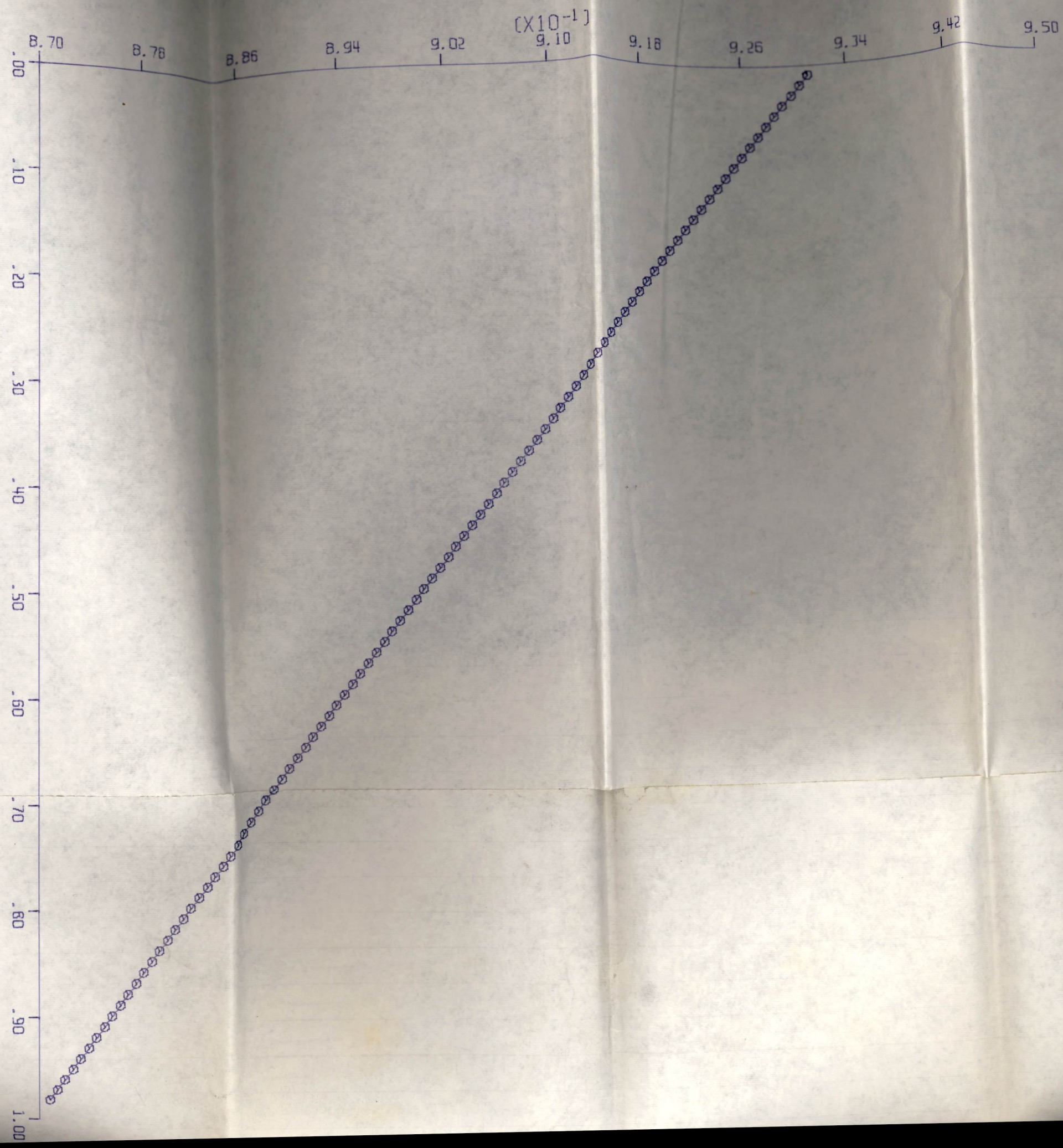
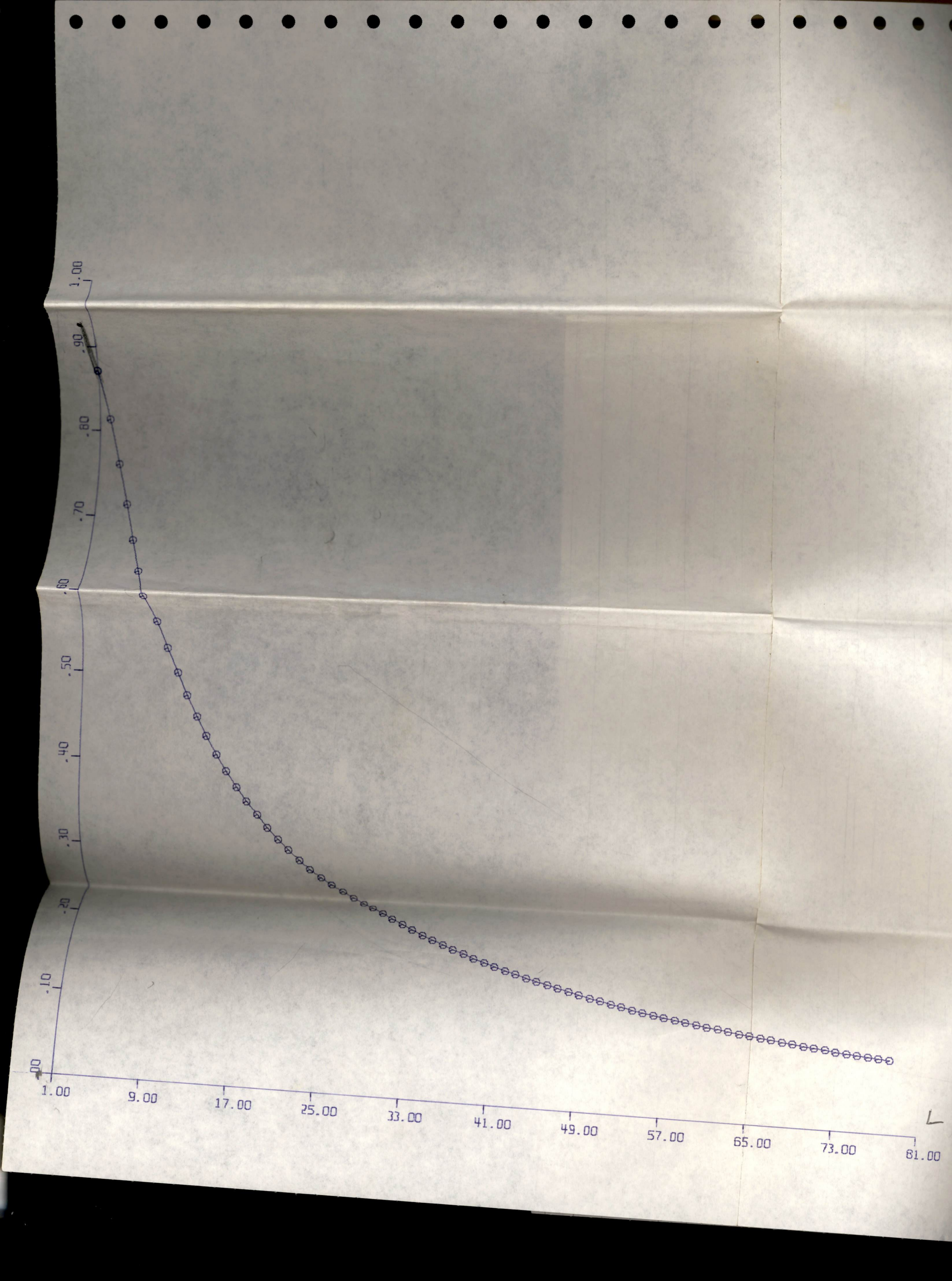
Marig. PV of Mine


[X10⁻¹]
8.70
8.78
8.86
8.94
9.02
9.10
9.18
9.26
9.34
9.42
9.50
.00
-.10
-.20
-.30
-.40
-.50
-.60
-.70
-.80
-.90
-1.00


1.00
.90
.80
.70
.60
.50
.40
.30
.20
.10
.00
1.00
9.00
17.00
25.00
33.00
41.00
49.00
57.00
65.00
73.00
81.00
L

Michèle on finding pt. of inflection

ASSUME $i > 0$

$PV = \sum_{1}^{L} A \cdot \frac{1-(1+i)^{-L}}{i}$

let $1+i \equiv K$ $\quad \ln(K) \equiv P$

$\partial PV/\partial L = \left[\frac{A}{i}\right]\cdot\left[\frac{(L)(K^{-L})(\ln K) - (1-K^{-L})}{L^2}\right] = \left[\frac{A}{i}\right]\left[\frac{LK^{-L}\cdot P - (1-K^{-L})}{L^2}\right] = \left[\frac{A}{i}\right]\left[\frac{K^{-L}\cdot P}{L} + \frac{K^{-L}-1}{L^2}\right]$

$\partial^2 PV/\partial L^2 = \left[\frac{A}{i}\right]\cdot\left[\frac{(L)(K^{-L})(\ln K)(-P) - (K^{-L})(P)}{L^2} + \frac{L^2(K^{-L})(\ln K)(-1) - (2L)(K^{-L}-1)}{L^4}\right]$

$\quad = \left[\frac{A}{i}\right]\left[\frac{K^{-L}P^2}{L} - \frac{K^{-L}P}{L^2} - \frac{K^{-L}P}{L^2} - 2\left(\frac{K^{-L}-1}{L^3}\right)\right]$

$\quad = \left[\frac{A}{i}\right]\left[\frac{(2)(1-K^{-L})}{L^3} - \frac{(2)(K^{-L})(P)}{L^2} - \frac{(K^{-L})(P^2)}{L}\right]$ ✓

$\partial^2 PV/\partial L^2 = 0$ if $(1-K^{-L}) - (K^{-L})(P)(L) - (K^{-L})(P^2)(L^2/2) = 0$

or if $1 = K^{-L}(1 + PL + P^2L^2/2)$

or if $K^L = 1 + P\cdot L + P^2\frac{L^2}{2}$

or if $(L)(\ln K) = \ln(1 + P\cdot L + P^2L^2/2)$

" " $(L)(P) = \ln(1 + L\cdot P + (L\cdot P)^2/2)$

" " $(L)(P) - \ln(1 + L\cdot P + (L\cdot P)^2/2) = 0$ ✱

CLAIM ✱ $= 0$ if and only if $L = 0$ or there is no point of inflection other than (at most) 0.

AT $L = 0$, ✱ $= 0 - \ln(1) = 0$

if $y = L\cdot P - \ln(1 + L\cdot P + (L\cdot P)^2/2)$

then $\partial y/\partial L = P - \frac{P + LP^2}{1 + L\cdot P + (L\cdot P)^2/2} = \frac{P + LP^2 + \left(\frac{(LP)^2}{2}\right)(P) - (P + LP^2)}{1 + LP + (LP)^2/2}$

$= \frac{\frac{(LP)^2}{2}\cdot P}{1 + LP + \frac{LP^2}{2}}$ is always positive if $L > 0$ (given $i > 0$)

Thus ✱ has a positive derivative for all values of $L > 0$

And ✱ is 0 at $L = 0$ $\therefore$ ✱ $\neq 0$ for any $L > 0$ QED

TAB FORMS MANUFACTURED BY O. L. SCHLIPARTH & CO. MILWAUKEE, WISCONSIN MADE IN U. S. A.

$PV = \frac{\$A}{L} \cdot \frac{1-(1+i)^{-L}}{i}$

$i > 0$ $\quad 1+i \equiv K$

$A > 0$ $\quad \ln(1+i) = p$

$$\partial^2 PV/\partial L^2 = \left[\frac{A}{i}\right]\left[\frac{(2)(1-K^{-L})}{L^3} - \frac{(2)(K^{-L})(p)}{L^2} - \frac{(K^{-L})(p^2)}{L}\right]$$

$$\partial^2 PV/\partial L^2 = 0 \text{ if } \left[(L)(p) - \ln\left(1+(L)(p)+\frac{[(L)(p)]^2}{2}\right)\right] = 0$$

let $x = L \cdot p$

then $\partial^2 PV/\partial L^2 = 0$ if $x - \ln\left(1+x+x^2/2\right) = 0$

$$f(x) = x - \ln\left(1+x+x^2/2\right)$$

$$f'(x) = 1 - \frac{1+x}{1+x+x^2/2} = \frac{x^2/2}{1+x+x^2/2}$$

$f'(x) > 0$ for all $x > 0$

thus $f(x)$ is an increasing fn

and $f(x) = 0$ if $x = 0$

Thus for all $x > 0$, $f(x) > 0$ so

$\partial^2 PV/\partial L^2$ is not 0 if $x > 0$ or if $L \cdot p > 0$

Thus, since $p > \ln(1)$, $p > 0$

And $\partial^2 PV/\partial L^2$ is never 0 when $L > 0$

8%

L'Hospital's Theorem.

L	$\frac{1}{m}\frac{(1-(1+i)^{-L})}{i} \div L =$	
0		1.000 > .07
1	.9259	.9259 > .0342
2	1.7833	.89165 > .03262
3	2.5771	.85903 > .03101
4	3.3121	.82802 > .02948
5	3.9927	.79854 >

$$\frac{1}{.08}\cdot\frac{1-(1+i)^{-L}}{L}$$

$$\frac{dL}{dL}=1$$

$$\frac{d}{dL}\left[-(1+i)^{-L}\right] = -1\frac{d}{dL}(1+i)^{-L} = +1\ln(1+i)(1+i)^{-L}$$

When $L=0$, $=\ln(1+i)=\rho\doteq .08$

So $f_{\infty}\to 1.^{00}$, ~~perhaps $\frac{\rho}{i}$~~.

Which common sense tells anyway. You get the entire A right now, no discounting.

PV approach:

$$\text{Max } PV(R-C) = \frac{A}{L}\frac{1-(1+i)^{-L}}{i} - C_0$$

Annuity approach:

$$\text{Max } a - c = \frac{A}{L} - C_0\frac{i}{1-(1+i)^{-L}}$$

$$= \frac{i}{1-(1+i)^{-L}}\left[\frac{A}{L}\frac{1-(1+i)^{-L}}{i} - C_0\right]$$

The annuity reaches its max. earlier than the p.v. So they are not the same.

c) ~~Other~~ As lower no. of shafts to one, Δ cost of xptg. are to foot of shaft, raise op'g costs. (Offsetting, Δ xpt. costs above shaft head or mouth, but that is cheaper than underground xptn.)

Therefore, the curve of PVC takes ~~the typical~~ half-U or scoop or ski-slope shape, falling fast at first and then flattening out, as shown in Fig. 1

~~Since~~ PVR continues to fall as l rises, while PVC stops falling, they must finally cross, as at ~~B~~ C in Fig. 1. But ~~before then~~ the max. PV(R-C) arrives, well before then, at A.

B is pt. of inflection. It seems likely A will precede B, ~~but~~ also not absolutely nearly.

Thus in most cases an opt. life of — years or less is probable. ~~Or quite a lot less than~~ That corresponds to a mean res. output ratio of half as much, or — years --- much less than what presently obtains in most mineral industries

An alternative way of looking at matter is lowering yearly costs. Where we double life, (as by sinking one well instead of 2) we roughly halve yields and halve capital costs. capital costs per unit of yield remain about the same The advantage is in lowering the annual equivalent of the capital costs, by spreading them over more years. ~~The disadvantage is in deferring the revenues~~ The disadvantages are in smaller scale of operations, and/or more distant spacing of wells; ~~and~~ costs of adding longevity; and deferring net revenues.

After about 20 years life the advantage becomes negligible The "capital recovery factor" (the annuity whose p.v. is unity) approaches asymptotically to $C_0 i$, and at $l = 20$

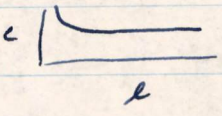
is not far above it. But the disadvantages grow, and after a few years the annuity value of yields, ~~falls~~ ~~less costs~~ falls below its max.

But this method is clumsy, bcs. the length of the annuity is in gen. as well as its height; and this is handled by the g.c. approach which considers both factors.

Opt. life of mine, where annuity value of revs ^{a}/// − annuity value of costs ^{c}/// is max.

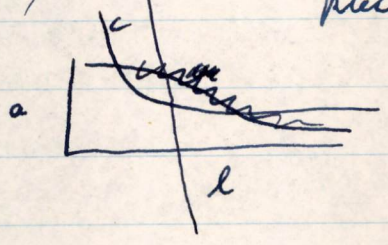
Where C_o is assumed const. with A, ~~but increases with~~

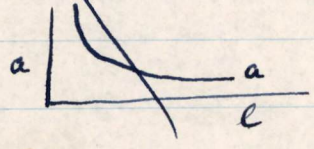
$$c = C_o \frac{i}{1-(1+i)^{-\ell}}$$



$$a = \frac{A}{\ell} \quad \frac{A(1+i)^{-\ell}}{\ell}$$

where A is the total mineral ~~reserves~~ recovered, ~~mining~~, presumed a constant with ℓ.





Max. a − c

If A rises with ℓ up to a few years, a curve has an upward reach, extends life.

But in general, a curve is falling. So, after the c curve flattens out, life should end.

---

c = capl. recovery annuity; cost per unit of rev.; a.

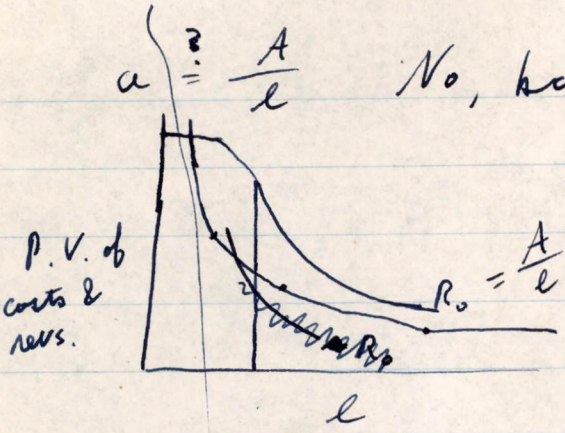
$_{\ell}C_{o\ell} \equiv C_o$ for life ℓ

$C \equiv \alpha A$;

Assume economies of scale in shaft, so double capac. w/o doubling costs. Also double life w/o doubling cost, but may ? double annuity cost when include interest.

When ∆ shaft capac. by parallel shafts, cut down xpts. from sites to foot of shaft, thus get higher % recovery & lower op'g costs.

$a \overset{?}{=} \frac{A}{\ell}$ No, bcs 2 a for 10 yrs > a for 20 yrs.



P.V. of costs fall bcs. you build less capacity; & defer variable costs. ∓
P.V. of revs. fall bcs. revs. come later.

$R_0 \equiv$ P.V. of revs $= \frac{A}{\ell} \frac{1-(1+i)^{-\ell}}{i}$, where A is total mineral recovered, (× price,) assumed constant w.r.t ℓ.

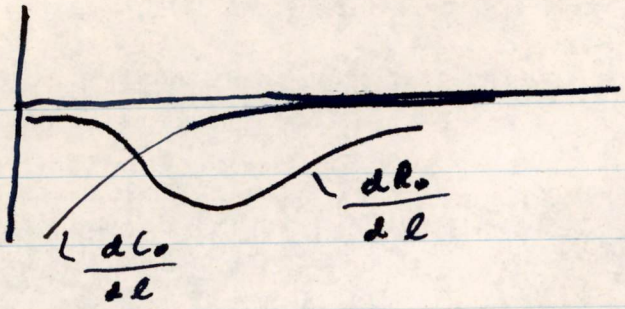
C_0 = initial capl. outlay plus p.v. of later outlays.

Max. $R_0 - C_0$

[The annuity approach is not so good bcs. the max. net annuity is not sufficient criterion. A lesser annuity over longer time has higher p.v. So use p.v. approach? ~~The~~ annuity approach has benefit that you can give a figure of ca. 20 yrs. when a flattens out. Try to use that, or match it in p.v. approach]

✓ [N.B. this is practically only case where naive p.v. approach can apply, bcs. is no reuse cycle. Single time horizon. But even here reuse is possible, today recovery. allow a little for that]

∓ But as you build less capacity you meet 2 diseconomies:
a) Small scale; b) Longevity and even replacement need.
So p.v. of ℓ stops falling



Geometrically probly obvious that $\frac{dR_0}{dl}$ ~~inflects to~~ has min.

to right of where $= \frac{dC_0}{dl}$

to find inflectn. of $\frac{A}{l}\frac{1-(1+i)^{-l}}{i}$ and conclude

opt. life is probly less than that.

$$\frac{d}{dl}R_0 = \frac{A}{i}\frac{d}{dl}\frac{1-(1+i)^{-l}}{l}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dl} = \frac{l(i)(1+i)^{-l} - [1-(1+i)^{-l}]}{l^2}$$

$$\frac{d^2}{dl} = l^2\left[i\left(l(-p(1+i)^{-l}) + (1+i)^{-l}\right) - -p(1+i)^{-l}\right] - \underbrace{\left[li(1+i)^{-l} - (1-(1+i)^{-l})\right]}_{\times 2l}$$

$\frac{d}{dl}(1+i)^l = p(1+i)^l$

$\frac{d}{dl}(1+i)^{-l} \stackrel{?}{=} -p(1+i)^{-l}$

$\frac{d}{dl}\frac{1}{(1+i)^l} = \frac{0 - p(1+i)^l}{(1+i)^{2l}} =$

$-p(1+i)^{-l}$ Q.E.D.

$\frac{d^2}{dl} = 0$ when

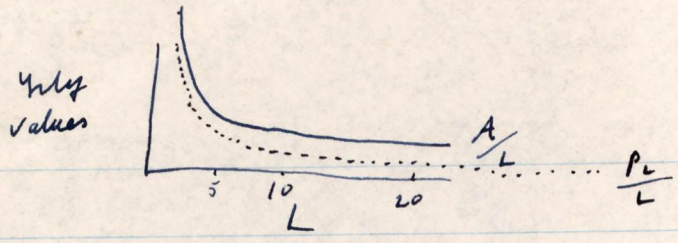
$$l^2\left[i\left[l\left(-p(1+i)^{-l}\right) + (1+i)^{-l}\right] - p(1+i)^{-l}\right] = 2\left[li(1+i)^{-l} - \left[1-(1+i)^{-l}\right]\right]$$

$$l\,(1+i)^{-l}\left[-ilp + i - p\right] = 2(1+i)^{-l}\left[li - \left[(1+i)^l - 1\right]\right]$$

$$-l^2 i^2 = 2li - 2\left[(1+i)^l - 1\right]$$

$$\frac{\left[(1+i)^l - 1\right]}{li} \cong 1 + \frac{li}{2}$$

This equality will come at fairly low values of l, about 10 ? guess. So opt. life is short.



$$\frac{A}{L}\,\frac{1-(1+i)^{-L}}{i} = \text{P.V. of annuity} \equiv P_L$$

$\frac{P_L}{L}$ is one way of annualizing P_L

[The other way is multiply by cap. rec. $\frac{i}{1-(1+i)^{-L}}$ which leaves $\frac{A}{L}$]

Dubious.

1	2	3	4
		at 6%	
L	$\frac{1}{L}$	$\frac{1-(1+i)^{-L}}{i}$	(2)·(3)
1		.94	.94
2		1.83	.91
3		2.67	.89
4		3.46	.865
5		4.2	.84
8		6.2	.775
10		7.4	.74
16		10.1	.62
20		11.5	
32		14.1	.44
40		15.0	
64		16.4	.25
∞			

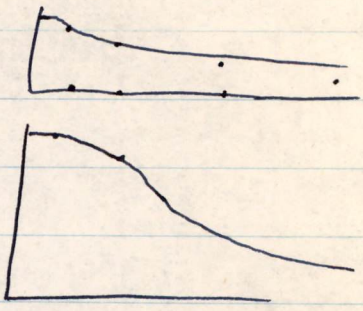
$6\overline{)100}$ → 17

17 × .7 = 11.9

17 × .9 = 153

$15\overline{)9.7}$ → .647 (90, 70)

(1)	(2)	(3) $17 \times \frac{[1-(1+i)^{-L}]}{i}$	(4) (2)×(3)
L	$\frac{1}{L}$		
~~0~~ 1	1	.94	1.00 .94
5^2	$\frac{1}{2}$ $\frac{1}{5}$	1.8 4.25	.90 .85
10	$\frac{1}{10}$	~~8~~ 7.4	.74 ~~.80~~
15	$\frac{1}{15}$	9.7	.65
20	$\frac{1}{20}$	~~12~~ 11.5	.57 ~~.6~~



40 $\frac{1}{40}$ 15.0 .375

47
54
176
220
.2376

64 $\frac{1}{64}$ $\frac{100}{6} \cdot [1 - 1.06^{-64}] = 16.7 \cdot .977 \simeq 16.4$

$1.06^{-64} = 1.06^{-50} \cdot 1.06^{-14} = .057 \times .44 = .023$

$32\overline{)14.1}$ → .44 (128, 130)

16.4
128
360
320
400

(1)

$$y = \frac{A}{i}\,\frac{1-(1+i)^{-t}}{t} =$$

i cont. %

t: time

$$1+i = a$$

$$y = \frac{A}{i}\,\frac{1-a^{-t}}{t}$$

1. Dom the gm

2. inflex. pm

$$\frac{dy}{dt} = \frac{k}{t}\cdot\frac{t(+a^{-t}\log a) - (1-a^{-t})\cdot 1}{t^2}$$

$$= \frac{k}{t}\,\frac{t(+a^{-t}\ln a) - 1(1-a^{-t})}{t^2}$$

decreasing

$$= \frac{k}{t^2}\cdot\; a^{-t}(t\ln a + 1) - 1 = 0.$$

$$a^{-t}(t\ln a) = 1$$

$$t\ln a = a^{t}$$

$$t = \log_a(t\ln a)$$

$$t = \log_a t + \log_a \ln a$$

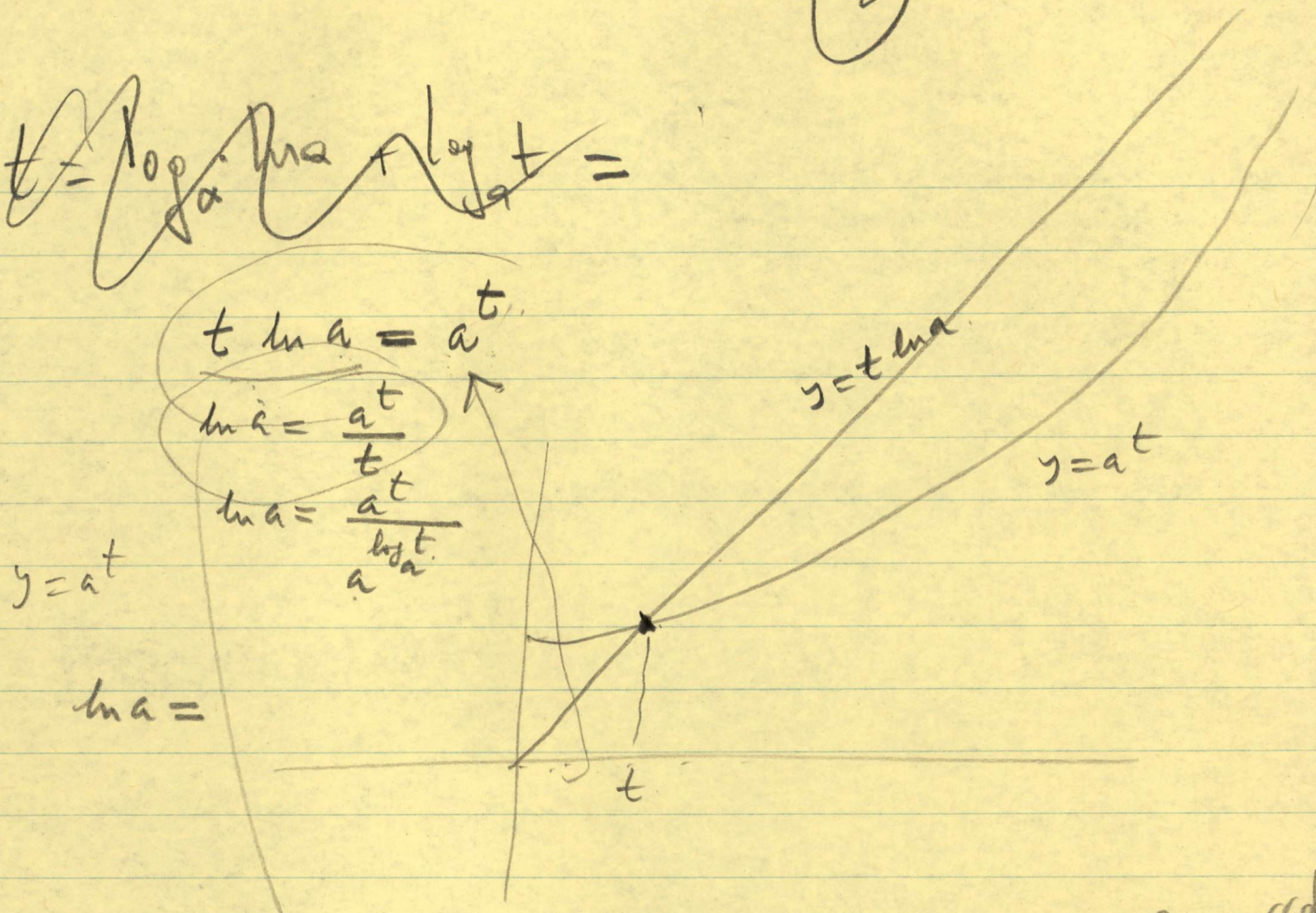
$$t - \log_a t = \log_a \ln a$$

$t = \log_a t \ln a + \log_a t =$

$t \ln a = a^t$

$\ln a = \frac{a^t}{t}$

$\ln a = \frac{a^t}{a^{\log_a t}}$



$\ln a =$

$$\ln a = \frac{a^t}{t}$$ - average of the assd

$$\frac{dy}{dt} = k\,\frac{(t\bar{a}^t \ln a - 1 + \bar{a}^t)}{t^2}$$

$$\frac{d^2y}{dt^2} = k \cdot \frac{t^2[\bar{a}^t \ln a - t\,\bar{a}^t (\ln a)^2 - \bar{a}^t \ln a] - 2t(t\bar{a}^t \ln a - 1 + \bar{a}^t)}{t^4} = 0$$

$$\bar{a}^t[t^2 \ln a - t^3 (\ln a)^2 - t \ln a - 2t^2 \ln a - 2t) + 2t = 0$$

$$\bar{a}^t[t \ln a - t^2 (\ln a)^2 - 2t \ln a - 2] + 2 = 0$$

$$-t^2 (\ln a)^2 - 2t \ln a - 2 = -2a^t$$

(3)

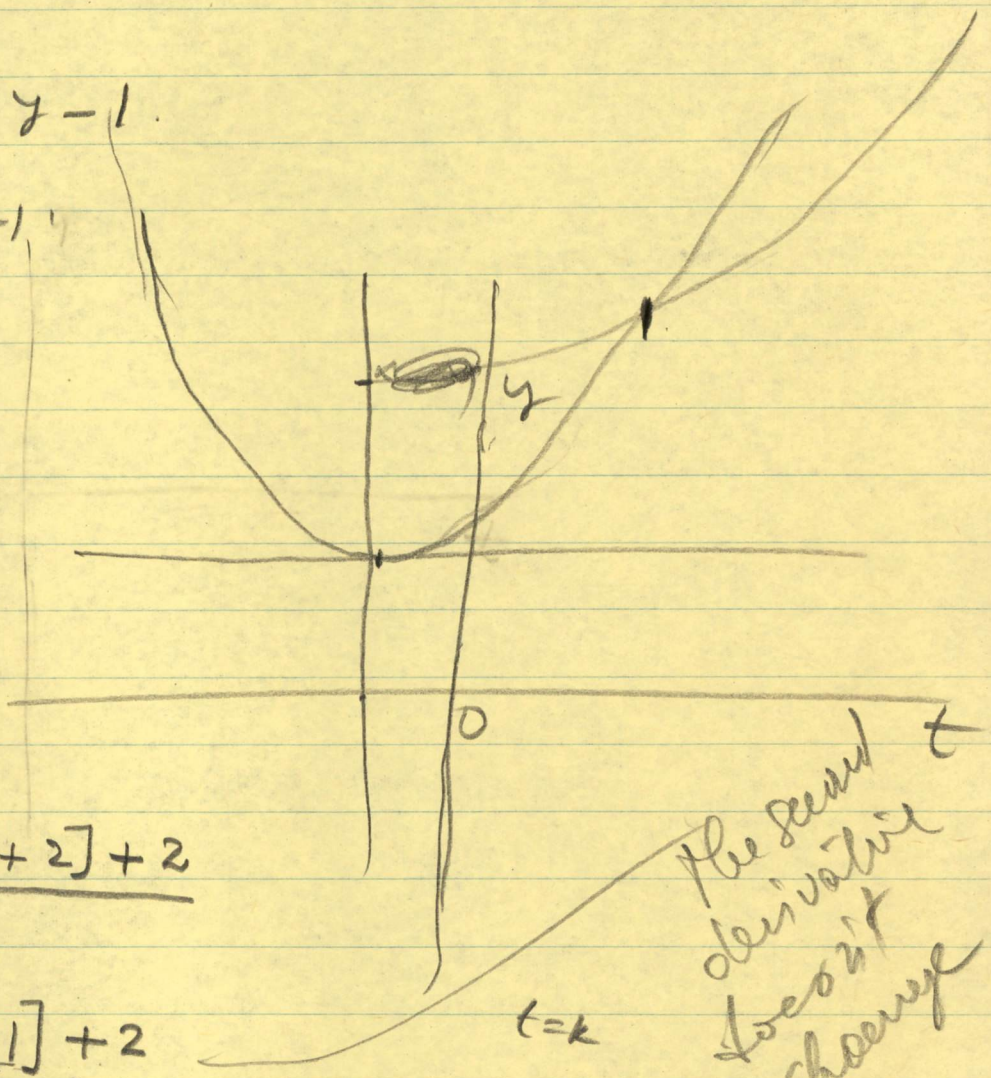
$$t^2(\ln a)^2 + 2t\ln a + 2 = 2a^t$$

$$y = t^2(\ln a)^2 + 2t(\ln a) + 2 \qquad y = 2a^t$$

$$(t\ln a + 1)^2 = y - 1$$

$$(\ln a)^2\left(t + \frac{1}{\ln a}\right)^2 = y - 1$$

$$\left(t + \frac{1}{\ln a}\right)^2 = \left(\frac{1}{\ln a}\right)^2 (y-1)$$



$$\frac{d^2y}{dt^2} = \frac{-a^t\left[t^2(\ln a)^2 + 2t\ln a + 2\right] + 2}{t^3}$$

$$= \frac{-a^t\left[(t\ln a + 1)^2 + 1\right] + 2}{t^3}$$

the second derivative doesn't change

$t = k$

$t < k$

$$y = \frac{A}{i}\left(1 - a^{-t}\right)$$

$$\frac{a-1}{a} = 1 + 0.05$$

$$y = K\frac{1-a^{-t}}{t}$$

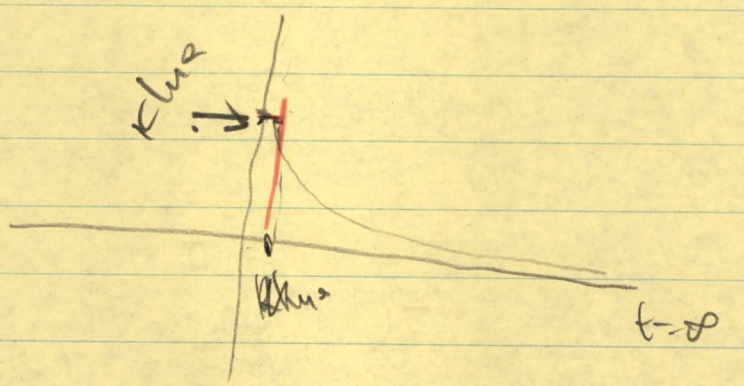
$$\lim_{t\to\infty} k\frac{1-a^{-t}}{t} = 0$$

$\Rightarrow$ $y=0$ is an asymptote to the curve.

$$y = k\frac{1-a^{-t}}{t}$$

$$\lim_{t\to 0} k\frac{1-a^{-t}}{t} \quad (\%_0 h)$$

$$= \lim_{t\to 0} k\frac{a^{-t}\ln a}{1} = \underline{k\ln a}$$



$$e^{\rho} = 1+i$$

$$e^{\rho L} = (1+i)^{L}$$

$$\frac{d\, e^{iL}}{dL} = \ln(1+i) = \rho$$

$$\frac{d\, e^{x}}{dx} = e^{x}$$

$$\frac{d\, e^{\rho t}}{dt} = \rho\, e^{\rho t}$$

$$\frac{d\,(1+i)^{x}}{dx} = (1+i)^{x} \ln(1+i) = \rho$$

$$i \cong \rho$$