



Rate of Return: Finding 'i' When You Know 'a', P, and 'n'

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Rate of Return: Finding "i" When You Know "a", P, and "n"

1. Simplest case: ROR on Finite Flow with no Redemption Value

Sometimes we know P and a and n, but not i. Solve for it. When you solve for it, i is called the Rate of Return (ROR), or Internal Rate of Return (IRR).¹

We already have ROR in the PIPO case (Reading III, D). It is harder to solve for i in the DCFF because you cannot collect terms. Here is what you get when you try:

$$i = a \frac{1 - I^{-n}}{P} \quad (1)$$

"i" is on both sides; (remember, "I" stands for (1+i)). You cannot collect terms, try as you may. With i on both sides, there is a "tail-chasing effect" when you plug in values of i: raising the left side raises the right side, etc.

Don't despair, Isaac Newton faced this problem long ago. The left and right sides grow at different rates, so they have to meet. Graphically, it's easy and obvious (explained below). Not one to accept defeat, Newton came up with a method of homing in on i by systematic trial and error. Newton's Method is built into many hand calculators which are used routinely in business to solve (1), and more complex variants as well. You can do it yourself with any programmable calculator, too: program in the right side, and proceed by trial and error. With a little practice you can get it to four significant figures in just a few passes.

Whenever na=P you'll get a zero rate of return, obviously, and it is correct (although not gainful). Otherwise, zero is always one of two solutions that satisfy the equation, but it is meaningless economically and may be ignored.²

¹You have also seen this called the "marginal efficiency of capital" (a term popularized by Keynes, in macro-economics). Wicksell called it the "natural rate of interest." Economists have a weakness for using different words to mean the same thing, [and also, alas, sometimes the same word to mean different things].

²We'll let fretters, philosophers and mathematicians meditate on its deeper meaning, and they do: some even tell us the IRR is ambiguous and should not be used. Meantime, the world goes right on using it, and so will we.

Sketch out a graph of the right side and left side of (1) as functions of i . The left side, of course, is a straight line sloping up at 45 degrees from the origin. The right side starts out steeper, then bows over and crosses the 45 degree line. That crossing is the right answer. That tells you that if your right side $> i$, your i is too low, so try a higher i . And vice versa.

Your right side can include a lot more factors than are shown in (1): salvage values, taxes, irregular revenues and costs at various dates, revenues declining according to various regular patterns, etc. Once you program those in, it only takes micro-seconds more for the calculator to grind out the answer, so you may end up doing that some day.

Today's more powerful calculators increasingly include a program for ROR. When you teach yourself by studying the manual of some simpler but programmable calculators, one of the first things they show you is how to write and store your own program for finding ROR.

In exams, I will not assume you have any such device available, although you may. Rather, I will give you 3-4 values of i and ask you to select the one that is correct. That way you don't have a long hunt. The correct i is, of course, the one that makes both sides equal.

When $n=\infty$, $i=a/P$. Otherwise, " i " is always less than a/P ,³ and may be negative. It is always an increasing function of n , although decreasingly so as n gets larger. It appears to vary directly with a/P but actually is even more sensitive than that because of the tail-chasing effect between i on the left and I on the right side.

Table 2: ROR for various n and a/P

$n =$	<u>2.50</u>	<u>5</u>	<u>10</u>	<u>15</u>	<u>20</u>	<u>30</u>	<u>50</u>
<u>a/P</u>							
.1	-.52	-.20	.00	.055	.077	.093	.099
.4	.000	.288	.385	.397	.400	.400	.400

³Unless a is rising. This is covered in III, G, "Valuation When Rents are Rising."

2. Redeemable Bonds and Orange Groves

Refer back to III, G, 7, "Valuation: Redeemable Bonds and Orange Groves."

This model combines the annuity case and the PIP0 case in one equation. This gives you, when you master it, a more comprehensive analytical power and grasp than either case taken alone.

What do bonds and groves have in common? The model applies literally to redeemable bonds (defined below). However, the math framework of it applies as well to speculative real estate, like a Riverside orange grove. This explains the heading. It is to let you know this framework applies to much more than the work of a narrow clique of Nicholas Leeson and other backroom money-changers and arbitrageurs shuffling papers. It applies to any investment that couples a cash flow with a resale value.

Literal "resale" isn't even necessary: a re-use value will do. For example, you might plan to rent out a vacation home for several years, and then retire there. The re-use value is what you reckon you would have to pay for a newly-bought home at that time. You might plan to use a lot by your store for parking over several years, then later build on it. You can think of other examples.

Most bonds are "redeemable." This simply means they are repayable debts: the bond pays cash interest (called the "coupon") over a finite life, and then you get back a lump sum at the end, usually in units of \$1,000. As a rule, that is also the original par value paid in, although not always. In either case the bond has a current market price, above or below \$1,000, that fluctuates daily, depending mainly on current interest rates available on alternative investments. Once the bond is issued and traded, the current price is cut loose from the original cost, and depends entirely on discounting future expectations.

Notation:

F = Future Redemption value after n years.

P = Price, or Present Value of bond.

a = annual cash flow from bond, the "coupon."⁴

a. Valuation

$$P = a \cdot \frac{1 - i^{-n}}{i} + FI^{-n} \quad (2)$$

Eqn. (2) combines the annuity case and the PIPO case in one equation. As I said, this gives you more power and grasp than either case taken alone. The Force will be with you. But you must concentrate, young Skywalker, if you want to become a Jedi Master!

b. Rate of Return on Bond, or "Yield-to-Maturity" (YTM)

This rate is also called the "yield to maturity" (YTM). It combines two yields: the "current yield," a/P ; and the bond appreciation from now to year n . You find it by solving Eqn. (1) for i .

$$i = a \cdot \frac{1 - i^{-n}}{P - FI^{-n}} \quad (3)$$

Review Eqn. (1). Equation (3) here, like Eqn. (1), is solved using Newton's method of systematic trial and error. This Eqn. (3) is marginally more complex than Eqn. (1), but if your calculator has to labor another nano-second, so what?

YTMs on redeemable bonds are published daily in the Wall St. Journal. Lots of people are interested. Many papers just publish what are called the "current yields," meaning just $i = a/P$. Anyone who invests based purely on current yield is cruising for a losing. He may often luck out, because for many bonds the appreciation element is minor; but for others it is major, and then you need Eqn. (3).

If $P = F$, the equation collapses into the current yield, $i = a/P$. Run through the algebra as an exercise - be sure you understand how this works. It's not hard, and it's fun to see all the hard

⁴Bond coupons are usually paid more often than once a year, but here we use annual periods, for simplicity.

stuff cancel out and disappear. Getting back exactly what you put in is the same as having a perpetual income stream - the premise of the simple capitalization formula. Recall that this is one of our three basic building blocks.

Ask yourself these questions - and, of course, answer them!

What happens when $F > P$? Why?

What happens when $F < P$? Why?

If $a = 0$, Eqn. (3) looks like big trouble, but you can step around the trouble by going back to Eqn. (2). The first term disappears, and Eqn. (2) collapses into the old familiar PIPO discount formula (see paper III,D). There are such bonds, today called "zero-coupon bonds." During and just after World War II they were called "E-Bonds." 15 years hence they will have a new name: salesmen are creative. So are revenue agents: E-Bond accrual was not taxable until you cashed the bonds in at the term; "zero-coupon" accruals are taxable as they accrue. Otherwise, however, *plus ça change, plus ça reste la même chose*.

c. ROR as Affected by Carrying Land over Time

In III, E (Faustmann's Formula) we saw that a simple solution for the ROR, in a PIPO case, overstates the true ROR by omitting the cost of the "catalyst," land, that has to be carried over the period of investment, and emerges unimpaired by use at the end. The same is true with this orange grove model.

However, we defer that to another paper, "Land and Capital." There is also an optional reading on Reserve, "Land Gains, Fast Write-off, and Incentives to Build." That paper combines the Land and Capital model with the income tax, and various plans for accelerating depreciation, and giving special treatment to capital gains. It is just optional in this course, but I often use it in Public Finance, and students catch on quickly. It shows what amazing complexities you can master, once you master the basic math as laid out here.