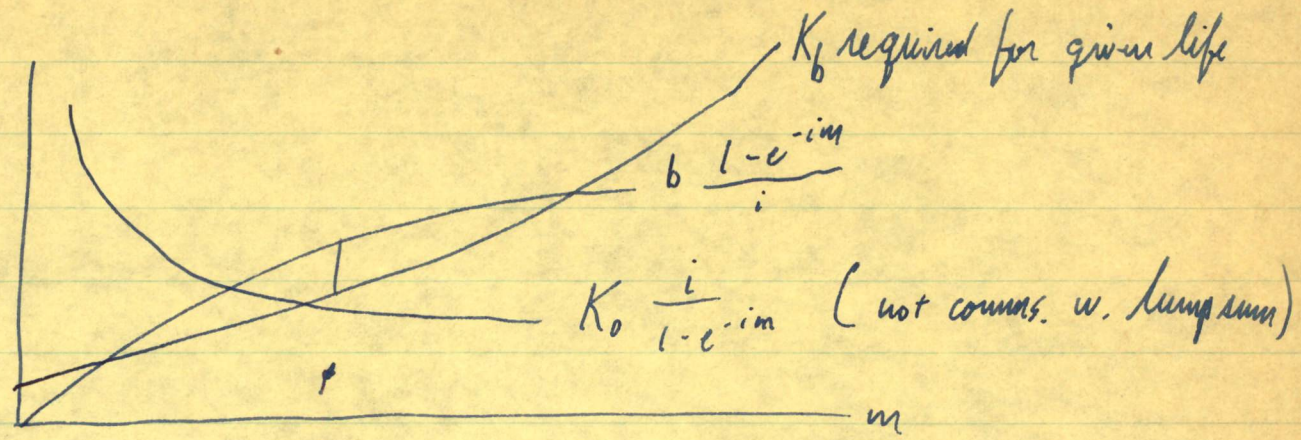
$$a \text{ also} = \frac{\Sigma \left[R_n e^{-in} \right]}{1 - e^{-im}} \, i$$

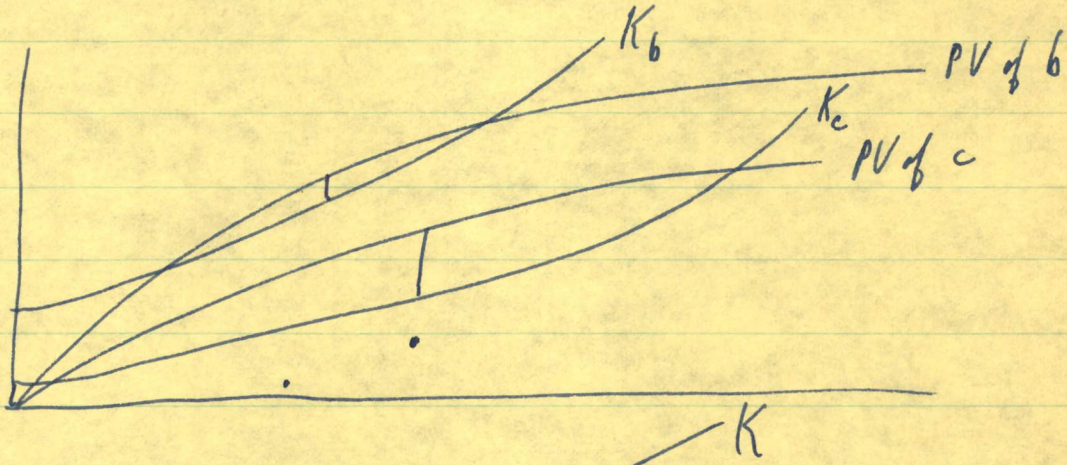
Change R to $(R - C)_n$

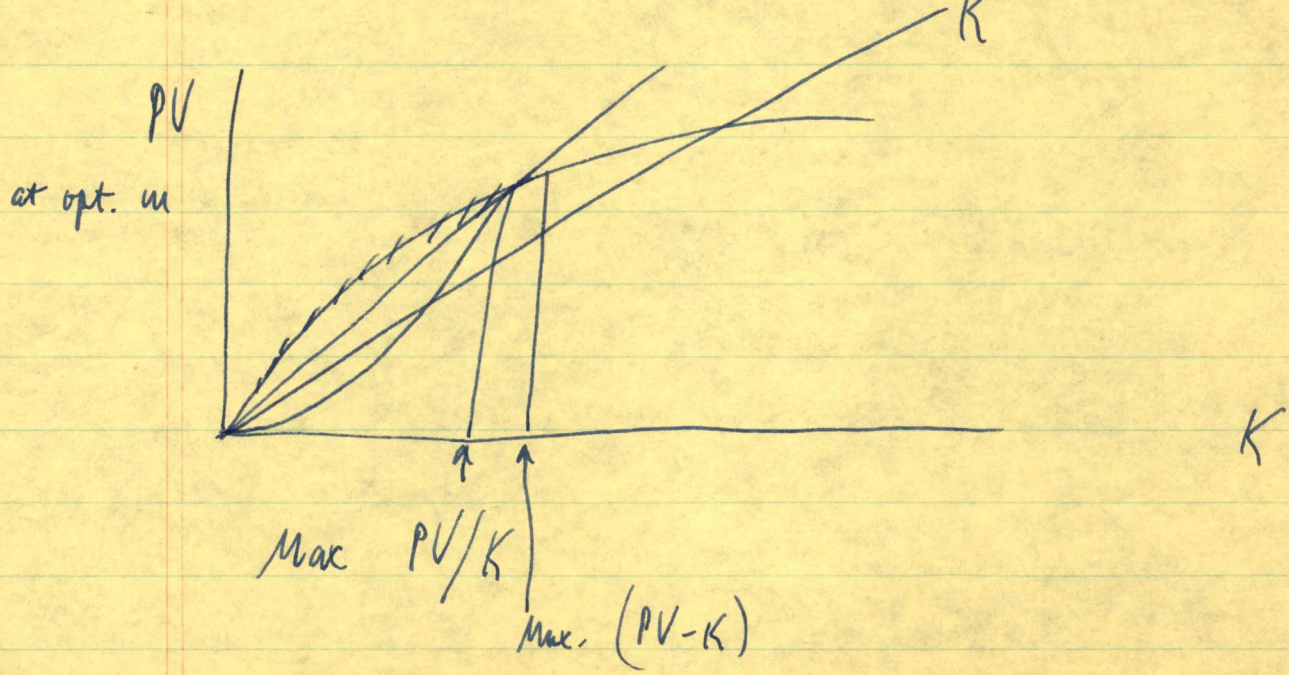
Handy to treat cap. costs separately as K_0

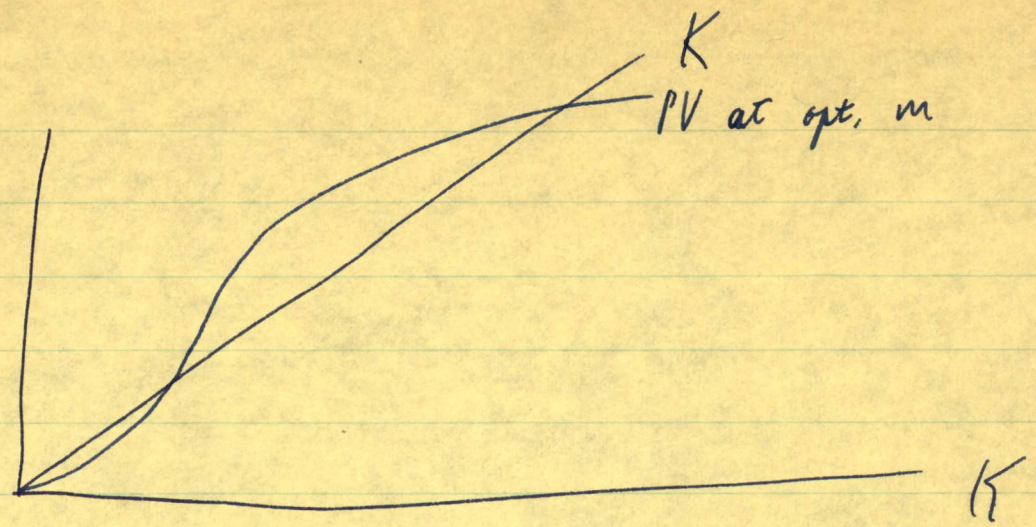
Higher K means higher R (& / or longer R_n).
Show some fns., by tables or graphs, to show tradeoffs.

.909
$11\overline{)10.0}$
99
100


K_b required for given life
$b\frac{1-e^{-im}}{i}$
$K_0\frac{i}{1-e^{-im}}$ (not comms. w. lump sum)
m

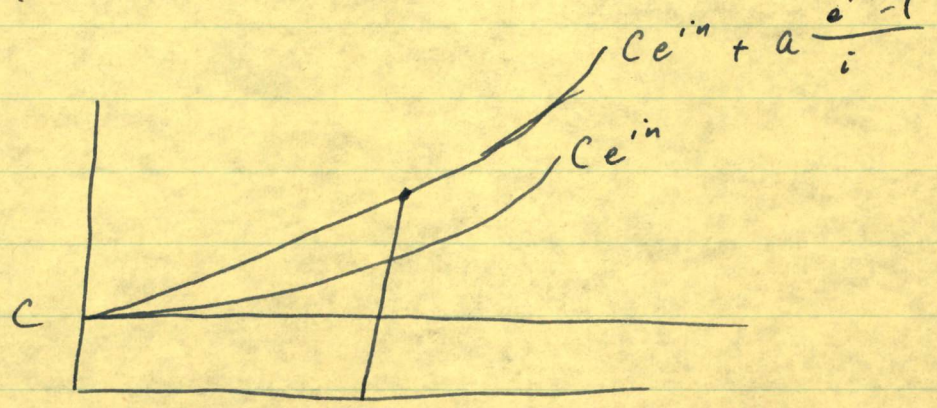

K_b
PV of b
K_c
PV of c
K


PV
at opt. m
K
Max PV/K
Max. (PV-K)

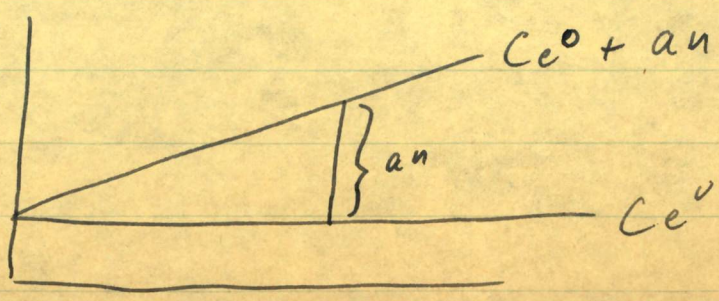


$$\frac{i}{e^{in}-1} \quad \text{as } i \to 0$$

$$\frac{\frac{di}{di}}{ne^{in}} = \frac{1}{ne^{0}} = \frac{1}{n}$$



as $i \to 0$, above becomes



$$I = (C+s)e^{in} - S$$

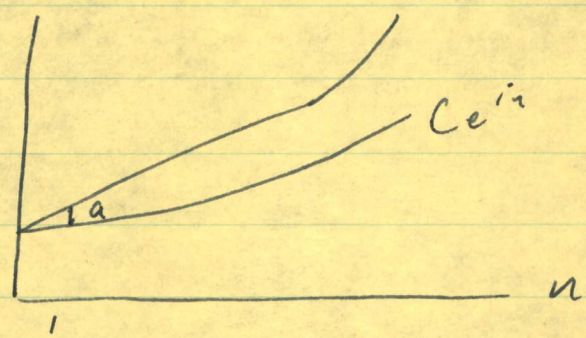
$$\frac{dI}{dn} = i(C+s)e^{in}$$

When $n = 0$

$$\frac{dI}{dn} = i(C+s) = iC + a$$

$$\frac{dI}{dn} - Ci = a$$

As:



Opt. life of K.

This is where idea of time-ave. is useful. Given you something to max.: "$\underline{a}$".

$$a = \frac{R_m \frac{i}{e^{im}-1}}{} = \sum_0^m \left[(R_n - C_n) e^{i(m-n)} \right] \frac{i}{e^{im}-1}$$

$$\frac{i}{1-e^{-im}} \quad \text{lim. as } i \to 0$$

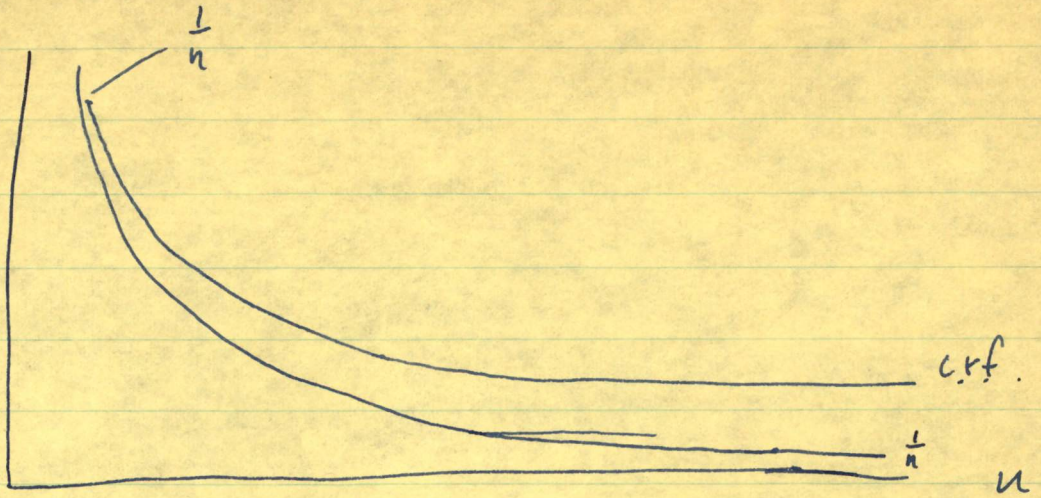
$$\frac{0}{1-\frac{1}{1}} = \frac{0}{0}$$

$$\frac{d}{di}\left(-e^{-im}\right) = -1\left(-me^{-im}\right) = me^{-im} = me^{0} = m$$

So lim. $= \frac{1}{m}$, same as for $\frac{i}{e^{im}-1}$

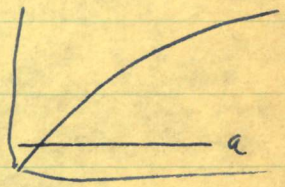
Time-averages

n	$\frac{1}{n}$	$\frac{i}{e^{in}-1}$		$\frac{i}{1-e^{-in}}$	
		$i = .01$	$i = .05$	$i = .01$	$i = .05$
1	1	1.0	1.0	1.01	1.05
2	.50	.50	.49	.5075	.538
3	.333	.33	.32	.3400	.367
4	.250	.25	.23	.2562	.282
5	.200	.20	.18	.2060	.231
10	.100	.10	.08	.1055	.130
20	.050	.05	.03	.0554	.081
40	.025	.02	.01	.0304	.058
80	.0125	.01		.0182	
100	.010	.01		.0159	



If $b = k$, then

$PV = b\,\frac{1 - e^{-in}}{i}$ and keeps growing.

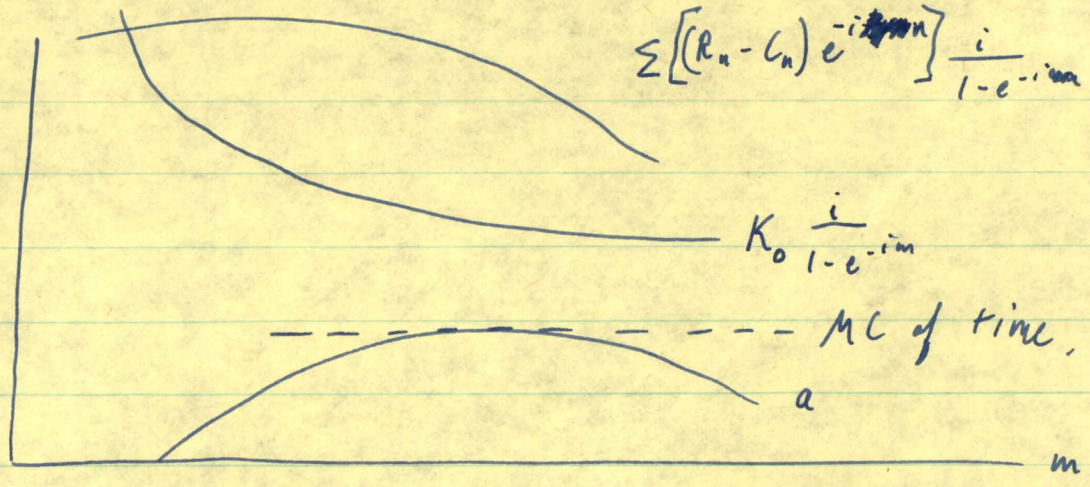


If $a = PV\,\frac{i}{1 - e^{-in}}$ then a is const. so long as b remains const.

b_n is the mgnl. Δ; a is the ave.

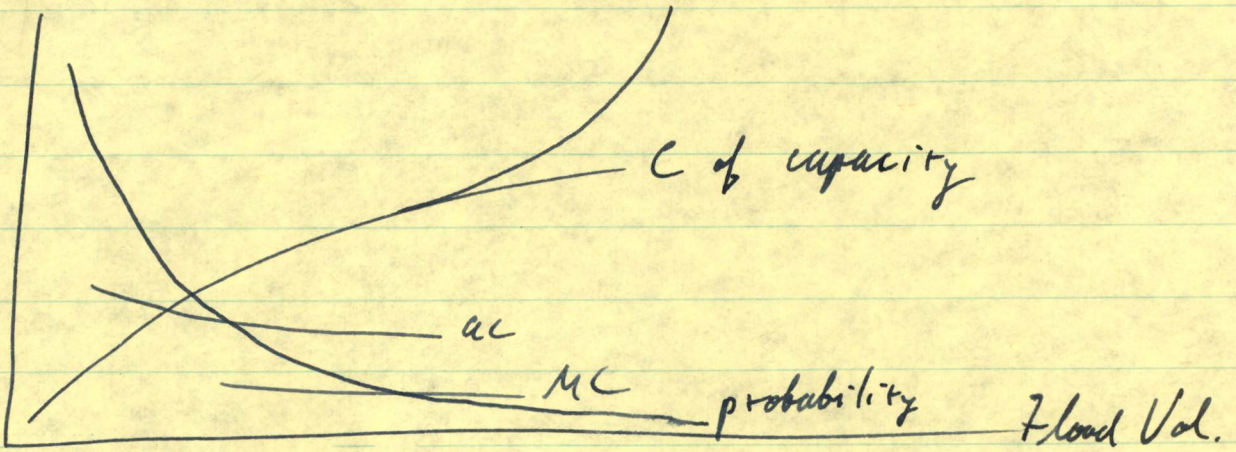
Ben. of future cycles, a is also mgnl. cost.

Ben. of hi K costs, a is not const.


$\Sigma\left[(R_n - C_n)\, e^{-i\omega n}\right] \frac{i}{1-e^{-i\omega}}$
$K_0 \frac{i}{1-e^{-i\omega}}$
MC of time.
a
m

Admin. 200

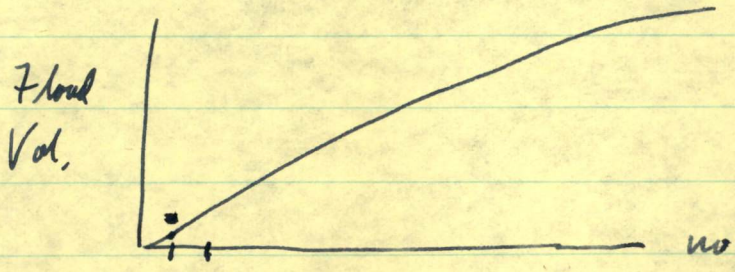
The Culvert Paradox



Washout ends life of culvert

So ∴ what is life? Capacity adds to life.

But that's part of value. Larger culvert protects both road and itself. Value of that is D, cost of ops losses plus replacing road & culvert itself. But as culvert gets larger, replacing same costs more, reducing net gain.



Capac. for the p-year flood will wash out every (p+1) years. So life is (p+1)

An engineering fallacy? wd. be to include in benefits (averted damages) the cost of the larger culvert.

ΔC & ΔB approach avoids this.

ΔB = PV of damage (D) averted.

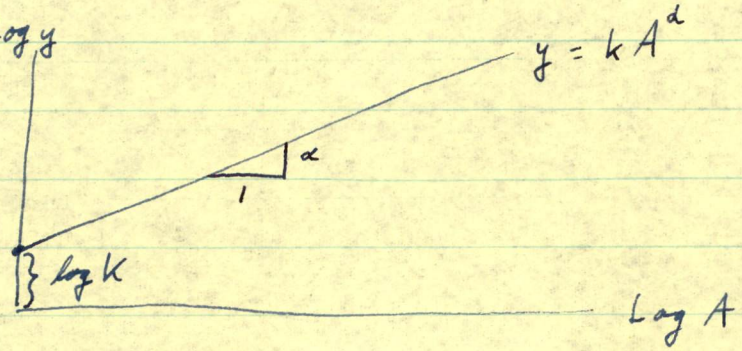
$PV = \frac{D}{P} \frac{1 - e^{-in}}{i}$ but what is n?

Larger culvert adds to its own life, too. So D is an Δ_2 fn. of cost?

I. $y = k A^{\alpha}$

$\log y = \log k + \alpha \log A$

Log-log paper



On the log scale, constant distances are constant % Δs. Show this by doubling.

So when $\frac{\Delta A}{A} = x\%$, $\frac{\Delta y}{y} = \alpha x\%$

Elasticities of all kinds in econ. are ratios of % changes. They are useful for many purposes. They are best analyzed, graphically, on log-log scales. They show up several places:

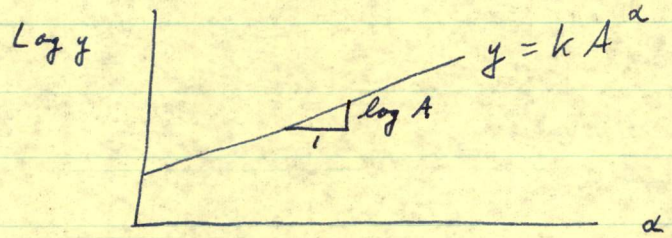
1. Elast. of prodn., as above
2. Elast. of demand
3. Valuation, max. P.V., by selecting opt. life, usually involves maxing a ratio whose num. & denom. are + fns. of life. You find the max. by equating % changes of num. & denom., ~~i.e. the~~ This is not an elasticity, in any simple sense, bcs. denom. is not life but a fn. of life (exponential at that). But you can say the PV is max when elastic. of num. w. respect to denom. = 1.

(If denom. is $(1+i)^L$, then $\% A = i$ when $\Delta L = 1$.)

4.

II.

If it is $\underline{\alpha}$ rather than $\underline{A}$ that varies, as in compounding, ~~then~~ then any given absolute $\Delta\alpha$, (such as 1), always gives same % increase of y. This is not an elasticity relationship. This is best shown on semi-log:



$$\text{Log } y = \log k + \alpha \log A$$

Every time $\Delta\alpha = 1$, $\log y$ increases by $\log A$, a const. (In interest problems, $\log A \doteq i$.)

III. In Cobb-Douglas, α is elast. of prodn. w.r.t. A.

How about $\alpha + \beta$? Since the increases of the two are %s, they are thus made commensurable — a huge advantage of the % approach. $\alpha + \beta$ must be the elast. of prodn. response to all inputs. Show that.

$$\text{Log } y = \log k + \alpha \log A + \beta \log B$$

If I $\Delta \log A$ by 1, and $\Delta \log B$ by 1, then I Δ Log y by $\alpha + \beta$. If $\alpha + \beta > 1$, I $\therefore$ $\Delta \log y$ by > 1, and the % $\Delta y >$ % Δ inputs

IV. There is an (alleged?) problem of measuring capl. input.
Why not just msre it by its cost? Weigh its cost vs. its output.
But don't try to define its quantity as det. by its output.
There are other problems with capital, later.

V. So far, anal. has been timeless or simul. Lends element of unreality. Esply. re circ. capl.
Over the life of any product, it is circ. capl. whose return is meaningfully "residual": It pays other inputs, what they get is their product. Capl. carries the product thru time. The popular word 'profit' conveys this, but is overstated by failing to deduct implicit costs, usually land rent, ~~quasi-rent~~ return on fixed capl., return of fixed capl., wages of mgmt and other owner labor, and int. on owner circ. capl.
These implicit costs are valued at o.c., interpreted in most liberal possible manner, in every dimension. (In s.r., fixed capl. has a s.r.o.c. or salvage value far below l.r. o.c., so s.r. profit, gross of capl. service, > l.r. profit. Ea. def. has its uses.)
Having done that,

How do we then behave to, ~~min~~ avoid "risk"? Min. the circ. capl. input, i.e. stay away from mgmt. invts.? Two meanings of 'risk': ~~absolute~~
1) ~~to~~ dispersion of gross cash flow, --- this is not reduced by avoiding mgmt. invts.;
2) dispersion of net cash over costs, as a % of net. -- This is reduced by avoiding mgmt. invts. But this doesn't work either, bes. then you are undercutting other assets you pay for, shooting for less than max. return to them.
So this brings us to where we were before: to min. risk, let max. % of costs be mgr costs or vble. costs; let min. % be fixed costs -- land & durable capl.

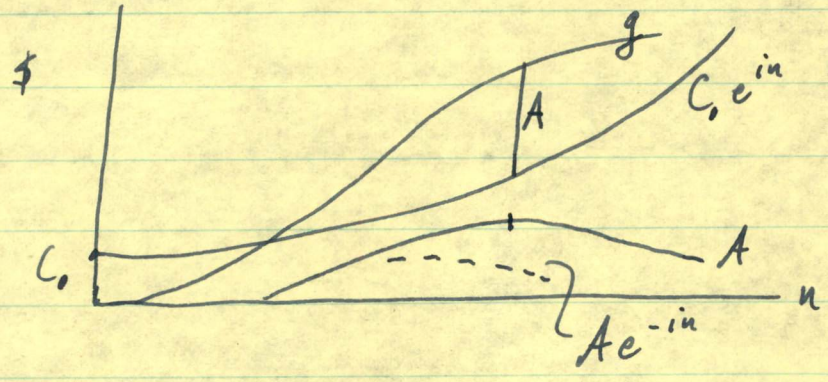
The confusion comes via credit rationing, when you own fixed assets outright, cost becomes implicit; and fixed-assets are collateral. So the risk of fncl. trouble is min. by undermining owned fixed assets.

Thus the greatest social risk becomes the least pvt. risk, and resources get misallocated. Too much goes to create fixed assets.

Admin. 203. Monday, March 14, 1977

Exam Fri. March 18, 5:30 P

1. Review math. of fncl. maturity



a. Max. A not r.t., bcs: 1) we're not discounting it.

~~2.~~ 2) $g' < g_i$

b. Max. Ae^{-in} not right, bcs.

1) $A = g - Ce^{in}$

$$Ae^{-in} = \frac{g}{e^{in}} - C$$

Is same as max. $\frac{g}{e^{in}}$

This is independent of C, i.e. replacement cost. Replacement policy shd. depend on replmt. cost.

Is same as $g' = g_i$. Doesn't cover rent.

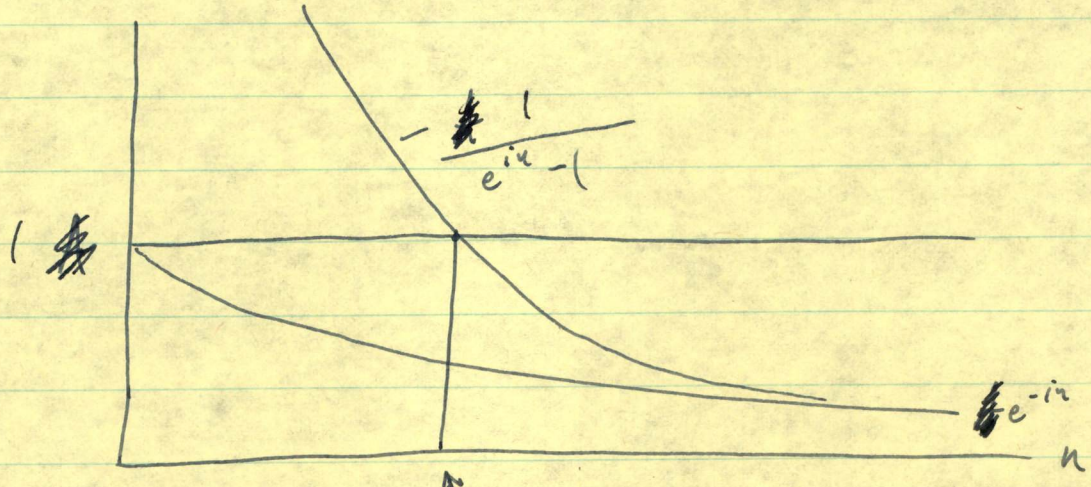
c. Right answer

Convert A to annuity, using that

$$a = A \frac{i}{e^{in} - 1}$$

Since the annuity can be repeated in perpetuity, it is ~~independent of n.~~ a fair way to compare rival plans that have different lives. Single-cycle DCF is not.

Since annuity is perpetual, can convert to S_e by simple capn: $\div i$ (The limit of PV formula as $n \to \infty$). $S_e = \frac{A}{e^{in} - 1}$

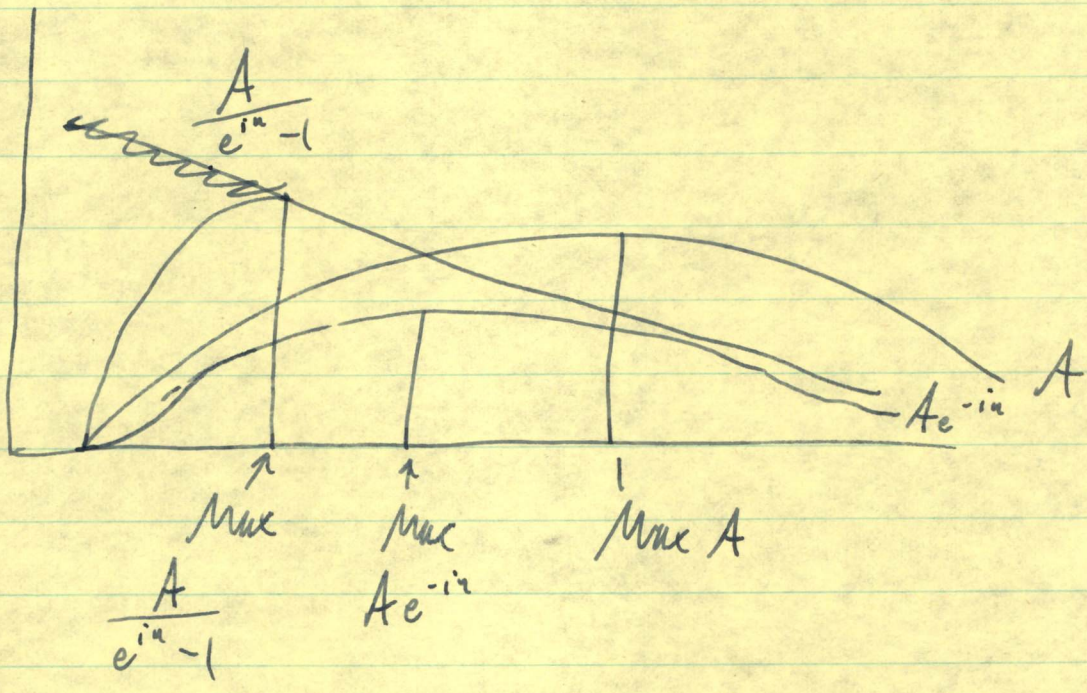


$e^{in} = 2$, or $n \doteq 7$

n	$(1+i)^n - 1$	$1/[(1+i)^n - 1]$
1	i	$1/i$
2	$2i + i^2$	$1/(2+i)i$

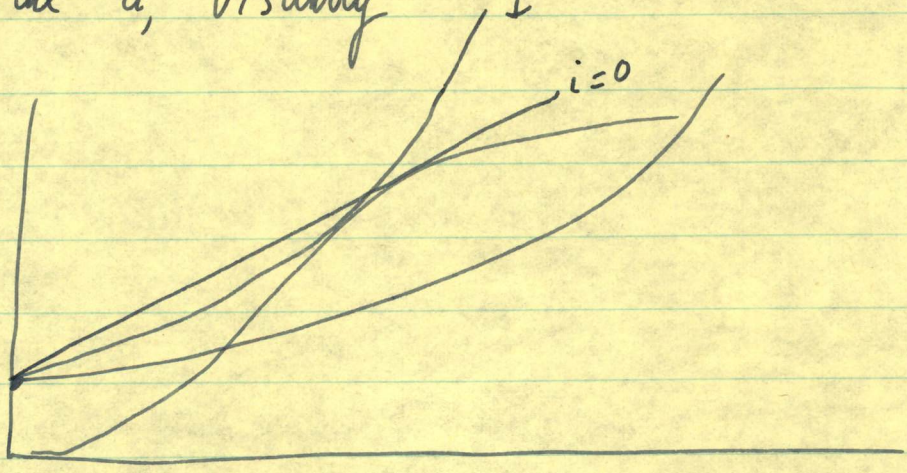
When n large, the two maxima are about the same.

For most cases, they are very different: wine, cattle, chickens



3A

Max a, visually I = Inventory value $= Ce^{in} + a\frac{e^{in}-1}{i}$



(C_0 is so small cfd. to g_1 that c_{mai} is not affected)

Also find S_e as sum of series

$$S = Ae^{-in} + Ae^{-2in} + \dots + Ae^{-2i\infty}$$

$$= \frac{A}{1-r} \quad \text{where } r = e^{-in}, \text{ & 1st term is } Ae^{-in}$$

$$\frac{Ae^{-in}}{1-e^{-in}} = \frac{A}{e^{in}-1}$$

This is the right approach to Replacement Policy - can be generalized.

Is right approach to inventory control policy.

Mgnt. land

a. One tree of every age.

No. of trees = n.

$$K = e^{i} + e^{2i} + \dots + e^{ni} = \sum_{n=1}^{n} e^{ni} = \frac{e^{ni}-1}{i}$$

($\frac{1-r^n}{1-r}$

Skip $r = e^i$

$\sum = \not{e}^{i} \frac{1-e^{ni}}{1-e^{i}} = \not{e}^{i} \frac{e^{ni}-1}{i}$)

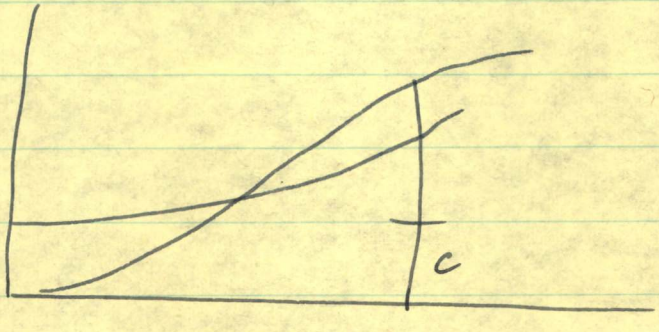
Holding cost = $Ki = e^{ni} - 1$

Cash flow = e^{ni}

$\frac{Ki}{e^{ni}} = 1 - e^{-ni}$, an increasing fn. of n.

If $n = \infty$, share = 100%.

Same as single cycle



Ec. 162, Memo. Nov. 10, 1981, by M. Huffman

Dedicated to Priscilla Douglas, who asked the right question.

This is for more serious students who want really and fully to understand replacement planning, and valuation of durable property, and the basic math. tools needed.

1. An oversimplified model of a typical asset.

We are considering buying or building a structure to generate cash flow on a site we own.

a = annual cash flow
(Cash flow = gross receipts less running costs.)
n = end of life
C = Capital cost, time zero.
G_1 = PV of Generation #1
S = Site Value in perpetuity

$$G_1 = -C + a\frac{1-(1+i)^{-n}}{i} \qquad (1)$$

$$S = G_1\left[\frac{1}{1-(1+i)^{-n}}\right]$$

$$= \frac{-C}{1-(1+i)^{-n}} + \frac{a}{i} \qquad (2)$$

The trouble with this oversimplified model of S_n is the same as with G_i alone, viz., it has no finite maximum w.r.t. $\underline{n}$. It just says that if $\underline{a}$ will run on indefinitely, then the less often we replace, the better.

To correct for this, we modify the model and let cash flow decline, or "decay", with n.

2. A simple but adequate model of a typical asset with decaying cash flow.

d = decay factor, such that

$$\text{cash flow} = a\left[\frac{1}{1+d} + \frac{1}{(1+d)^2} + \dots + \frac{1}{(1+d)^n}\right]$$

$$G_i = -C + a\,\frac{1-[(1+i)(1+d)]^{-n}}{d+i(1+d)} \qquad (3)$$

$$S = \frac{-C}{1-(1+i)^{-n}} + \frac{a}{d+i(1+d)} \times \frac{1-[(1+i)(1+d)]^{-n}}{1-(1+i)^{-n}} \qquad (4)$$

a skilled mathematician could inspect (4) carefully, and probably see S rising to a max and then falling, with n. However, let us nail it down firmer.

3. Finding the max of a ratio.

It is intuitive, and you can demonstrate yourself, that a ratio rises in value when the numerator grows by a larger % than the denominator; and stops growing when the two % changes are the same.

$$\frac{u}{v} = \text{a ratio}$$

Ratio constant when:

$$\frac{du/dx}{u} = \frac{dv/dx}{v} \qquad (5)$$

Remember or look up the derivative of a ratio. Usually you find:

$$\frac{d\,\frac{u}{v}}{dx} = \frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2} \qquad (6)$$

But we may rewrite (6) so it immediately confirms (5).

$$\frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2} = \frac{u}{v}\left[\frac{\frac{du}{dx}}{u} - \frac{\frac{dv}{dx}}{v}\right] \qquad (7)$$

This may make you feel better. Our intuitive device is just simplified differential calculus.

4. Maximizing S

To maximize S, solve (5), using (4a)

$$S = \frac{-C + a\,\frac{1-[(1+i)(1+d)]^{-n}}{d+i(1+d)}}{1-(1+i)^{-n}} \qquad (4a)$$

$$\frac{du}{dn} = \frac{-a}{d+i(1+d)}\left[\quad -[(1+i)(1+d)]^{-n}[d+i(1+d)]\right]$$

$$= a[(1+i)(1+d)]^{-n} \qquad (8)$$

$$\frac{dv}{dn} = i(1+i)^{-n} \qquad (9)$$

$$\frac{du/dn}{u} = \frac{dv/dn}{v} \quad \text{when}$$

$$\frac{a\left[(1+i)(1+d)\right]^{-n}}{-C + a\dfrac{1-\left[(1+i)(1+d)\right]^{-n}}{d+i(1+d)}} = \frac{i(1+i)^{-n}}{1-(1+i)^{-n}} = \frac{i}{(1+i)^n - 1}$$

$$\frac{a\dfrac{(1+i)^n - 1}{\left[(1+i)(1+d)\right]^n}}{a\dfrac{1-\left[(1+i)(1+d)\right]^{-n}}{d+i(1+d)} - C} = i \qquad (10)$$

The left side represents $\dfrac{du/dn}{u}$, and (once the denominator > 0) the derivative is positive so long as the left side is $> i$. It clearly is when $v = 0$, and for a reach of n-values greater than that, as it grows slowly. But then it clearly must peak and fall, because:

a) $v \to$ an asymptote or constant value

b) $u \to \frac{a}{(1+d)^n}$ and keeps getting smaller.

5. a simple, marginal view of how to max w.r.t. n

The tangle of (10) can become suddenly clear and evident when we rearrange terms in such a way as to make it clear that the condition says that G_1 should be replaced in the year when the decaying cash flow ceases to earn interest on site value.

The denominator of (10) on the left side is G_1. Substituting,

$$a\frac{(1+i)^n-1}{(1+i)^n(1+d)^n} = G_1 \cdot i \qquad (11)$$

$$\frac{a}{(1+d)^n} = i\frac{G_1}{1-(1+i)^{-n}} = i \cdot S \qquad (12)$$

In (12), the condition for max S is seen as a requirement that the current cash flow, the left side, just cover interest on the site. This makes intuitive sense in its own right.

So what most people do (if they think this

through at all) is to figure they will demolish + replace capital when its cash flow no longer pays interest on the value of S. This is a useful rule of thumb to remember.

6. The ultimate need to max S w.r.t. n

The trouble is, logically it is circular, because S is unknown, it is what we are determining. Most people just take market value for S, and make out all right, but someone has to do the original thinking for others to follow. This person earns extra income by being first. She or he can spot bargains, avoid turkeys, etc.

This leads us back to (10). We need to solve it for n. The best way is to set up some standard tables, based on typical parameter values. Let:

$i = .05$ ≠

$c = 100$

$a = 20$

$d = .08$

Now we figure left side as function of n

≠ Using a low or real rate of interest has the same effect as letting a rise annually due to inflation. There are alternative adjustments for inflation

n	1.05^n	$(1.05 \times 1.08)^n$	u	v	u/v
22	2.925	15.90	2.42	39.87	.0607
23	3.072	18.03	2.30	40.98	.0561
24	3.225	20.45	2.18	41.96	.0519
24.5	3.305	21.78	2.12	42.40	.0499

So S is max at $n = 24.5$

$$S = \frac{-100 + 20\,\frac{1 - \frac{1}{21.78}}{.134}}{1 - \frac{1}{3.305}} =$$

$$= \frac{-100 + 20\,\frac{.95}{.134}}{1 - .303} = \frac{42.4}{.697} = 60.8$$

As a check, recall from #5, above, that at max S,

$$S \cdot i = \frac{a}{(1+d)^n} \qquad (12)$$

Here, $60.8 \times .05 = 3.040$

$20\,(1.08)^{-24.5} = 3.035$

That's pretty close, considering rounding and other approximations used here.

7. Sensitivity analysis: how is optimal replacement time (n) affected by changes in parameters (a, C, i, and d)?

I append some rudimentary sensitivities. I did this by programming (10) on a HP 38E, so I could proceed quickly by trial & error. With practise & experience & better algebra, no doubt we could find an easier way.

The outstanding finding is that n is much more sensitive to a/C than to i or d.

In a few rooms, one could work up a set of tables covering all the cases you would meet in a lifetime of making investment & replacement decisions.

Sensitivity Analysis

Case No.	a/c	i	d	n
1	.20	.05	.08	24 +
2	.25	.05	.10	19 −
3a	.30	.05	.15	15 +
b	.30	.04	.15	~~15~~ −
c.	.30	.03	.15	14
d	.30	.02	.15	13 +
e	.30	.01	.15	~~13~~
4a	.30	.05	.15	~~15.4~~
			.14	15.3
			.13	15.2
			.12	15.2
			.11	15.3
			.10	15.5
			.09	15.8
			.08	16.3
			.07	~~16.8~~
			.06	17.5

Case

5a

a/c	i	d	n
.30	.05	.15	15.4
.28			17
.26			19+
.24			22+
.22			29
.21			41
.20			∞

Conclusion: n is insensitive to i + d but highly sensitive to a/c.

(But when $\frac{a}{c} \to (i+d)$, n becomes sensitive to i + d)

Ec. 100B, Prof. Taffney

Class notes. Wed. Jan. 28, 1981. Installment #5

Payback time, (concluded).

1. Errors, omissions, and clarifications, in notes of Jan. 21.

a. An omission. Put "Installment #4" on heading.

b. An error, p. 3, Eq. (2), should read:

$$K = \sum_{n=0}^{12} (1+i)^n = \text{etc.} \qquad (2)$$

This is because the first term of the progression is 1, rather than $(1+i)$. Remember the formula from which we derived the result:

$$S = 1 + r + r^2 + \dots + r^{(n-1)}$$

We have to watch the end points of the progression, to avoid pesky minor errors.

The object of beginning with 1 is to arrive at the simplest possible final form. If we have to or want to begin with $(1+i)$, however, we adjust by multiplying (2) by $(1+i)$. Or, we can add $[(1+i)^{12} - 1]$, which gives the same result

Payback Time (concluded), p. 2

This is mainly of interest to purists & technicians, and changes nothing of importance.

c. Likewise, Eq. (3) should read:

$$K = \sum_{0}^{n-1} (1+i)^n = \text{etc} \qquad (3)$$

We are now valuing capital at the beginning (not the end) of each week. For cohort #1, the value is 1, not $(1+i)$; etc.

d. Clarification of Eqn. (4)

When we divide K by V, the $V = (1+i)^n$ (and not $V = (1+i)^{13}$). I mentioned this in class, but some may have missed it.

e. On p. 7, Eq. (9), I use "S" (Sales) as the divisor. This is the same as "V" (Volume) used in (4) and (7). For consistency, change S to V in (9).

2. A second approach to showing that Payback Period = Capital ÷ Volume $(P = K/V)$

On Jan. 21 we used a "going concern" model to show this. You might ask, what if we don't have a going concern? So let's look at just one cohort of broilers going through its cycle from investment to liquidation.

To avoid imposing a lot of detail on you, we'll just do case 1, as on p. 3 of Installment #4. This is called "PIPO" (Point-Input, Point-Output). The name refers only to inputs other than capital, which are concentrated at one point of time, when $n = 0$. We have seen that there are also regular doses of capital inputs each year, as unpaid interest is plowed back. We'll keep the name "PIPO" to remind us of how easy it would be to make the mistake of thinking the payback period is $\underline{n}$, when really it is on average less than $\underline{n}$.

a picture will help:

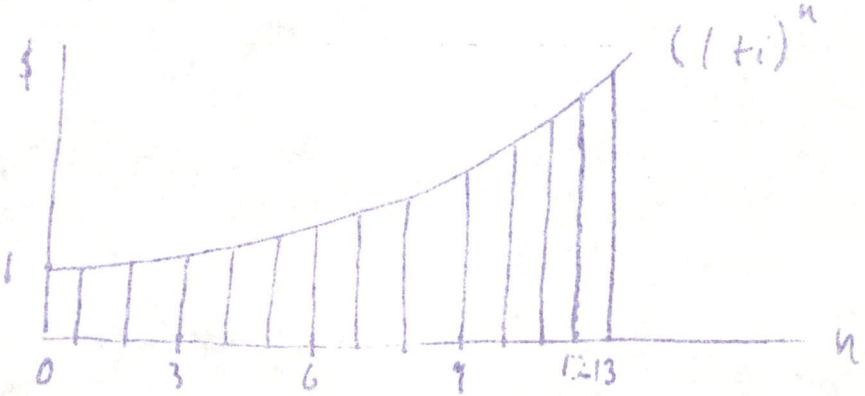


Fig. 1

Eq. (2) is the sum of the columns under the curve. In Eq. (4) we divided this sum by the last column (the right side of the last column,

to be exact, or $(1+i)^{13}$.) We said this ratio of K/V was the average payback period.

The sum of the columns under the curve is the area under the curve. This area can also be calculated as the sum of horizontal bars under the curve, where each bar is one dose of input times its respective payback period, or $k_i n_i$ (the subscript "i" here is <u>not</u> the interest rate; it means the individual unit.)

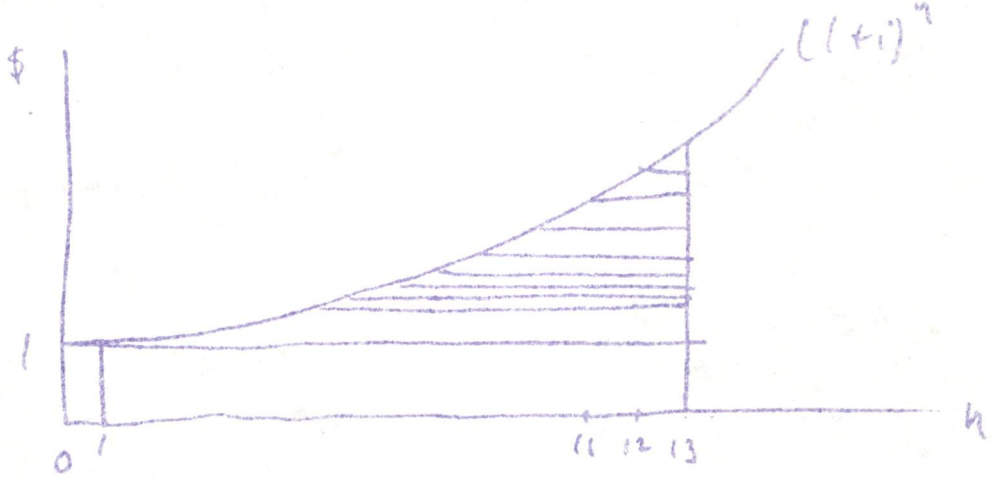


Fig. 2

I have left out the first few bars — they are too close together to draw, except for the very bottom one. [The little triangle in the top right is excluded, because our summation formula values capital only at the beginning of each year. In integral calculus they correct for this minor inaccuracy. Here, we

accept the ~~straight~~ approximation based on columns with squared-off tops -- it is plenty accurate for most purposes.]

Now observe that the area under the curve is the sum of the horizontal bars, or $\sum_{i=0}^{12} k_i n_i$, where i is the date of each dose of capital, and n_i is its individual payback time. We already know the area under the curve, so we have:

$$\sum_{i=0}^{12} (k_i n_i) = \frac{(1+i)^{13} - 1}{i} \qquad (10)$$

We can also see that $\sum_{0}^{12} k_i$ is the height of the last column, which we know is $(1+i)^{13}$.

$$\sum_{0}^{12} k_i = (1+i)^{13} \qquad (11)$$

(10) is the sum of payback periods weighted by the amount of capital with that period. (11) is the sum of all capital. Dividing

(10) by (11) (which is the sum of the weights) therefore gives the average payback period of all the capital:

$$\frac{(10)}{(11)} = \frac{\sum_{0}^{13}(k_i n_i)}{\sum_{0}^{13} k_i} = \frac{(1+i)^{13} - 1}{i(1+i)^{13}} = \frac{1 - (1+i)^{-13}}{i} \qquad (12)$$

But (12) is the same as (4), Q.E.D. The average payback period equals the capital/output ratio, or K/V.

3. Shorter periods are more labor-intensive.

In Eq. (1), p. 3, Installment #4; and Fig. 1, p. 3, Installment #5, we see that output is $(1+i)^{13}$, or generally $(1+i)^n$.
In the PIPO case, labor is applied once, when $n = 0$, and is equal to \$1 in value. Labor's share of output is therefore:

$$\frac{\text{Wage Bill}}{\text{Output}} = \frac{1}{(1+i)^n} = (1+i)^{-n} \qquad (13)$$

Capital's share is:

$$\frac{(1+i)^n - 1}{(1+i)^n} = 1 - (1+i)^{-n} \qquad (14)$$

In a going concern model, the result is the same. Capital's share is:

$$\frac{Ki}{V} = \frac{(1+i)^n - 1}{i} \times \frac{i}{(1+i)^n} = 1 - (1+i)^{-n} \qquad (15)$$

Note that capital's share increases with n; labor's share falls ~~with~~ as n rises. This is not because the rates of wages or interest change: they don't. It is because capital adds a higher share of the value of output as n rises.

We can now perceive the power of

short payback periods to employ idle labor in two ways:

a) Shorter payback means more payrolls are advanced by the same capital;

b) Shorter payback means production is more labor-intensive.

4. Effect of higher interest rates on choice of payback periods.

The interest rate is the economic constraint or discipline that makes us economize on capital when it is scarce. The following diagrams illustrate how this works. We will discuss them in class.

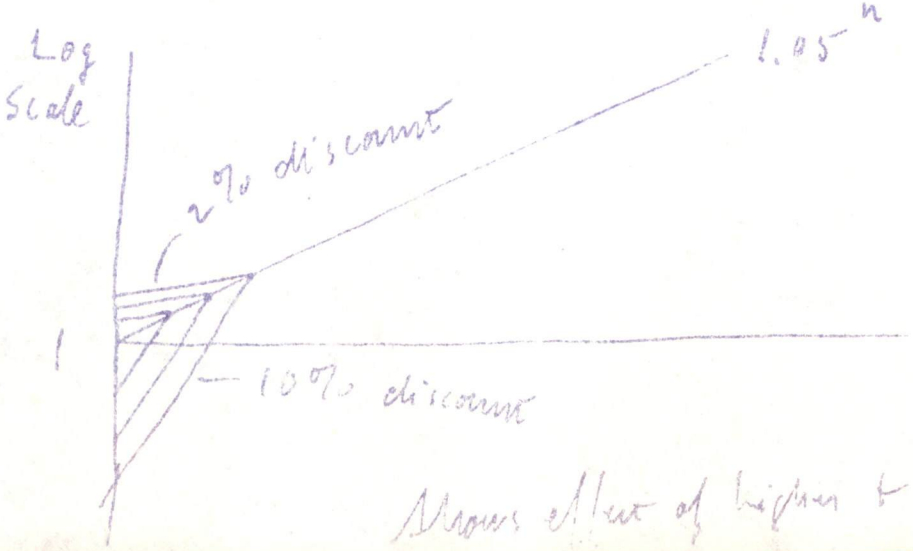


Fig. 3

Shows effect of higher & lower i.



Fig. 4

Shows effect of higher wage rates

Mgmt. Ecs.

$100 = \frac{F}{1.31} - 100$

n	i	$(1+i)^n$	R_n	C_0	Cycle / DCF	L.V.	Rent
4	.07	1.31	~~231.05~~ 262.-	100	100	409.5	28.66
5	.07	1.40	288.-	100	105.7	358.0	25.05
5	.07	1.40	294.-	100	110	372.5	26.07

E.g: Growing Hybrid Poplar for energy or pulp.

Spacing is another factor. Space the stems rotations closer. This increases C_0; avoids the drop in R_n.

a	4	.07	1.31	300	100	129.-	528.2	37.0
b	5	.07	1.40	300	90	124.3	420.9	29.5
c	5	.07	1.40	300	80	134.3	454.8	31.8

Going from c to a means a 25% ΔC_0 and no increase in R_n — but an increase in rent of ~~[illegible]~~ $\frac{37.0}{31.8} - 1 = 16\%$.

To get this gain all in one number, let there be no ΔC_0, so entire gain shows in Δ rent.

n	i	$(1+i)^n$	R_n	$\%\Delta R_n$	C_0	PV_1	$\%\Delta PV_1$	L.V.	$\%\Delta$ L.V
4	.07	1.31	300	–	100	129	–	528.2	–
|	|	|	330	10	100	152	17.8	622.4	17.8
			360	9.1	100	175	15.0	716.6	15.1

Leverage is main thing shown here. Note that leverage is not just related to debt, but to basic profit and rent.

4	|	1.31	300	–	100	129.9	–	528.2	–
8		1.72	600	100	|	248.8	93%	580.2	9.8%
16		2.95	1200	100	|	306.5	23%	454.9	-21.6%

Interpretation needed!

8. B Joe sinks the $/in when a credit crunch hits. No one but the Mafia juice men will lend to him. Not to worry, says Joe, my assets are as great as ever. What adjustments must be make when his payback time stretches out to 5 mos?

9. ~~XX~~. "Cost accts may look at fwd. data, but lenders look at pwd. data." Can you explain?

10. A cash flow annul. looks so:

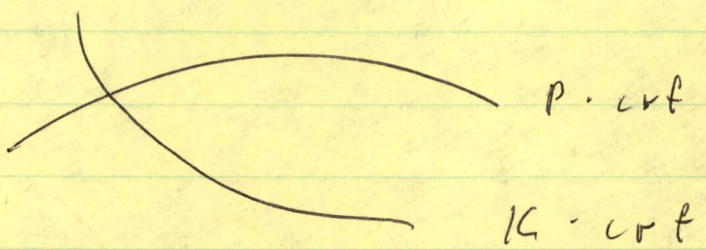
Yr:	0	1	2	3	4	5	6
Unrecov. Cap.	0	11	17.6	16.6	13.26	10.59	8.64
$ in	10	5	0	—	—	—	—
Acc. Cap.	10	16	17.6	16.6	13.26	10.59	8.64
Int. on ..	1	1.6	1.76	1.66	1.326	1.06	.86
Cash flow	0	0	2.76	5	4	3	2
Unrec. balance	11	17.6	16.6	13.26	10.59	8.64	7.51

Complete the table.
Is it consistent w. the logic of fwd. cash?
What is PV: -2.78 @ 10%
When wd. tabular c.f. can. be better?

$$\left[GPV - K + L I^{-n}\right] \frac{i}{1 - I^{-n}}$$

$$= \quad - \quad + \frac{Li}{I^n - 1}$$

$$L = \frac{DCF}{1 - I^{-n}}$$



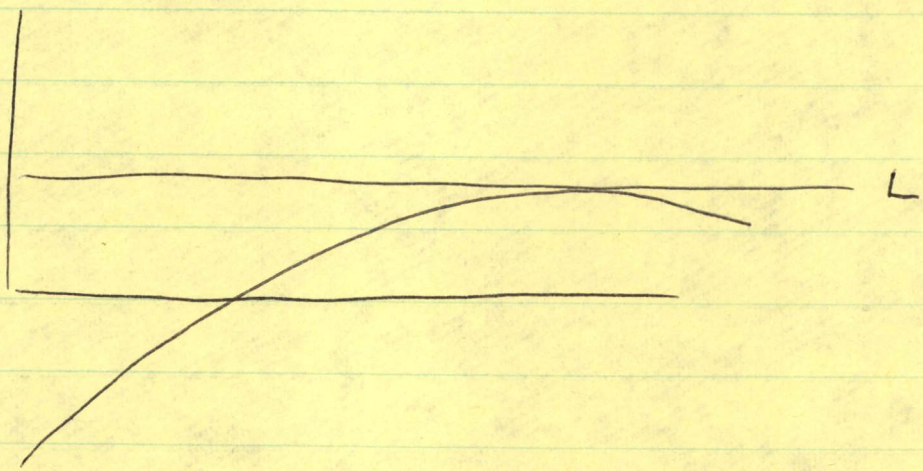
$$(P - K)\,crf = i\,\frac{P - K}{1 - I^{-n}}, \quad \underline{\text{Annuities LV}}$$

1. As Total Solution

Max $DCF + L I^{-n}$

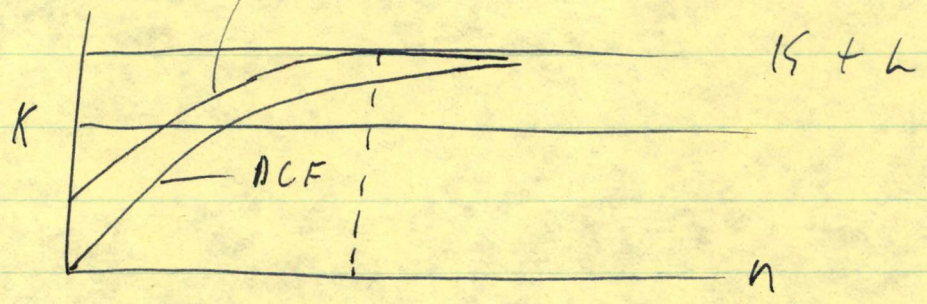
$DCF = GPV - K$

$\therefore$ also max $GPV + L I^{-n} - K$



Thurs. Missing Link from Tues.

$$DCF + L I^{-n} = \frac{DCF}{1 - I^{-n}}$$



$$DCF + L I^{-n} = DCF\left(1 + \frac{I^{-n}}{1 - I^{-n}}\right) = DCF\left(\frac{1 - I^{-n} + I^{-n}}{1 - I^{-n}}\right)$$

$$= \frac{DCF}{1 - I^{-n}}$$

Why is this opt? As n rises, PV of DCF keeps rising, but PV of land salvage keeps falling. Max of the sum gives opt. n.

E.g. let DCF:

$$DCF = I^{-1} + \frac{1-d}{I^{-2}} + \frac{(1-d)^2}{I^{-3}} + \cdots + \frac{(1-d)^{n-1}}{I^{-n}}$$

$$= I^{-1}\,\frac{1 - \left(\frac{1-d}{1+i}\right)^n}{1 - \frac{1-d}{1+i}} = \frac{1 - \left(\frac{1-d}{1+i}\right)^n}{I - \frac{1-d}{1+i}} = \frac{1 - \left(\frac{1-d}{1+i}\right)^n}{i + d}$$

$$= \frac{1 - (\quad)^n}{I\,\frac{1+i-1+d}{I}} = \frac{1 - \left(\frac{1-d}{1+i}\right)^n}{i + d}$$

$$\frac{\text{PVIF}}{1 - I^{-n}} = \frac{1 - \left(\frac{1-d}{1+i}\right)^{n}}{(i+d)\left(1 - \frac{1}{1+i}\right)^{n}}$$

Marginal solution (3) is intuitively meaningful in an obvious way. It says the marginal cost of time is Si, interest on the cash foregone by holding the uncut timber for one more year. Solution (4) has a comparable interpretation. We can show that (4) says that the marginal cost of time is $(S+L)i$.

Rewriting ( ):

$$S' = Si \frac{1}{1 - I^{-n}}$$

Purpose: to factor out Si, to identify the other element on which solution 4 requires us to earn interest.

Method: add and subtract I^{-n} to and from the numerator, letting us factor out "1".

$$S' = Si \frac{1 - I^{-n} + I^{-n}}{1 - I^{-n}} = Si\left(1 + \frac{I^{-n}}{1 - I^{-n}}\right) = Si + \frac{Si}{I^n - 1}$$

$$= i\left(S + \frac{S}{I^n - 1}\right)$$

But $\frac{S}{I^n - 1}$ is Land Value, L.

So solution (4) boils down to this:

$$S' = i(S + L)$$

S' is the marginal product of time — the yearly growth of S.

$i(S + L)$ is the marginal cost of time — the foregone gain of tieing up the Stumpage and the Land for another year. These have an opportunity cost. The Stumpage could be liquidated for cash, and reinvested. As to the land, it could be put to its next use yielding an annual rent of the amount $\underline{a}$. But $a = Si$.

$i = .05^-$ $\quad n = 25^-$

	(1) $\frac{1-d}{1+i}$	(2) $1-i-d$	(3) $\frac{1}{1+i+d}$	(4) $\frac{1}{(1+i)(1+d)}$	(5) $\left((4)/(1)\right)^{25^-}$	(6) $\frac{(3)}{(1)}$	(7) $\left(\frac{(2)}{(1)}\right)^{25^-}$
$d = .02$	.933	.930	.935$^-$	.934	1.010	1.034	.914
$d = .05^-$	.905$^-$	.900	.909	.907	1.065$^-$	1.127	.876
$d = .10$	.857	.850	.870	.866	1.286	1.433	.811
$d = .20$	.762	.750	.800	.794	2.775$^-$	3.386	.675$^-$

If more working K goes to pay land + K, there is less for labor; and/or not enough for full empl. Cf. Marx. Bk. III Ch. VII

Value of i to make $PV = \frac{n}{2}$

n	$100\,i$
60	~~2.6~~ 2.7
50	3.1
40	3.9
30	5.2
20	7.7
15	10.2
10	15.0
5	28.5
2	63.0

$$PV = \frac{1 - I^{-n}}{i}$$

$$\frac{n}{2} = \frac{1 - I^{-n}}{i}$$

$$i = \frac{2(1 - I^{-n})}{n}$$

~~PV~~ Table of PV as we double i and n

n \ i	.01	.02	.04	.08	.16	.32
1	.990	.980	.962	.926	.862	.758
2	1.970	1.942	1.886	1.783	1.605	1.331
4	3.902	3.808	3.630	3.312	2.798	2.096
8	7.652	7.325	6.733	5.747	4.344	2.786
16						
32						
64	47.10	35.92	22.97	12.41	6.25	3.12

FVF end points.

A. Basic case: 1st pmt at end of 1 yr; draws int. for n-1 yrs.
Last pmt. draws no int.
n = no. of pmts.

$$F_3 = I^2 + I^1 + 1$$
$$= 1.21 + 1.1 + 1 = 3.31$$

B. 1st pmt. at beginning of yr. Same no. of pmts, n.

$$IF_3 = I^3 + I^2 + I = 3.641$$

C. Add a pmt. to A; add a year

$$F_4 = I^3 + I^2 + I + 1$$
$$= 1.331 + 1.21 + 1.1 + 1 = 4.641$$

D. Odd case: squeeze 4 yrs into 3
1st pmt. at beginning of yr; also add pmt at end

$$F = I^3 + I^2 + I + 1 = 4.641 \quad \text{same as } F_4$$

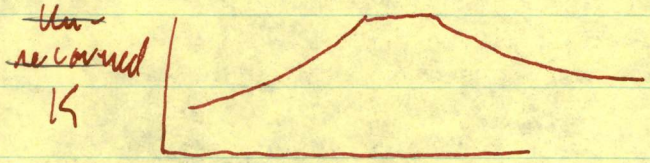
Ec. 162

Uses of FVF

1. Long constru. pds., cap'g interest. "Cash flow analysis"

Forgone

~~the~~ recovered



Pub. Works

Nuclear plants

Utility rate bases. CWIP. Inc. reporting. Inc. taxing.

Constru. loans. High i.r. ~~Phantom rates.~~

~~Spoil~~

Bond fncg, it all becomes funded debt.

Values of FVF (1st pmt is at end of yr – no int.)

n	i				
	.08	.10	.12	.14	.16
1	1	1	1	1	1
7	8.92	9.49			
14	24.21	27.97	32.39	37.58	43.67
21	50.42				
28	95.34				
35	172.32				
42	304.24				

$$\frac{d}{dn}\left(\frac{I^n - 1}{i}\right) = \frac{1}{i} \cdot i I^n = I^n$$

$$\frac{d}{dn} I^n \cong i\, I^n$$

Growth is $\left[i \cdot \frac{I^n - 1}{i} + 1 \right] = I^n$

Cost of delay

Cost Overrun

Values of FVF and FV/n

	i = .08		.10		.12		.14		.16	
n	FV	FV/n	FV	FV/n	FV	FV/n	F	F/n	F	F/n
1	1	1	1	1						
7	8.92		9.49	1.36	10.1	1.44	10.7	1.53		
14	24.2	1.73	28.0	2.00	32.4	2.31	37.6	2.68	43.7	3.12
21	50.4		64.0	3.05	81.7	3.89	104.8	4.99		
28	95.3		134.2	4.79	191	6.81		9.75		
35	172.3		271	7.74	432	12.3		19.82		
42	304.2		538	12.8	964	23		41.6		75.7

When $n = 7$ and $i = .12$, 44% of your ~~cost~~ investment is capd. int.

But, on a current basis, the growth of your sunk cost is mostly int:

Current interest $= I^n - 1 = 2.21 - 1 = 1.21$

That is 121 x 1 ; 1 is current outlay.

So $\frac{1.21}{1.21+1}$ of ΔK is int., or 55%.

Hiding mistakes

Ways to hide this int. burden: make it implicit. I.e., plow in captive funds, change int.

A. Pay int. from current surplus.

1. Corps – plow back profits
2. PUB – change int. on CWIP to current rate payers, raise rates.

E.g. San Onofre

(a) Before came on line. This never got accrd. for – until I demanded it. They were ready to hide it; it no one asked. (b) After came on line, but cost more than other sources. This excess is also a cap. cost. i.e. the true value of power. Is never accrd. for.

a. When they do acct. for it, use the lowest i.r. can get over. Cf. SE, where they use higher one!

In this case ~~they used~~ $n = 14$, i.e. was up to 13% on 50. $1.13^{14} = 5.53$

Current int. was 4.73 x current revenue.

Except was ~~as~~ higher, bcs. current spending was stepping off. Stalling tactic: "SONGS II is 99% complete". Trouble is, last 1% took 2 more years.

Thus we were adding 140 a year to the ~~GDS~~ K, just by standing still. Came to $12m/yr, $1m/mo.

Now ~~K is down~~ acknowledged K is up to $130 m.

~~Ed~~ Mgrs. who goof like that want desperately to cover up. Hire same promoters. "Tell me again — I get so discouraged."

(Note how little of the u.i. Thus created makes jobs.)

B. Capture pension fund
   B.C. Hydro
   U.S. Govt. capture of Soc. Sec. funds.
C. Borrow from U.S.

SEE $\quad a = \frac{i}{I^n - 1}$

a. Uses - prepare for a big expense
E.g. replacement, or college.
(Used to be more common. Now that int. is tax able & deductible, is more common to fly now, pay later.)

Decommissioning.
Sheep vs. Trees.

$$a = \frac{10{,}000 \times .12}{1.12^{60} - 1} = \$1.34$$

b. Sensitivity to $\underline{n}$ and $\underline{i}$

Values of $\underline{a}$ in %

$\underline{n}$ \ $\underline{i}$	.05	.08	.10	.12	.14	.16
10	8.0	6.9	6.3	5.7	5.2	4.7
20	3.0	(2.2)	1.7	1.4	1.1	0.9
30	1.5	0.9	0.6	0.4	0.3	0.2
40	0.8	0.4	[0.2]	0.1	0.07	0.04

Bank moved from 8% and 20 years to 10% and 40 yrs. Thus cut this expense to $\frac{1}{11}$.

If you know this math. you can see how they are shading things, see behind the bafflegab, see motives.

Summary. Rule of 72

$\frac{72}{i} = n$, where n is doubling time.

1. For $i = .08$, most accurate

2. For $i < .08$, gives too high an n

$1.01^{72} = 2.047$ or 2.35% too high

$1.02^{36} = 2.040$ or 2.0% " "

$1.07^{10.29} = 2.006$

$1.08^{9} = 1.9990$

$1.09^{8} = 1.992$

$1.1^{7.2} = 1.986$

$1.12^{6} = 1.974$

$1.16^{4.5} = 1.95$

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under 3% use 70

So: under 5% use 71

over 11% use 73

" 12% " 74

" 15% " 75

"Rule of 72"

Money doubles in a number of years roughly equal to:

$$n = \frac{.72}{i} = \frac{72}{100\,i}$$

where i = interest rate (as a decimal)
$100\,i$ = " " (as a %)

Rule is approximate, for low values of i, below about 15%.

Table 1: Accuracy of Rule of 72

i	$\frac{.72}{i}$	Correct n ★	$\frac{\text{Correct } n}{\text{Est. } n}$	$\frac{\text{Est. } n}{\text{Correct } n}$
.01	72.0	69.7	.97	1.03
.05	14.4	14.2	.99	1.01
.10	7.20	7.27	1.01	.99
.15	4.80	4.96	1.03	.97
.20	3.60	3.80	1.06	.94

Table 1 shows est. n is 3% too high at $i = .01$; very close to the mark between $i = .04$ and $i = .12$; and increasingly too low for higher values of i.

★ From (1, A), p. 2

Basis of Rule of 72

$$(1+i)^n = 2 \qquad (1)$$

Solve for n, by taking ln. of both sides.

$$n \ln(1+i) = \ln 2$$

$$n = \frac{\ln 2}{\ln(1+i)} = \frac{.693\ldots}{\ln(1+i)} \qquad (1, A)$$

The trick here is that $\ln(1+i) \cong i$.
$\ln(1+i)$ is called "the force of interest", and the standard notation is Greek rho, or "ρ".

But ρ is slightly less than i.

Table 2: ρ compared to i

i	ρ	ρ/i	i/ρ
.01	.00995	.995	1.005
.05	.04879	.976	1.025
.10	.09531	.953	1.049
.15	.13976	.932	1.073
.20	.18232	.912	1.096

To compensate for this error The Rule of 72 raises .693... up to .72, or by 3.9%. This is correct at $i = 8\%$, and roughly accurate for values around 8%.

Doubling Time at 14.4%

The Rule says money doubles in 5 years at 14.4%. But we are into a range of 3% inaccuracy: (Table 1). Accurately:

$$n = \frac{\ln 2}{\ln 1.144} = 5.1523\ldots \qquad (2)$$

It takes more than 5 years for money to double at 14.4%. In 5 years at 14.4% money less than doubles:

$$1.144^5 = 1.9594\ldots \qquad (3)$$

What you did with your calculators was equivalent to raising (3) to the 6th power:

$$1.9594^6 = 56.595\ldots \qquad (4)$$

Hence you got about 57, instead of the 64 which The Rule predicts. Small errors raised to the 6th power become larger errors. My example was pushing on the outer limits of the usefulness of The Rule of 72. You were sharp to pick this up, and question it.

Usefulness of Rule of 72

The Rule is useful for making quick horseback estimates, and for raising your consciousness about the power of compound interest.

But for accuracy, buy a hand calculator and use (I, A).

The Rule also focuses your mind on the fact that the product of $n \times i$ is a constant (roughly). So doubling i means you cut n in half. This rough rule becomes an absolutely accurate rule when compounding is continuous, as in calculus, (and in some bank accounts). But that's another rule for another day.