



Basic Mathematics of Finance, Skeletal Outline

By Dr. Mason Gaffney

Article citation:

Gaffney, Mason. "Basic Mathematics of Finance, Skeletal Outline." September 16, 1991. Study guide.

Fair Use Disclaimer

The Robert Schalkenbach Foundation, a non profit private foundation, is publishing this article, at no cost, pursuant to the Fair Use provisions of Section 107 of the Copyright Act which states that under certain circumstances, the fair use of a copyrighted material for purposes of criticism, comment, teaching, scholarship and research is not an infringement of copyright. If you have any concerns about the publication of this article, please contact us at info@schalkenbach.org.

The Robert Schalkenbach Foundation's Digital Collections and Archives is an extension of the Robert Schalkenbach Foundation's Engel Georgist Library. It collects, organizes, preserves, and provides free online access to researchers and the general public of Georgist literature including books, newspapers, pamphlets, scholarly articles and archival materials.

RSF is particularly proud to offer free online access to the works of the late Georgist scholar Mason Gaffney, made possible through the generosity of the Danica Foundation and with the consent of the Mason and Letitia Gaffney Trust.

Robert Schalkenbach Foundation: <https://schalkenbach.org/>

Engel Georgist Library: <https://www.librarycat.org/lib/EngelGeorgistLibrary>

RSF Digital Collections and Archives: <https://schalkenbach.starter1ua.preservica.com/>

A very few emendations noted, 16 Sept 91
Pp. 44 ff. need editing

1

II. Basic Mathematics of Finance, skeletal outline

(Notes by Mason Gaffney, 28 April 86, Rev. 16 Sept 91)

A. Comparing present and future Point Values: the "PIPO" case

A value at a single point in time is a "Point Value," aka a "lump-sum" value, "balloon payment," "single payment value" (spv), salvage value, redemption value, liquidation value, resale value, etc. It is standard, but wrong, practise to call it a "fund" (as opposed to "flow") value, but many point values are not really funds, e.g. land values, and common stock values, as discussed later.

Two such single Point Values together form a "PIPO" pair. PIPO is the acronym for "Point-input, point-output."

Notation:

F = Future Point Value at a time, n , meaning the end of the n th year. If there is any doubt about what year is meant, I add a subscript to F. To avoid clutter, I only do it when needed.

P = Present Point Value at time 0, meaning the start of the first year.

i = rate of interest, annual. (You can use periods other than years, too, and adjust n and i accordingly. Months are commonly used in normal business practise today; for simplicity, I use years in these notes.)

$I = 1+i$. This seemingly redundant term is used to cut down the clutter in equations which make repeated use of $(1+i)$.

n = number of periods, usually years, between time zero and the time that F is evaluated. n can be a whole number or a fraction.

1. Compounding

$$F = PI^n \quad (1,A)$$

This is called "compounding." I^n is the "Compounding Factor" (CF). Remember that $I=1+i$. This is the last time I will remind you, and in some later strings of equations you will see "I"

suddenly change, without further explanation, to $(1+i)$, where that helps find a solution.

Compounding is the basic building block of all the math that follows. Be sure you have a firm grasp of it. I'll repeat this shortly.

Compounding looks a little mysterious at first, but it all goes back to simple addition. "Money makes money, and the money money makes makes more money," is how smart old Ben Franklin put it. That's about as compact and concise as language gets. After that, we need a little math notation.

You put \$1 in the bank at the start of a year; at the end you have $1+i$, or I . Next year-end you have $I+iI = I \times (1+i) = I^2$. Next year-end you have $I^2+iI^2 = I^2 \times (1+i) = I^3$.

It goes on like that, so in general, at the end of any year, n , you have I^n dollars. Again, compounding is the basic building block of all the math that follows. Be sure you have a firm grasp of it.

Firmly to fix your understanding, make a table with three columns. Column 1 is n . Column 2 shows what you have at the start of each year; column 3 at the end. I could do this for you, but you learn by doing, so do.

I habitually do this from top to bottom. You could do it either way, as the spirit moves, but do it my way to facilitate comparing data.

First, do 4-5 rows algebraically. Then, since you now understand what you are doing, do it again numerically, using $.03 < i < .15$. Extend column 2 down in ten-year jumps to about $n=50$. Plot the points on a graph, and join them with a smooth curve. This is a curve of "exponential growth" (since the exponent, n , is the independent variable). It should look like Bob Hope's ski-slope nose, with Bob facing to your left.

2. Discounting

Sometimes you know F , i and n , and want to find P . Solve for P .

$$P = FI^{-n} \quad (1,B)$$

Eqn. 1,B is called "discounting"; I^{-n} is the discount factor. It is the opposite of compounding. Since compounding is adding

interest to P, discounting has to mean subtracting interest from F. Mathematically, to be sure, it involves division, but this is just a short cut. Logically it is a matter of subtracting: P may be viewed as F net of the interest cost of getting from now to then.¹

When F is the deferred product of some other inputs, P is called their "discounted product," meaning their product net of the interest costs of holding the product to maturity. There is also a discounted marginal product. This is a handy way to impute to a current input like labor its worth to the employer. Cf. earlier section on "Imputation."

Discounting is used to put present values on future sums. It is the basis of valuation of all durable wealth.

Make a table with three columns. Column 1 shows n , starting with $n=1$. Column 2 shows the discount factor. Column 3 is cumulative: add each discount factor to the sum of those before it. Use a high interest rate, about .13 or more, so you won't need to work out a lot of sums to see the shape of the curve.

Plot the points from Column 2. This is a curve of "exponential decay." This time Bob Hope is facing to your right, but it may be Cyrano de Bergerac, or perhaps Michael Doonesbury, because the slope is distinctively different. It starts out slower, and flattens out more.

Shade in the first five columns on your graph. The shaded area is the sum of five discounted values: this sum is the value of an asset that yields five payments in successive years. Save this graph to review when we do (5,B), the Present Value Factor.

Plot the points from Column 3. This is a "cume curve," (cume for "cumulative"). If your integral calculus is operational, you'll recognize it as the integral of Column 2. If it isn't, no matter, "cume curve" is self-explanatory. It shows how the present value grows with the life or durability of an asset. Note that it grows, but at a decreasing rate. Put this aside for the moment: we use and develop it extensively a few pages hence.

3. Rate of Return

¹There are other time-costs too, like ground rent, dealt with later in (16,A to 16,F).

Sometimes you know P , F , and n , and want to know i .

$$I = (F/P)^{1/n};$$

therefore

$$i = (F/P)^{1/n} - 1 \quad (1,C)$$

" $1/n$ " is the reciprocal or inverse of n . As an exponent it means the n th root. Do not be intimidated: that's why you have a calculator. With this wonder-working little device, you can take the 53d root as easily as you can add and subtract. Practise a little so you can do it smoothly on an exam: you'll need to.

When you find i like this, as the dependent variable, it is called the "rate of return" (ROR).² One also sees IRR (internal rate of return); and RRBT and RRAT, for rates of return before taxes and after taxes.

ROR is more general than $(1,C)$, which is just the simplest form of ROR, when an investment yields its return only in one lump sum. In general whenever you solve a financial equation for i the result is called ROR. Usually the data are more complex than in $(1,C)$.

4. Period of time

Sometimes you know everything but n . Here we need to use logarithms. Don't panic, it is time you learned how simple that is with a modern calculator in your hand.³

²Keynes called this the "marginal efficiency of capital," and it is so designated in many standard macro texts and writings. Wicksell called it the "natural rate of interest," which many standard macro texts and writings also discuss. (Economists have a weakness for using different words to mean the same thing, [and also, alas, sometimes the same word to mean different things]).

³I was raised on long log tables that were still too short to give accurate answers without "interpolating," which is an invitation to errors; and on the technical marvel of the slide rule, with which you interpolated by eye, and guessed out to three significant figures on the left side of the rule, and two if you had to work on the right side. It helped to be myopic.

"LNx" means "the natural logarithm of x." It is usually written lower case, but I use Upper Case to make equations easier to read: wouldn't want you confusing ln for 1 x n, for example.

Solving for n :

$$nLNI = LNF - LNP \quad (1,D)$$

To simplify, set $P=1$.⁴ Then $LNP=0$, and:

$$n = LNF/LNI \quad (1,E)$$

Simplifying more, it can be shown that LNI nearly equals i . For a proof, see Appendix II-1. Meantime, check it out on your calculator. You'll see that LNI is slightly lower than i . LNI is called the "force of interest." It is the rate that corresponds to i when compounding is continuous. In practise hardly anyone uses continuous compounding in business, except occasionally as a sales gimmick. Monthly compounding has nearly the same quantitative effect as continuous.⁵

So finally we have:

$$n = LNF/i \quad (1,F)$$

You can also use (1,F) to solve for i , when you know F/P and n . We already solved for i by taking the n th root of F/P (1,C). Dividing a log by n is another way of taking a root.⁶ I mention this only in passing, for the record, for your interest, and for your better understanding. In this course we will use the method of (1,C) to find i , and the method of (1,F) to find n .

"LNF" is still a mental block for many, and understandably so: we don't use them every day. This is where the method of tabulation is useful. Let's just have a look at some numbers, to get an idea where they fall.

⁴When P is not equal to one, use the ratio of F/P instead of F in (1,E and 1,F). This has the same effect.

⁵Continuous compounding is still very useful to simplify derivations, resolve conundrums, and give added insight: we will be using it later.

⁶In the ancient years B.C. (Before Calculators), taking roots was one of the main uses of logs.

Table 1: Values of F, and corresponding logs

F:	1.0	1.5	2.0	3.0	4.0	8.0	16.0	32.0
LN F:	.0	.41	.69	1.10	1.39	2.08	2.77	3.47

Play with the "LN" button on your hand calculator until you have a feel for logarithms. Don't mess with the "LOG" button, which gives logs to the base 10: we don't use that in this course.

5. The Rule of 72.

The Rule of 72 is used to estimate doubling times. It is a rule of thumb, not 100% accurate, but very useful for two things:

a) quick and dirty estimates;⁷ b) getting a better feel for and understanding of exponential growth. (Each doubling is a growth of 100%.)

The rule comes from the fact that $\ln 2 = .69$. The .69 is raised to .72 to allow for the error in the approximation setting $\ln I = i$.⁸

To estimate the doubling time, divide .72 by the interest rate, expressing the interest rate as a decimal; or divide 72 by the rate expressed as a percentage. Thus, at 3% money doubles in 24 years; at 6% in 12 years; at 9% in 8 years; at 12% in 6 years; and so on.

It is useful to have on the tip of your tongue that money doubles every 10 years at 7.2%. It follows from that that in 60 years each original dollar has doubled 6 times, and grown to $\$2^6 = \64 . It's enough to make a young person start an early savings program. It only works, however, if you faithfully plow back all the interest, never withdrawing a bit for that new Toyota pickup, wedding bash, or anything.

6. Using semi-log scales.

⁷The rule is most accurate when $i = .08$. When $i \gg .08$, the rule gives too high a value for n . When $i < .08$, the rule starts giving too low a value for n . (So? check that again. What about when $i = 0$?)

⁸In fact, $\ln I$ is a shade lower than i . If we used continuous interest, it would be a rule of 69, and would be 100% accurate. However, we use annual interest rates, so it is a rule of 72, and tolerably accurate for economic and managerial purposes, where we are dealing with a narrow range of interest rates, and pretty fuzzy data most of the time anyway.

An excellent way to visualize compounding, or exponential growth,⁹ is to plot the growth curves on a semi-log scale, with n on the abscissa and F on the ordinate. The origin of the scale is where $n=0$, but $F=1$.¹⁰ On such a scale, equal vertical rises represent equal percentage rises. That means every curve of constant percentage growth, such as $(1,A)$, is a straight line. Compare this with the Bob Hope ski-nose curve you drew earlier, for $(1,A)$.

You can make your own semi-log scale from a common scale. Just label the ordinate origin as 1, and equal rises on the ordinate as 2, 4, 8, 16, 32, 64, etc. When you get up in the clouds of 1024, 2048, etc., you'll get an awesome feel for the power of compounding. Money doubles from 2048 to 4096 just as fast as from 1 to 2 (if you should live that long, and save that much).

The slope of the line is in proportion to the percentage growth rate. Thus, doubling the percentage growth rate doubles the slope. That means you reach any given F -value in half the time if you double the growth rate. Growth curves with different growth rates, starting from the common origin, diverge like rays of a sunburst as n rises. The vertical distance between them increases with n . This shows that their ratios -- not just their absolute differences -- change with n . A higher ray becomes an ever higher multiple of a lower ray. (For constant multiples, draw parallel lines. They all have a constant ratio: e.g., $.5y$ is parallel to y ; so is $2y$.)

Discounting on the scale is the mirror image of compounding. Instead of growing the curves "decay" exponentially, forming straight lines sloping downwards to the right, starting from the origin. The changes are negative, shown by the negative slope. The absolute changes are much smaller, but the percentage changes are the same, at corresponding points.

Compare these downsloping rays with the discount curve you drew to illustrate $(1,B)$, Column 2. (Don't try to add up the areas under these curves on semi-log paper.)

⁹Remember, "exponential growth" is the growth of an exponential function when the exponent is the independent variable, and the base is constant.

¹⁰You get the same effect by plotting the logs of the growth curves on a common scale, but here we just treat the semi-log scale.

These decay curves show the effect of interest rates on present values. Higher interest rates selectively downvalue future values in proportion to their futurity. The higher is n , the more sensitive values are to i .

For more on semi-log scales, see Appendix II-2. Take note of "The Arrow Theorem" there, it tells you a lot.

END OF PIPO

B. Converting flow values to point values, and vice versa

Finding the point value, (or lump sum, or single-payment) that corresponds to a known flow (or "stream") of payments is called "capitalization." Valuation, appraisal, and assessment are also standard terms. The inverse operation is "annualization."

Point values are often called "funds," but it is a poorly chosen word. "Point value" is more general than fund. It means a value at a single point in time.¹¹ A standard text will tell you all values are either funds or flows, but the term "fund" is wrong because there are many point values that are not funds. Land and corporate stock prices, for example, are point values, but not funds.¹² We will use point value, or just "point," rather than "fund."

Durable capital, likewise, is not a pure fund. Like a fund it is stored up, and it may be drawn down; but unlike a true fund, it can not be drawn down quickly, and not at a speed you can control or affect much. It is not on tap, like beer.¹³ The value in it may be liquidated or recovered only at a rate predetermined by the way it was constructed. It is "illiquid," meaning the capital in it is frozen solid and you cannot thaw it at will: there is no universal microwave to handle this defrosting job. In a liquidity crisis (like now?) its future cash flows cannot be advanced to

¹¹In vulgar usage, e.g. by John Wesley Dean of Watergate fame, "point in time" is redundant, bureaucratic and affected. The present usage is expressive and functional.

¹²Land values are not created by storing-up, that is by saving and investing. Disinvesting is not possible, in the aggregate: they cannot be drawn upon like canned goods in the cupboard. One individual can sell to another, but this is a zero-sum transfer, not net social disinvestment or investment.

One could carry this further, and find another word instead of "capitalization." If I had a license to mint words, I would call it "pointilization." Discounting and cumulating are what is being done. These are valuation operations only: they do not transmute land into capital; they just make it commensurable with capital, to facilitate exchange.

However, to keep the parlance common, I will use the conventional term "capitalization."

¹³The German Lager means a store or stock. Auf Lager means on tap.

the present, however great the need for current cash, and however bleak the prospects of future demand.¹⁴

The flow values are variously called cash flows, incomes, earnings, profits, and rents, each with a significantly different meaning. Many authors, sad to say, neglect to define the flows, paper over the differences among different words, and fall into errors. Let's you and I stay clean. Generally, "cash flow" is the right term, and will be used or implied unless otherwise specified.

The flows may be of service rather than cash, as in the case of houses and grounds, pleasure cars and yachts. "Service flow" is a standard term. Do not think they are nebulous or hypothetical. Service flows are capitalized into cash prices of residences, golf club memberships, pleasure craft, and so on.

Notation:

"a" is used for annual flow. The equations are kept simplest by assuming that a is received in one payment at the end of the year. P is value at the start of year zero, i.e. one year before the first payment.

My convention is always to use lower case for flow values, and Upper Case for point values. Dates are indicated by P, for present, and F, for future. This often avoids the use of cluttering subscripts; but where necessary, I use subscripts to show dates.

1. "Capitalization in perpetuity."

Converting a flow value to the equivalent point value is simple where the future flow is infinite. This mainly applies to land, so this point value will later be denoted as "G" (for Ground).¹⁵

¹⁴Theorists speak too blithely of "opportunity costs." Capital loses its short run opportunity cost the moment it is sunk into hard, or durable forms. Oddly land, the most durable asset, retains its opportunity cost because of limited quantity and constant demand for the same old land for new uses.

¹⁵G also can be remembered from the Greek Geo. It also works in German, using Grund. Why German? Because the use of K for Kapital is standard, so it is fitting to use a German word for land. Why not L? Because L is normally used for Labor. Why not T, for terra, tierra, terre, etc? Sometimes you just flip a coin:

Here, where we are simply getting our math straight, I will use P , for Present Value.¹⁶

$$P = a/i \quad (2)$$

(2) is called simple capitalization in perpetuity. It is simple because a is assumed to continue the same, or "level." "In perpetuity" is, of course, legalese for "forever": this also keeps the math simpler.

The reasoning is, if your money in the bank earns i , then other cash flows will earn the same. The market process that makes them the same is "arbitrage": buying and selling securities and titles, pushing up P until rates of return are equalized.

Some people have trouble getting used to this reasoning, and understandably so, because cause and effect are inverted. When you put money in the bank to earn interest, P is the cause and a is the effect. When capitalizing cash flows, however, a is the cause, and P the effect. The question of what causes a is left unanswered. Sometimes the answers are not very good, and have been papered over.¹⁷ In this course we take a -values as given, without endorsing all their origins.¹⁸

When a is growing you get fascinating variations on (2). See Appendix II-3.

The derivation of (2) is so simple we didn't even use the basic building block, the Compounding Factor (1,A). To see how things fit together, however, let's derive (2) by using (1,A). The

"G" won.

¹⁶ P is useful because, in addition to Present value, it stands for Principal, Price, Point, Planting cost, Prime (as in Priming the Pump), Prior, Put up, Put out, and other things that seem fitting.

¹⁷Perhaps the original owner jumped a claim, shot the first squatters, massacred some Chinese and Indians nearby, bribed a land office agent using money earned in the slave trade, bought an election, and perjured himself to acquire the property.

¹⁸We will also take as given the reality of taxes on property and its income. These taxes compensate in part for injustices in the origins of property, or did until 1978 when Prop. 13 cut them by two-thirds.

present value of a future flow of \$a is the sum of an infinite geometric series:

$$P = a[I^{-1} + I^{-2} + \dots + I^{-\infty}] \quad (3)$$

The sum of an infinite geometric progression is:

$$S = f \frac{1}{1-r} \quad (4)$$

where S is the sum, f is the first term, and r is the rate of progression. (This is high-school algebra; check your old textbook.) In (3), f is $a[I^{-1}]$, and r is I^{-1} . Substituting those values in (4), we have (2) again:

$$P = I^{-1} \frac{a}{1-I^{-1}} = \frac{a}{I-1} = \frac{a}{i} \quad (2, \text{ rpt.})$$

(2) is the basis for valuation of all real estate; also most common stocks, which are shares in the assets of corporations whose permanence rests on their ownership of real estate. (2) is highly oversimplified, but that doesn't stop most practitioners from using it most of the time as the basis of most valuations. You should know both it, so you know what others are thinking; and its limitations, so you can do a better job.

2. Present value of finite flows

Here, we build on (2). (2) gives the present value (PV) of all future time. Now we divide all future time into two parts. Part one is from now to $\underline{n}$, where $\underline{n}$ is the number of years (the "term") of the finite flow. That is what we want to find. Part two is from there ($\underline{n}$) to eternity. We find part one by subtracting the PV of part two from a/i (2). The PV of part two is its point value (which is always a/i) discounted back to the present over $\underline{n}$ years, using (1,B):

$$P = a/i - (a/i)I^{-\underline{n}} = a \frac{1 - I^{-\underline{n}}}{i} \quad (5,A)$$

The more common way to derive (5,A) is using the algebraic formula for the sum of a geometric progression, which is:

$$S = f \frac{1 - r^n}{1 - r} \quad (6)$$

where S is the sum, f = the first term, r = rate of progression, and n is the number of terms in the progression. In this case, f is aI^{-1} , and r is I^{-1} . Substituting in those values we have (5,A) again:

$$P = aI^{-1} \frac{1 - I^{-n}}{1 - I^{-1}} = a \frac{1 - I^{-n}}{I - 1} = a \frac{1 - I^{-n}}{i} \quad (5,A \text{ rpt.})$$

The ratio to the right of a is the Present Value Factor, or PVF. Learn it. Find a way to remember and retain it. Either memorize it, or remember how to derive it. It is your HOME BASE. It is both a launching pad and a shelter from the storm: most of what we do from here on is derived from it; and you may find all the other factors, even if you forget them, by starting with this PVF and rearranging it a little. REMEMBER THE PVF!!!!

You may program it into a programmable calculator if you like - that is a legal form of cribbing. You will need it from here on: it is another building block. This "block" is not like a brick so much as a LEGO part: it will be standing on its head and turning inside out to serve you, so long as you are ready to be served.

It devolves into a/i (2) when n is infinite; and into a/I when $n=1$. It applies strictly only to level cash flows that suddenly terminate after n years,¹⁹ but can be adjusted (with added complexity) to handle flows that decline in various regular ways.

Isn't it remarkable that a factor derived by the first method of (5,A), where we subtracted one infinite series from another, is the same as that derived by the second method of (5,A), where we chugged along prosaically one year at a time. Can these findings really be the same? Check it out on your calculator, which has a cumulative memory (on simple calculators, use the M+ button). Add I^{-1} , I^{-2} , ... out to I^{-n} . You choose the values of n and i . Use the full power of your calculator, out to 7 or 8 decimal places.

¹⁹Older writers call this the "One-Hoss Shay" model, because of its resemblance to a vehicle described in The Deacon's Masterpiece, a classic poem by Oliver Wendell Holmes. We must find a modern metaphor based on something in Star Wars, Dances with Wolves, or Hagar the Horrible.

Store or record your answer. Now calculate the PVF from (5,A). You should get the same answer, to the next to last digit,²⁰ and be convinced.

To visualize all that, retrieve the graph you drew earlier when we studied the PIFO Discount Factor, (1,B). This is the graph of Cyrano's nose (or Doonesbury's), facing to your right. You shaded in the first five columns; the shaded area is the present value of all five together. This is the value that the PVF gives you, numerically, precisely.

You also drew a cume curve. Each point on the cume curve is numerically equal to the area under the discount curve out to that value of n. Extend that cume curve now out to $n=50$ and $n=60$. Note how it flattens out: life after 60 doesn't add much to present value at time zero.²¹

Test your mathematical ingenuity. Divide the PV of a 60-year cash flow into three parts of 20 years @. Show how much, and what share, of PV comes from each reach of 20 years. Now how would you go about that? Hmmm - over to you. Think!²²

3. Solving PVF for its different elements

a. Solve for a

$$a = P \frac{i}{1 - I^{-n}} \quad (5,B)$$

The ratio on the right of (5,B) is the Installment Plan Factor, or IPF. (Some call it the "capital recovery factor"). This is the

²⁰The last digit is rounded.

²¹Reasoning thus, one might follow ex-Gov. Lamm of Colorado, who invited all folks over 60 to move over, pull the plug, and let him scatter their ashes over the Rockies. However, in year 60 itself, life after 60 has a much higher PV than it does at time zero. It is also possible that the annual service flow may be higher, depending on individual cases. Gov. Lamm may also be right about some folks (other than me, of course).

²²A call of nature at this time is not avoidance, but a sign your body knows you are really getting into it. A sudden urge to visit the beach, on the other hand, is avoidance.

factor or coefficient by which you multiply P to get the equivalent a .

This is how you buy on time in level periodic installments. " a " paid regularly over n periods is equivalent to P . From the lender's viewpoint, he recovers both Principal and interest.

(5,B) is something akin to a mean per year. That is, it is akin to P/n . However, it is always higher, because it covers interest as well as Principal. It is just a little higher than P/n when n is low, and a lot higher when n is high.

Exercise: check the extreme values algebraically. Show that $IPF=1$ when $n=1$, and $IPF=i$ when $n=\infty$.

Next, use the numerical, tabular method. Make four columns, showing n , $1/n$, i , and IPF , for various values of n . Use $i=.10$ the first time, so we can check our results together. Work out 3-4 intermediate values of IPF : it will always be higher than either $1/n$, or i . (On exams, you never know what crazy value for i I will ask about, so don't groove on .10 permanently.)²³

Next, use the graphical method. Using your table, put your few points on a graph. Interpolate by eyeball to sketch out a smooth curve of the whole function, and visualize how these functions behave. Put n on the abscissa, and decimals from .10 to 1.10 on the ordinate.

Think of IPF as a combination of dividing by n , as in a simple yearly average; and multiplying by the interest rate, i . IPF approaches $1/n$ when n and i are both small; and approaches i when n and i are large, but it always remains above both $1/n$ and i .

When you retire or "amortize" a debt on the installment plan, using level annual payments, IPF is used to figure the annual payment. To convert to monthly payments, which are commonly used, multiply n by 12 and divide i by 12.

What if you want to convert to weekly, daily, hourly, or minutely payments? No problem, with your calculator. Play around with it,

²³After you have worked out 3-4 values, you will understand better why people write programs. You who know how to program your calculators may want to do so at this point, although it is optional. If you do, you can easily find many intermediate values, and get a detailed picture of the whole function.

and you will end up rediscovering the magic number e , the base of natural logs. You will find it a lot easier than poor old John Napier, in the 1590s or so: he did it all by hand.²⁴

Just for fun, figure out how much you would have today if John had put aside a tuppence for you then, and no one had touched it or its income of 7.2% annually over those years. 2^{40} gives a quick and dirty estimate, and makes it fairly clear that no one has ever had a string of descendants with that kind of will power or luck.

What if you know the monthly payments, and want to figure out what i they are charging? Patience, that comes later when we solve for i .

What if you've made your payments faithfully for many months or years, and they tell you your unpaid balance is higher than you think? Patience, that comes later, too, under "Amortization."

b. Convert P to F

Sometimes you want to know a future point value. To find the factor, just multiply the PVF times the Compounding Factor (CF), from (1). This has the effect of multiplying each of the many individual terms in the finite series by CF, converting each to a future value: quite a labor-saving procedure.

$$F = PI^n = aI^n \frac{1-I^{-n}}{i} = a \frac{I^n-1}{i} \quad (7,A)$$

The ratio on the right of (7,A) is the Future Value Factor (FVF).

The FVF is used when you want to know how much a stream of payments in or out will come to at some future date, with accumulated interest. It may be money coming in: good news. Say your generous grandmother is putting \$5,000/year into a trust fund for you to control when you reach age 40, and the fund is earning 9%, how much will you have then?

²⁴ That was towards the end of the long reign of Queen Elizabeth I, as she was mulling over whether to make James Stuart her successor (she did, just before she died in 1603). There were no calculators easily available at that time. It was the year they settled Jamestown, 17 years before the Pilgrims landed at Plymouth Rock, if that helps you place it.

Suppose you put aside \$100/month for your retirement, and can earn 8%. How much will you have at age 65? $1.08^{45} = 31.92$. You take it from there.

Often it is bad news. Say it takes \$13 millions to construct a plant, but the cost is stretched out over 13 years, financed at 14%. How much will you have tied up in the plant by the time it goes on stream? (7,A) will give you the bad news, and it is a shocker: \$32,088,654. Check it out. Some \$19 millions of that, or 59.5%, is accumulated interest.

When your plant finally goes on stream its cash flow must earn a return on the whole \$32+ millions, not just the \$13 millions. The plant won't last forever, either, so the cash flow must not only earn a return on \$32+ millions, but also a return of \$32+ millions: i.e. enough more to recover the capital, which now includes the accumulated interest. That's why it is called "capitalized interest": it's added to the capital cost.²⁵

Review problem: how much cash flow is needed to pay off \$32 millions over an assumed plant useful life of 25 years? Use IPF, and $i = .11$. Your answer is ???

How do I come off lowering i from the original .14 to .11? Because construction loans generally come higher than "take-out" loans you can get after going on stream.

Compound review problem. After 25 years of throwing back cash (at the rate you just found), your plant retires. Summing up at that time, how much, and what share, of its gross cash flow has gone to pay interest? Who made out better, the hard-hats or the pin-stripe briefcase brigade?

c. The "Comsimp Ratio"

To find the ratio of total cost to construction cost, divide (7,A) by na . The a 's cancel out, leaving you with the FVF/n . This is also the ratio of compound interest to simple interest, for

²⁵Regulated utilities, like SCE, are guaranteed a fixed return on their invested capital, which is called the "rate base." AFDC is an acronym for Allowance for Funds During Construction: it is capitalized interest on plants that take a long time to build. It goes into the rate base when they go on stream. It is one reason why the richest economists are those who work on utility rate cases.

which reason I call it the "Comsimp Ratio."²⁶ It and its reciprocal crop up in managerial problems from time to time, so remember The Comsimp Ratio (8):

$$F/na = a \frac{I^{n-1}}{nai} = \frac{I^{n-1}}{ni} \quad (8)$$

The Comsimp ratio = 1 when $n=1$, and then rises, slowly at first, then ever faster. For an exercise, figure out the ratio for the French Canal du Midi, linking the Mediterranean with the Bay of Biscay: it took 200 years to complete. (Don't complain that I forgot to give you an interest rate; go get one yourself, they're in the middle cupboard.) Closer to home, try the Tenn-Tom Canal, linking the Tennessee River and the Tombigbee in Alabama. It was started when Andrew Jackson was President (I won't insult your education and culture by telling you when that was), and completed in about 1985.²⁷

d. Solve FVF for a

Sometimes you need to know how much to set aside regularly in order to meet a future payment or need. Then you solve (7,A) for a.

$$a = F \frac{i}{I^{n-1}} \quad (7,B)$$

The ratio on the right of (7,B) is called the Sinking Fund Factor, SFF. It is used to determine the annual payment necessary to meet a known future need.

²⁶"Comsimp" is a mnemonic, i.e. a name that helps you remember something. During the name-calling era of the 1950s, comsimp was a pejorative used by various inflammatory speakers and journalists to mean communist sympathizer. Symp was misspelled simp either to suggest simpleton or because the speller was one: which, is still a matter of opinion and dispute. Either way, remember the Comsimp Ratio.

²⁷Many structures still yield low or negative cash flows after they are "completed," so the capitalized interest keeps building up. We will draw the curtain of charity over the rest of these horror stories so we can move on.

For example, nuclear generating stations need to be removed for burial in remote locations at the end of their lives (about 25 years), and the anticipated cost is enormous. The Nuclear Regulatory Commission, expecting that the owners may plead poverty when the time comes, requires them to set aside in trust an amount each year which, invested at compound interest, will meet the anticipated need.

The amount required, a in (7,B), depends on two speculative forecasts: F , and i . a is obviously an increasing function of F . How about i ? If you were SCE, arguing to hold down a , would you argue for a higher or lower value of i ?²⁸

Some debts come due all at once, in what's called now a "balloon" payment. The lender has what's called a "redeemable" note or bond: your balloon is his redemption.²⁹ To meet that future debt, you invest a regular amount. The fund you put the money in is called a "sinking fund."³⁰ Some state constitutions require the state to set up a sinking fund every time it borrows (unless the debt is fully amortized).

SFF is also used to decide between competing resource uses. E.g., in Northern California there are lands which can grow timber or graze sheep. A timber harvest of \$10,000/acre, 60 years in the future, cannot compete for the land against the rental from sheep at \$10/year, unless the interest rate is below 8%. Looks incredible, but figure it out: another example of the power of compound interest.

Let's pause here for breath, and sum up. We now have the PVF in four different forms: straight (5,A); the IPF (5,B); the FVF (7,A); and the SFF (7,B). We have seen these are all just different arrangements of the same basic equation: if you know

²⁸A higher value of i . This makes a lower. This figures: if your principal earns more interest, you need less principal to reach a given F .

²⁹That's not a theological term, although the concept of redemption originated in debt, which is as old as human society. Not to be sacrilegious, the bondholder feels better knowing that his redeemer liveth. Actually, though, originally it was the debtor who was redeemed. He redeemed his body, or the bodies of his family, from debt slavery.

³⁰It's an unfortunate term: you want this fund to float and rise, not sink.

one, you know'em all. This is why I told you to master the original PVF: a little bit goes a long, long way, and we're not done yet.

Things get a little trickier after this, though, so be sure you are thoroughly on top of what we have done so far. Review; work problems. Then pump yourself up, jog in place, and here we go on the next lap!

e. Internal Rate of Return (ROR, aka IRR)

Sometimes we know P and a and n , but not i . Solve for it. When you solve for it, i is called the Rate of Return (ROR), or Internal Rate of Return (IRR). You have also seen this called the "marginal efficiency of capital" (a term popularized by Keynes, in macro-economics).

We already have ROR in the PIPO case (1,C). It is harder to solve for i in Eqns. (5,A and 7,A) because you cannot collect terms. Here is what you get when you try:

$$i = a \frac{1 - i^{-n}}{P} \quad (5,C)$$

" i " is on both sides; you cannot collect terms. There is a "tail-chasing effect" when you attempt values of i : raising the left side raises the right side, etc.

Don't despair, Isaac Newton faced this problem long ago. Not one to accept defeat, he came up with a method of homing in on i by systematic trial and error. Newton's Method is built into certain hand calculators (e.g. the HP12C) which are used routinely in business to solve (5,C) and more complex variants as well. You can do it yourself with any programmable calculator, too: program in the right side, and proceed by trial and error. With a little practise you can get it to four significant figures in just a few passes.

Whenever $na=P$ you'll get a zero rate of return, obviously, and it is correct (although not gainful). Otherwise, zero is always one of two solutions that satisfy the equation, but it is meaningless economically and may be ignored. We'll let fretters, philosophers and mathematicians meditate on its deeper meaning, and they do: some even tell us the IRR is ambiguous and should not be used. Meantime, the world goes right on using it, and so will we.

Sketch out a graph of the right side and left side of (5,C) as functions of i . The left side, of course, is a straight line sloping up at 45 degrees from the origin. The right side starts out steeper, then bows over and crosses the 45 degree line. That crossing is the right answer. That tells you that if your right side $> i$, your i is too low, so try a higher i . And vice versa.

Your right side can include a lot more factors than are shown in (5,C): salvage values, taxes, irregular revenues and costs at various dates, revenues declining according to various regular patterns, etc. Once you program those in, it only takes micro-seconds more for the calculator to grind out the answer, so you may end up doing that some day. See Appendix II-4.

Today's more powerful calculators increasingly include a program for ROR. When you teach yourself by studying the manual of the HP28S, one of the first things they show you is how to write and store your own program for finding ROR.

In exams, I will not assume you have any such device available, although you may. Rather, I will give you 3-4 values of i and ask you to select the one that is correct. That way you don't have a long hunt. The correct i is, of course, the one that makes both sides equal.

When $n=\infty$, $i=a/P$. Otherwise, " i " is always less than a/P ,³¹ and may be negative. It is always an increasing function of n , although decreasingly so as n gets larger. It appears to vary directly with a/P but actually is even more sensitive than that because of the tail-chasing effect between i on the left and I on the right side. See Table ~~x~~. 2

Table 2: ROR for various n and a/P

$n =$ a/P	<u>2.50</u>	<u>5</u>	<u>10</u>	<u>15</u>	<u>20</u>	<u>30</u>	<u>50</u>
.1	-.52	-.20	.00	.055	.077	.093	.099
.4	.000	.288	.385	.397	.400	.400	.400

5. Amortization or payback

³¹Unless a is rising. This is covered in Appendix ~~I-10~~. II-3

Amortization is the process of payback, that is retiring debt by borrowers; and the counterpart, recovering capital by lenders. We are also interested here in the payback of equity capital: how fast do direct investors in durable wealth get back their money. "Capital recovery" is the term for that. We will treat them together, and interchange the terminology.

In naive usage, payback time means simply P/a , as though all cash flow from an investment could be used to retire principal. You hear that a lot. People selling energy-saving devices during the energy crunch used it to death: anything in a good cause. I support conservation, too, but this was ridiculous. You should be able to see right through it though. Here we will develop correct ways to figure payback time.

When a business gains revenues, the creditors line up in a certain priority, both logically and legally. Current operating expenses come first; the residue after that is called cash flow. In this course we generally finesse that step, and begin with cash flow.

Interest comes next. Interest is always based on the unpaid balance of a debt. Likewise, for your equity investments, you owe yourself interest on their unrecovered balance.³² Only after you have paid current interest can you apply any overage to reduce the balance. For equity capital, such payments are called "capital consumption allowances," and also "depreciation" allowances. For debts, they are payments on principal, or redemptions. Payments including both interest and redemption are called "debt service."

Ground rents and taxes have their priorities, too, but we'll bypass that for now to keep down the complexity.

With PIPO investments we said nothing about amortization, because it is obvious you recover your money all at once, at the term (i.e. the end, the time of "point-output").³³

³²If you try to skip an interest payment, and pay back principal instead, what happens? The unpaid interest is capitalized, i.e. added to the principal. Might as well pay interest first and avoid such mickey-mouse.

³³It's not so obvious when you put your money in. Capitalized interest, an invisible input, goes in one year at a time. If we were willing to make it that complex, we could take an average value of n based on when each bit of interest was

In cash flow management we are especially interested in how fast capital comes back to be reinvested. It is that flow of cash recovery that pays the bills, meets the payroll, keeps the wheels turning, and saves you from insolvency. Macroeconomically, reinvestment in payments to workers generates most of the national income and makes jobs, so it is important to the whole fleet as well as to your little put-put.

We deal here with debts being retired on the installment plan, i.e. with level payments over n years. The level payment, a , is determined using the IPF, Eqn. (5,B).

a. Interest vs. Recovery in year one

In year one the interest is obviously Pi . To find interest as a share of the total payment, divide Pi by $[P \times IPF]$:

$$Pi/[P \times IPF] = 1 - I^{-n} \quad (9)$$

Repayment (R) as a share of the total is obviously the complement of Pi :

$$R/IPF = I^{-n} \quad (10)$$

For example, on a 30 year note at 12%, $I^{-n} = 1.12^{-30} = .033$. That means only 3.3% of the first year's payment goes to reduce the balance owing. The rest is all interest.

I bet you'll forget. You'll think that in the first year you reduce the balance by 3.3% - that's WRONG! What I said is that only 3.3% of the first year's payment reduces the balance. To get the percentage drop of the balance, multiply I^{-n} by a/P . Explain why that gives you the percentage drop in the balance. Here is what you get:

capitalized. In addition, imputed ground rent has to be capitalized each year.

For a really simple case we would use the "cow model": the cow yields a flow of milk for n years, and then a salvage value which, for real simplicity, we can make equal to the initial cost of siring, birthing and rearing a cow. The cow neither appreciates like a tree, nor depreciates like a building: it is like a simple savings account from which you withdraw the interest regularly. This case is so simple I'm only referring to it in this note.

$$I^{-n} \times [a/P] = I^{-n} \frac{i}{1-I^{-n}} = i/[I^n-1] \quad (7,B, \text{ rpt.})$$

Now that is a small, small number indeed. Figure it out, with $i=.12$, and $n=30$. I get .004144. Do you? What percentage is that?

Note that the rightmost term is our old friend, the SFF. That's why it's numbered "7,B, rpt." These patterns have a way of repeating, and cropping up all over.

b. Interest vs. Recovery in later years

(9 and 10) hold also for later years, but in place of the original n of 30 years you use the remaining term, t , or $30-c$. " c " (for current year) is the number of years elapsed. $t < n$; that makes R larger.

It is a comforting note that when $c=29$, $t=1$, and I^{-1} of your last payment goes to retire the debt. You do finally get out from under. CHECK THAT.

In place of the original P , use the current unpaid Balance, B_c . To find B_c , it has to be the PV of the remaining stream of payments. Use the PVF, with t where originally you used n , or 30.

c. Balance remaining unpaid after half the total life

The remaining balance at $n/2$ is the present value of the level payment, a , over $n/2$ years. (Surprised? Remember that the original P is the present value of the level payment over n years, too; and if you think about it, this must always be true.)

$$B_{n/2} = a \frac{1-I^{-n/2}}{i} \quad (11)$$

The unpaid balance as a share of P is:

$$\frac{B_{n/2}}{P_0} = \frac{1-I^{-n/2}}{1-I^{-n}} \quad (12)$$

(12) is one of those inscrutable, unfamiliar functions it is helpful to tabulate, to get a feel for how it behaves. Table 3 shows the complement of (12), meaning $[1-(12)]$, i.e. the share

that has been repaid after half the life, for various original values of n (n is the original whole life of the debt or investment).

Table 3: Share repaid after half the full life, for various full lives (n), and various interest rates, i .

n :	2	4	10	20	30	60	80
i							
.04	.49	.48	.45	.40	.36	.24	.17
.08	.48	.46	.40	.32	.24	.09	.04
.12	.47	.44	.36	.24	.15	.03	.01
.16	.46	.43	.32	.18	.10	.01	.003

d. "Half-life": Time required to pay half the debt

This is different from half the time, which is $n/2$. As you have seen, debt is repaid slowly at first, then ever faster to the end. To visualize this distinction, draw a curve that slopes negatively, but that is nearly flat at first. It slopes more and more, and finally reaches a value of zero while sloping at an angle of roughly 135° . Label the graph, the Unpaid Balance of Debt Amortized in Level Installments. Label the curve B_c .

At $n/2$ on the abscissa draw a vertical line. Where it intersects the curve of B_c is the unpaid balance when $c=n/2$.

Now go up the ordinate from zero halfway to the y-intercept of B_c . From here draw a horizontal straight line. Where it intersects B_c is the half-life, the year when half the debt is repaid, or half the capital recovered.

Now you have a clear picture of what we are doing, let's do it with algebra, which will then yield us numbers: nice, comforting, "real" numbers. For this we must learn to solve PVF for n . This requires logs, but don't panic, whip out your trusty modern hand calculator and learn what wonders you can work with this modest investment.

Starting with $(5, A)$, here is how you solve for n .

$$Pi/a = 1 - I^{-n}$$

$$I^{-n} = 1 - Pi/a$$

Take the LN of both sides:

$$-nLNI = \text{LN}[1-Pi/a]$$

$$n = \frac{-\text{LN}[1-Pi/a]}{LNI} \quad (5,C)$$

(The Equation is numbered (5,C) because it is still the same Eqn. as (5,A) and (5,B). It just looks a lot different when you solve it for n.)

"How can n be negative!?" I hear you scream. Not to worry: $0 < Pi/a < 1$, so $[1-Pi/a]$ is positive and less than one. The log of a positive number less than one is negative. Everything is legal.

(5,C) must be easy to remember, because my students always do somehow. What I do is just remember the PVF, and derive everything else from that. You do whatever works for you.

Patience, now, (5,C) does not give us the half-life we are seeking. I know you want to jump the gun and get this over with, but (5,C) is just step one. Gently, gently ... we'll get there, step by step.

Half the debt is retired (aka amortized, repaid, recovered, worked off, worked down, etc.) when P is reduced to half. So just replace P in (5,C) by $P/2$, and solve the equation for n. But this n is not the original n, so we will call it "x." Note that x is not half of n, it is the number of years of life remaining when half the debt is retired.

Warning! "x" is still not the final answer. We are looking for the half-life, which is the number of years elapsed. x is not that, but rather the life remaining. The half-life is $n-x$.

Some of you will forget that, I know you will. I am already laying traps to catch you on the next exam. I am warning you all to watch it, but someone won't. I can warn, warn and warn, but there's another klutz matriculating every year. If not distracted by life in general, it'll get distracted by what we do next. But first, let's find x.

$$x = \frac{-\text{LN}[1-Pi/2a]}{LNI} \quad (13)$$

Now, what is x , again? Doesn't help to find it if you forget what it is. If you've forgotten, check back and refresh.

You don't need to know P or a to find x from (13). From (5,A), $P/a = 1 - I^{-n}$. Therefore:

$$1 - P/a = 1 - \frac{1 - I^{-n}}{2} = \frac{2 - 1 + I^{-n}}{2} = \frac{1 + I^{-n}}{2}$$

Substitute that in (13), and:

$$x = \frac{-\text{LN}\{[1 + I^{-n}]/2\}}{\text{LNI}} = \frac{\text{LN}2 - \text{LN}[1 + I^{-n}]}{\text{LNI}} \quad (13, \text{alt.})$$

As I warned you, of course, there is some klutz out there who is going to have been distracted by all that razzmatazz, and forget the last step. You, however, are not that klutz: you going to remember. Let h = the half-life, then: *me*

$$h = n - x \quad (14)$$

Now, at last, we have done what we set out to do. And what was that? In all the technicalities, it's so easy to forget! Check back to be sure you know. Then, to check yourself, see note.³⁴

It is interesting to see how the half-life compares with n , the whole life. Remember, half-life is not $n/2$, it is the year at which you repay half the debt. To get the half-life as a share of the whole life, divide by h by n :

$$h/n = (n - x)/n = 1 - x/n \quad (15)$$

We went through this hassle to get numbers, so let's get some. They are in Table 4.

³⁴Our objective is to find the half-life, h , corresponding to a whole-life, n , of a debt that is fully amortized on the installment plan.

Table 4: Half-life/whole-life, for various n and i .

$n:$	2	4	10	20	30	60	80
i							
.04	.51	.52	.55	.60	.64	.74	.79
.08	.52	.54	.59	.68	.74	.85	.89
.10	.52	.55	.61	.71	.78	.88	.91
.12	.53	.56	.63	.74	.81	.90	.92
.16	.54	.57	.67	.78	.85	.92	.94

Table 4 tells you that if you buy on time at 12%, over 30 years, it will take 81% of 30 years, (or 24.3 years), to retire half your debt.

Before organizing an indignant debtors' protest march, look at it from the lender's view. You are sitting on his money. He can't make any new loans with it until you return it. He can't generally mint more (in spite of what you have sometimes heard about bankers). If he wants more, he has to pay interest for it too, and/or render services.

Maybe you won't cry much for him, but look at it from the viewpoint of other borrowers: you are crowding them out. The lender can't lend them his capital until you return it.

Look at it from the equity investor's viewpoint, when he sinks his capital into durables that resemble a 30-year mortgage in their payout schedules. He can't reinvest his capital until he recovers it. His volume depends on capital turnover. This is a good reason to avoid building your version of The Trump Tower until you are very, very solvent.

They used to say there are lots of children running around without shoes on because their daddies forgot to lead out trump; but there are more today driving with bald tires because their daddies forgot to schedule the cash flow. "The Donald" may become one of those, along with a number of overextended local venturers whose names you may read in the press from time to time as their creditors close in.

Back to business, look at it from the employee's viewpoint. Payrolls vary with volume. You're sitting on his job. When God created Heaven and Earth he created compound interest, to keep capital busy turning and people in jobs.

e. Summary on payback

Here is a summary of what we have done with Eqns. (5,B; 10; 12; 14 and 15), w.r.t. a 30-year loan at 12%.

From (5,B) the level annual payment (pmt) is at 12.41% of the Principal, P. Thus it is just a sliver above the i of 12%. But if that sliver were reduced to zero, the loan would never be repaid.

From (10), the share of the first year's pmt that is not interest is only 3.3%. The symbol for this portion is R, which you may read as Redemption, Retirement, Recovery, Repayment, Recoupment, etc., for they all mean about the same thing. The rest is all interest, a pure flow of expense to the debtor, and to the lender, income.

Most income is normally consumed. The bulk of future investment has to come from R: most of this is normally recycled. The reason the people who have most of the capital still have it is because they habitually avoid impairing its substance. Prodigal folks will, of course, sometimes "eat the seed corn," but, having done so, they are out of it as capitalists. The ones that remain relevant are those that Recover and Recycle it.

From (12), the share repaid in 15 years is 15% of P. The other 85% remains unpaid, a stone around the neck of the debtor, and to the lender, frozen capital which he cannot yet recycle.

From (14 and 15), the half-life is 24.2 years, or 81% of the whole life.

Students with fine-tuned math instincts will sense that half-life as defined here is not identical with the true mean life of the capital, defined scrupulously. Gaffney's "half-life" is a bit unsophisticated and naive, they may sniff. Let them sneer: I plead guilty. My motive, however, is economic: the extra trouble of 100% accuracy is not worth the cost in time. For greater accuracy, see Appendix II-5, "Calculating Mean Periods of Investment, pp. 9-11.

END OF B, CONVERTING POINT VALUES TO FUNDS AND VICE VERSA

C. Land and Capital together

Notation:

K is capital (from the German Kapital). It has a finite life, through consumption of the corpus, or through depreciation and obsolescence.

G is land (from Ground, or the German Grund). L usually stands for labor, so is not used for land.³⁵ Land differs from capital in about 20 or more specific qualities, but the one most relevant here is its permanence. Land has infinite life, except for exhaustible resources, which are not treated here. See Appendix II-6 for references to exhaustible resources.

Both K and G are measured in value terms. Subscripts refer to dates. Lacking a subscript, time zero is meant.

What we have previously called P, for Present Value, is now divided into two parts, K and G.

1. PIPO cases, with G constant

G is here treated as constant over time, in order to develop the basic math in simplest form. Having done so, you can have fun playing with various scenarios about rising G. Falling G is not so much fun, but it happens and you'd better consider that, too.

PIPO cases generally deal with growing capital like a tree, or livestock, which yields its value at one point of future time. "F" is used to denote that Future Value, just as in (1,A). But remember that P is now gone, replaced by its two components, G and K.

a. Compounding

$$F + G_n = [K + G]I^n \quad (16,A)$$

You put in G at the start of the cycle, and take it out at the end. It is a kind of "dead weight," but it changes the necessary relationships among F, K, I, and n.

³⁵You often see S, for Site, Soil, or the Spanish suelo; T, from the French terre, Spanish tierra, and Latin terra; B, from the German Boden; E, for Earth; etc.

i. Solving for K, if G is given

See (16,C), below.

ii. Solving for G, if K is given

Solving for G opens up new vistas, and is treated separately below under Faustmann's Formula (16,F).

b. Discounting

$$K + G = [F + G]I^{-n} \quad (16,B)$$

i. Solving for K, if G is given

$$K = FI^{-n} - G[1-I^{-n}] \quad (16,C)$$

Recall that I mentioned earlier (under A,2, Discounting) that the costs of passage through time include more than interest. I promised to treat them later. This is later, and I am here to keep my promise. Aren't you glad?

^e These other time-costs must be subtracted from F to arrive at the discounted product of whatever produces K. A big time-cost is rent, or land-cost. Substitute a/i for G (they are equal, it's OK), and you'll see that the right term in (16,C) is [a x PVF], the present value of the rent stream from 0 to n. If one pays ground rent explicitly, this is obvious. If one owns the ground and charges imputed rent, it is the same.³⁶

ii. Solving for G if K is given

To repeat, solving for G opens up new vistas, and is treated separately below under Faustmann's Formula (16,F).

c. Rate of Return

i. With G constant

³⁶ Most analysts are purblind to imputed ground rents and often overlook them. So much the more opportunity for you. Most people say, "if others overlook them, that must be right, so I will too." That is herd thinking, and leads you right to the abattoir with the other lambs. Nothing gives you more power than to see what others don't, or won't, but you also need a little courage and self-respect. Try reading Emerson, "Self-Reliance."

Solving for $\underline{i}$, as in (1,C), gives a closely parallel answer, which may usefully be compared with (1,C).

$$\underline{i} = \left[\frac{F + G_n}{K + G} \right]^{1/n} - 1 \quad (16,D)$$

(16,D) shows how the presence of the deadweight cost of G acts to reduce $\underline{i}$, the ROR. The higher the value of G , the closer the ratio in brackets [like this] approaches one. A higher ratio of F/K is now required to earn a given return.

ii. With G rising

If you want to get a quick fix on how rising G affects the ROR, (16,D) is the place to start. If G_n (which is the G on top) is expected to rise above G_o on the bottom, that offsets part of the deadweight effect and pushes up ROR toward what it would be if G were zero.

If G_n/G_o gets up as high as F/K , then the rise of G completely offsets the deadweight effect, and ROR is the same as if G were zero. There is no more "deadweight," but not to cheer. This deadweight has a function, it is what makes people economize on land. It's like paying rent. When this cost is offset by rising G values, people treat valuable land like a free good, and stop economizing on it. Macro-economically, this effect is quite capricious and destabilizing. See Appendix II-7.

If G_n/G_o were to get greater than F/K , that would seem to raise the ROR. But it will never happen. Why? Now watch carefully. Because then investors would just buy more land at the price G_o , rather than put any K on land they already have. They would maximize their private ROR without using the land at all. They would become pure land speculators, since the ROR on just holding land is higher than on using land. This is a case where the private boat is not helping the fleet, but holding it up.

Private parties bidding for land will push up G_o until the ROR on holding land falls. This gets tricky, however, and cyclical and destabilizing, because: a) A rise of G_o may generate expectations of equal or greater rises of G_n ; b) A higher value for G_o will lower the ROR on using land, as in (16,D), which tends to turn investors back towards pure land speculation again. Theory alone can't unravel all the possible scenarios here, because expectations are a group product, often dominated by herd

psychology. Let's just say it's an interesting topic that needs a lot more study. ³¹

iii. With G falling

Your real estate man keeps pretty mum on this, but G moves down as well as up. Land gets overpriced in booms, then slides back down: 'way back down. "Overpricing" means, of course, that a lot of people made mistakes. By hindsight, they bought in error. They were not necessarily "irrational," because successful prediction would call for omniscience, which no human possesses. Most of them, however, were following the herd.

Herd behavior is interesting and relevant, but lies outside the scope of this short course. ³⁷ In a herd-like society, rationality and mathematics don't fully explain others' behavior, but are still useful to you: they help you know when to buck the herd and set your own course. (If you are skating through this course by cribbing and relying on other students, you are setting yourself up to spend a life following the herd. You know where herds go.)

Rather, we must look at the world around us today - and yesterday. It is not enough to look around just at "today," because we are dealing with changes in G over several years. For this purpose, a page of history is worth a volume of trade gossip or reportage: history has a time-dimension. Most students underrate the study of history: it gives you the wisdom of being 200 years old or more, without the infirmities. ³⁸

Given time, we would look at some history of speculative land bubbles which gives us an idea of the range of documented human behavior in the past. There is no need to go far back in history, however: California, 1978-92, is near at hand, and tells quite a story.

The story for this course is that G often has a life of its own, moving up and down for reasons quite outside your control, and partly outside your understanding. Just remember, if G is headed south, ROR is less than would be determined by the ratio $[F+G]/[K+G]$. This depresses investors and lenders, and aborts a

³⁷ In Econ. 146, Urban Economics, we look at this in some depth.

³⁸ They keep finding more and more smart things animals can do, that people used to think we alone could master. No other animal studies history, though, so far as I know.

lot of otherwise productive investing. Likewise, when it was headed north, it induced a lot of unproductive investing. The combination means boom followed by bust.

d. Period of time

We can solve (16,A) for n , just as we did in (1,D to 1,F).

$$nLNI = LN[F+G] - LN[K+G]$$

$$n = \frac{LN[F+G] - LN[K+G]}{LNI} \quad (16,E)$$

The effect of G is to bring the two logarithms in the numerator of (16,E) closer together, thus reducing the n required to achieve a given ROR from given values of F/P . In other words, you have to double your productive money (F/K) faster in order to make the same ROR.

e. Valuation of G: Faustmann's Formula

Solving (16,A) for G opens up a new world of more realistic understanding of managerial finance. What you have learned up to now is valuable for its own sake, but its best use is to equip you to master what follows.

Up to now, we have thought of PV in terms of one cycle of investment and recovery (except in (2), where cash flow was perpetual and steady). Actually, the world goes on forever, in sequential cycles, generation following generation. Faustmann's Formula deals explicitly with that.

Solving (16,A) for G :

$$G = \frac{-KI^n + F}{I^n - 1} \quad (16,F)$$

(16,F) is named the Faustmann Formula, after a German forester who discovered it in 1849, in the process of appraising forest land. Stringing together nouns in the German way, he called it Bodenwerte (land value). American foresters, piling on words for no reason, call it "Soil Expectation Value." I refer to it as Faustmann - that is easier to remember than (16,F), or the

German, or the prolix forestry name. It also honors a very bright and useful person. Martin Faustmann was a forester, but this formula (with minor adaptations) applies to all chains of repeating cycles.

Faustmann is premised on there being many cycles of investment and recovery after the first one. In fact, that is how Faustmann first derived (16,F), which we have reached by a different route.

i. The PV form of Faustmann

It is generally useful to put Faustmann in the PV form. G is already a PV, and this change won't affect G. We just multiply both top and bottom of Faustmann by the discount factor, I^{-n} . This makes Faustmann more directly comparable with other valuation equations, as we will see.

$$G = \frac{-K + FI^{-n}}{1 - I^{-n}} \quad (16,G)$$

The top of (16,G) is the PV of the first cycle of land use. The bottom is a divisor that converts the PV of cycle one into a land value derived from all future cycles (including #1).

Rather than divide, it is more accommodating to our habits, and therefore more useful, to take the reciprocal, tabulate it as a factor, and multiply. We'll even call it the "Land Value Multiplier" (LVM), which tells you exactly how to operate it.

Here are some LVM values, to give you a feel for it, when $i=.05$.

Table 5: Land Value Multipliers when $i=.05$.

n:	1	2	4	8	16	32	64
LVM:	21.0	10.8	5.6	3.1	1.8	1.3	1.05

For example, when $n=16$, cycles following the first one add 80% to the value of cycle one. You can see why Herr Forstmeister Faustmann didn't want to overlook this factor when valuing forest land. Today, in the southeastern U.S., many stands of Southern Yellow Pine reach harvest age at 16 years. Nominal interest rates are, to be sure, at 10% or more, which would lower the multiplier. However real inflation-adjusted interest rates, the ones that matter, are more like 4%.

ii. Value and Rent

Multiplying G by $\frac{1}{i}$ we get annual rent (cf. (2)). We need a notation for rent:

$$Gi = b = \text{annual ground rent}^{39} \quad (17)$$

Note that in the top of (16,G) all values are present values. Multiplying by $\frac{1}{i}$, the rest of it becomes the IPF, so we have:

$$b = [-K + FI^{-n}] \times \text{IPF} \quad (17)$$

What (17) does is to take the irregular stream of outlays and revenues that we actually get, and convert them to a level annual amount, b , which is the annual value of land, or rent.

This is a yardstick for making the results of all land uses commensurable and comparable, so you can decide which is best. One may have a cycle of 4 years, another of 23 years, but (17) reduces all to a common measure and performance standard.

iii. Optimal life, and replacement timing

A most important by-product of Faustmann is determining the optimal value of n , which is the length of the cycle of investment and recovery. Herr Faustmann discovered he could not determine G without also determining n , so he found them simultaneously. Optimal n is the value of n that maximizes G .

A condition for maximizing G is that the log derivative of the numerator of (16,F) equal that of the denominator. That means, recall, that their percentage rates of change are equal, so that

³⁹ b is from German Bodenrente. Your German prof may shriek, because German nouns begin with Upper Case. I sacrifice linguistic accuracy here to hew to my convention that flows are denoted by lower case letters. Upper Case letters are reserved for point values. It's a trade-off, and good German lost: I apologize to the foreign language department.

It would be nice to use g , since rent is annualized G . However, g is used elsewhere to mean a growth factor. One runs out of letters after a while. Some turn to Greek, but it is, well, Greek to most of us today. It also requires another font, which I lack.

the ratio doesn't change.⁴⁰ Let's see where that leads us. To reduce clutter, let $F' = dF/dn$. Then:

$$\frac{F' - KiI^n}{F - KI^n} = \frac{iI^n}{I^n - 1} \quad (18,A)$$

Cross-multiplying, and cancelling terms:

$$F'[1 - I^{-n}] = Fi - Ki \quad (18,B)$$

$$\frac{F'}{F - K} = \frac{i}{1 - I^{-n}} = IPF \quad (18,C)$$

(18,C) says that you harvest a tree when the percentage growth rate of the harvest value, net of restocking costs, is a little higher than the interest rate, i . Even before calculators, this was operational in practise. We already know how the IPF behaves; IPF values are all tabulated. Tree growth functions are known from field studies. There was a little tail-chasing, because with rising values of n , both the left and right sides fall, but with tables in your hand you could chase the tail and catch it, no problem. The left side falls faster, and the curves cross.

Notice, we did not assume any given value for G in solving this problem. We want G -max. We take the n -value from (18,C) and evaluate G , knowing now that this is its max.

There is a careless, common, simple-minded solution to this problem, found in many texts, that leaves out G . Its simplicity makes it popular - never mind that it's wrong. It says cut the tree when $F'/F=i$: that is, the percentage growth rate equals the interest rate. The trouble with that is, it has the last year's growth earn a return only on the capital in the tree, with nothing earned on the value of the site. It doesn't pay the rent.

iv. Marginal product of time and marginal costs of time

⁴⁰I've dispensed with formalities, but kept the essence of setting the derivative of a quotient equal to zero. Just be sure it's a max, not a min. Other evidence tells us this quotient maxes out, so let's go for it.

The excess of IPF over $\underline{i}$ can be shown to equal the annual value of G , so what (17) is saying is that a growing tree has to grow fast enough to pay interest on its own value, plus the ground rent, or else you harvest it.

To show that, let $\underline{x}$ be the excess of the required growth over F_i , and solve for $\underline{x}$.

$$F' = (F-K) \frac{\underline{i}}{1 - I^{-n}} = F_i + x$$

$$x = (F-K) \frac{\underline{i}}{1 - I^{-n}} - F_i = \frac{F_i - K_i - F_i + F_i I^{-n}}{1 - I^{-n}} =$$

$$(F_i I^{-n} - K) \times IPF = b \quad (19)$$

From (19), $x=b$, and optimal harvest timing is when

$$F' = F_i + b \quad (20)$$

(20) says that a growing tree or other capital has to grow in value fast enough to pay interest on its own value, and also pay rent on the land it preempts from alternative uses. The left side is the marginal product of time, and the right side is the marginal cost of time.

Of course, $b=G_i$: they are generally interchangeable. So you could find optimal $\underline{n}$ by maximizing annual ground rent, $\underline{b}$, just as you can by maximizing ground value, G . You can skin this cat, in fact, in several ways. One is enough for this course.

v. Wider applications

The greatest value of Faustmann is outside forestry. Trees have a high and growing harvest or salvage value, so adding $\underline{b}$ to the marginal cost of time speeds up harvest only a little. Old buildings, on the other hand, have a low, zero, negative and falling salvage value. If you ignored $\underline{b}$ you would never take them out. $\underline{b}$ is the entire marginal cost of time. Read on.

2. Level cash flows, with G and K treated separately: the Building Model

This "Building Model" applies to more capital than the PIPO model. Perhaps 20% of the world's capital is in growing forms that resemble livestock, or the tree model; 80% is in forms like buildings and machines that yield a flow of cash or service over life, and then have little salvage value except for the land they occupy.

You can apply Faustmann to buildings in two ways:

one) tear them down when $b > CF$. (CF is cash flow from the old junker.)

two) max G in two steps. First, capitalize future cash flows from each generation at its respective startup date. Second, apply Faustmann to max G.

We might pursue that approach here, but you already know about Faustmann, now, and may want to learn something else. We will therefore use the Redeemable Bond approach, which follows. It accomplishes the same ends, and gives yet another viewpoint, and has other applications as well. It is a little simpler, too. In the Redeemable Bond approach, we take all future cycles after #1 and express their value at one point, the end of Cycle #1.

a. Redeemable bond

This case is clean and classic, and serves here as an introduction to the basic math needed.

Most bonds pay out cash interest, (called the "coupon"), over a finite life, and then are redeemable, meaning you get back a lump sum at the end, usually in units of \$1,000. As a rule, that is also the original par value paid in, although not always. In either case the bond has a current market price, above or below \$1,000, that fluctuates daily, depending mainly on current interest rates available on alternative investments. Once the bond is issued and traded, the current price is cut loose from the original cost, and depends entirely on discounting future expectation.

Notation:

D = Redemption value after n years. (Usually R is used for this, but in these notes R has another use, so I picked the next consonant, D).

P = Price, or Present Value of bond.

a = annual cash flow from bond, the "coupon."

i. Present Value of Bond

$$P = a \frac{1 - I^{-n}}{i} + DI^{-n} \quad (21,A)$$

(21,A) combines the annuity case and the PIPO case in one equation. This gives you, when you master it, a more comprehensive analytical power and grasp than either case taken alone. The Force will be with you! (But you have to concentrate, young Skywalker, or you will never be a Jedi master.)

ii. Rate of Return on Bond

This is also called the "yield to maturity" (YTM). It combines two yields: the "current yield," a/P; and the bond appreciation from now to year n. You find it by solving for i.

$$i = a \frac{1 - I^{-n}}{P - DI^{-n}} \quad (21,B)$$

Like (5,C), this equation is solved using Newton's method of systematic trial and error. It is marginally more complex than (5,C), but if your calculator has to labor another micro-second or two, so what?

YTM's on bonds are published daily in the Wall St. Journal. Lots of people are interested. Most papers just publish the current yields, a/P. Anyone who invests based purely on current yield is cruising for a losing. He may luck out for a while because for many bonds the appreciation element is minor. But for others it is major, and then you need (21,B).

It is tempting to simplify by dividing (21,B) into two parts, finding ROR separately, and adding. Don't: it's wrong, and can be very wrong. See Appendix II-8 on "Carter's Cut: How to Lose Money."

If $P = D$, the equation collapses into (2). Getting back exactly what you put in is the same as having a perpetual income stream.

If $a = 0$ (21,B) looks like big trouble, but you can step around the trouble by going back to (21,A), which just collapses into (1,B), from which you go to (1,C).

b. Land and Capital

The analysis of most direct investing (i.e. actually making or owning real things, not just paper) involves the same math as (21,A), which is why I started you out with (21,A). Most direct investments consist of capital occupying and preempting land. When the capital is terminated, the land is released for another use. The land, therefore, is analogous to the Redemption Value, D , of the redeemable bond. The notation for land is G (remember? G is for Ground).

The other difference is that the original investment, P , is separated into two elements, K and G , for capital and land.

$$G + K = a \frac{1-I^{-n}}{i} + GI_h^{-n} \quad (22,A)$$

(21,A) shows how the parts are tied together. Like all equations it can be solved for any element, so you can reason from what you know to what you want to find out.

Usually you treat known, fixed values as parameters, and solve for what you don't know. This is not just for recreation, but to help you maximize its value by adjusting the amounts of other inputs used. It is a judgment call what value is most fixed exogenously, and what you can manipulate: the answer varies with circumstances. In order to make good judgment calls, which is what managers are for, you have to know the techniques, the options, and the potential gains and losses. That's why you're learning this material.

i. Value of Land

$$G = \left[-K + a \frac{1-I^{-n}}{i} \right] \times \frac{1}{1-I^{-n}} \quad (22,B)$$

The expression in brackets is the PV of Cycle One (C_1). To its right is the Land Value Multiplier (LVM) (16,G,).⁴¹ What we have here is similar to Faustmann in its PV form (16,G). The difference is that Faustmann applies to PIPO, and (22,B) applies to flows.

Unlike Faustmann, (22,B) does not give us a way to solve for optimal life, n . The terminal n is just assumed. "a" goes along at a level value and suddenly self-destructs. This (the part in brackets) is therefore known as the "One-hoss Shay" model, as noted earlier. I repeat things sometimes, to help you remember, and tie things together.⁴²

(22,B) can be adjusted (with added complexity) to show "a" declining, as is often true as capital ages. I have done so in a separate set of notes, "Declining Cash Flow Models." See Appendix II-9.

ii. Rate of Return

If G is exogenous (fixed, given), solve for i . Again, this is not pure recreation: you may want to maximize i .

$$i = \frac{a}{G + K \times \text{LVM}} \quad (22,C)$$

You must be getting tired of having me find all these expressions for you: I'll let you figure out how I got from (22,A) to (22,C). As you do, I hope you'll appreciate how nice and simple (22,C) is compared to what I might have thrown at you.

Values of LVM are in Table 5, in Section C,1,d,i.

(22,C) shows how the presence of G depresses i , the ROR. Compare this to (16,D). Everything said there applies here.

⁴¹This is discussed in section C,1,d,i.

⁴²Repeating more, The One-hoss Shay is from a classic poem, The Deacon's Masterpiece, by Oliver Wendell Holmes, about a wonderful vehicle that ran perfectly for 100 years and a day until it abruptly fell all apart, something like Cinderella's coach at midnight (except that turned into a pumpkin).

If $G=0$, (22,C) collapses into (5,C). If it didn't, something would be very wrong. Run through the algebra for an exercise.

— To explore scenarios about the effect of rising and falling G on ROR, start again from (21,A), only this time put subscripts on G_0 and G_n . Your expression will be a little more complex, but not daunting. For an extended analysis of this case, showing the effects of different income tax provisions, see Appendix II-10, "Land Gains, Fast Write-off, and Incentives to Build." My
 ? students in Public Finance master this material without
 , collapsing: so can you.

END OF C: LAND AND CAPITAL TOGETHER

D. Sensitivity

Sensitivity analysis is the art of expressing how one variable responds to another. The calculus derivative is a first pass at sensitivity, but it is useful to divide the derivative by the function, getting the fractional rate of change. Multiplying that by 100 you get the percentage rate of change, a common and familiar concept.

The derivative divided by the function is also the derivative of the natural log of the function, called the "log derivative." You have the choice of taking the log first, and then the derivative, if that saves work.

A. The PIPO case

Take the basic compounding factor, with $P=1$ for simplicity:

$$F = In \quad (1,A)$$

$$LNF = nLNI = ni$$

$$dLNF/dn = i \quad (11)$$

For example, if n increases by one, $\sim F/Fn = i$. Of course we already knew that, but it's reassuring when everything checks. In fact when anything doesn't check you have surely goofed.

But here is something we didn't know:

$$dLNF/di = n \quad (12)$$

Applying this is trickier, because i doesn't change by much, hardly ever by as much as one, which is 100%. It is more likely to change by .01, and although that doesn't look like much, empires have fallen from lesser causes -- consider the present scramble to refinance mortgages because rates have fallen a few points.

When i rises by .01, F rises by $n/100$, which is $n\%$. This is (almost) equally true if i rises from .01 to .02, or from .20 to .21. Thus if n is 25, every point rise in i makes F rise by about 25%.

This is called "The $n\%$ Rule of Thumb," and it brings out one of the vital facts of economics, that small differences of i result in large differences in future values, and the size of the

differences increases with the futurity, as measured by n . (It is a rule "of thumb" because it is less than 100% accurate, owing to the approximation that $LNI = i$; and it needs serious adjusting when n gets above 50 or so).

The $n\%$ Rule gets especially potent in discounting, when you divide by the compounding factor (or multiply by the discount factor, which amounts to the same thing). Higher interest rates depress present values in proportion to n , the futurity of the values being discounted. Let's show this directly. This time we let $F=1$.

$$P = I - n \quad (1, B)$$

$$LNP = -nLNI = -ni$$

$$dLNP/dn = -i \quad (13)$$

$$dLNP/di = -n \quad (14)$$

(14) says that a 1-point rise of i reduces P by $n\%$.

Another way to compare the growth of money at different rates, i and j , is to take their ratio. Let $J=1+j$, and $K=1+k$.

$$I_n/J_n = [I/J]^n = K^n \quad (15)$$

$$k = I/J - 1 = \text{-----} \quad (16)$$

(15) shows that the ratio of the sums growing at different rates is not constant, but keeps growing at the rate k , which (16) shows to be approximately the difference between i and j .

An excellent way to visualize this is to plot the growth curves on a semi-log scale, with n on the abscissa and F on the ordinate. The origin of the scale is where $n=0$, but $F=1$. (You get the same effect by plotting the logs of the growth curves on a common scale, but here we just treat the semi-log scale). On such a scale, equal vertical distances represent equal percentage increases (from the bottom of each vertical line), and every curve of constant percentage growth, such as (1,A), is a straight line.

The slope of the line is the percentage growth rate, so that doubling the growth rate doubles the slope. Growth curves with different rates, starting from the common origin, diverge like rays as n rises. The vertical distance between them increases

with $\underline{n}$. This shows that their ratios change with $\underline{n}$ -- not just their absolute differences. The highest curve becomes an ever higher multiple of the lower curves.

Discounting on the scale is the mirror image of compounding. Instead of growing the curves "decay" exponentially, forming straight lines sloping downwards to the right, starting from the origin.

These decay curves show the effect of interest rates on present values. Higher interest rates selectively downvalue future values in proportion to their futurity. The higher is $\underline{n}$, the more sensitive values are to $\underline{i}$.

B. The PVF case

Starting with (3,A), let $a=1$ for generality. We know (from the alternative derivation of (3,A) on p.4) that when $\underline{n}$ rises by one, P rises by $I-n$, which we shall call P' . Now we solve for the fractional increase, P'/P :

$$P'/P = \text{-----} = \text{-----} \quad (18)$$

(18) shows that P'/P is a sharply decreasing function of $\underline{n}$, approaching zero as an asymptote. This underscores the futility of building in durability beyond 50 years or so, especially at high interest rates.

C. The IPF case

For generality, set $P=1$. Take the log of both sides, and differentiate.

$$\text{LNa} = \text{LNi} - \text{LN}(1-I-n)$$

$$a'/a = d\text{LNa}/dn = \text{-----} \quad (18)$$

D. Elasticities

Elasticity is another useful measure of sensitivity. It is the ratio of the log derivatives of the dependent and independent variables. We already have the first, and the second is a piece of cake, it is simply the reciprocal of the independent variable, in this case $\underline{n}$. Dividing by $1/\underline{n}$ is the same as multiplying by $\underline{n}$, and there you have elasticity, generally described as the ratio of the percentage changes of two variables.

We can translate what we already have into elasticities immediately. Let E denote elasticity.

E of F wrt $n = n_i$

E of F wrt $i = i_n$ (same thing)

E of P wrt $n = -n_i$ (PIPO case P is referred to here) E of P wrt $i = -i_n$ "

Let PV denote the present value of a flow, as in (3,A), to distinguish it from P , used for the PIPO case.

E of PV wrt $n = \text{-----}$ (19)

E of IPF wrt $n = \text{-----}$ (20)

The ratio seen in (19) and (20) pops up often in finance. You can know how it behaves by realizing that it is the ratio of simple interest to compound interest. That means the top grows nearly as fast as the bottom for a few years, but then the bottom races far ahead and the ratio falls swiftly.

The trouble with these elasticities (and the log derivatives, too) is that they are true only at one point. If the changes are large, e.g. if you double n from 15 to 30, these point elasticities understate the response. They are accurate for small changes, however; and always useful to put you in the right ballpark.

Those are the basic tools. The applications and variations are endless. If the tools seem hard to learn, remember poor Mr. Carter. You can learn the cheap way from me, or the expensive way from your creditors. There are lots of folks out there eager to help you learn by taking your money.

May The Force be with you. Keep away from The Dark Side.

Appendix II-8: Carter's Cut, or How to Lose Money

A friend of mine named Carter tried a short-cut to (21,B) which sounds reasonable, but lost him a lot of money -- and he never even knew it! He reasoned that the YTM is the sum of a/P + appreciation, so why not just figure each one separately and add them together? Here is what Carter figured:

$$i = a/P + [D/P]^{1/n} - 1 \quad (25)$$

Carter's Cut just adds together Equations (2) and (1,C). It works out pretty well if D/P is close to one. Otherwise it leads you gently towards poverty.

It led Carter to invest heavily in "deep-discount" bonds -- bonds with $P \ll D$. He would compare his figures from (25) with those published in the Wall Street Journal, which uses (21,A), and notice that his figures were higher than theirs. He thought they were understating the true YTM, and he had figured out how to beat the market. He's still happy thinking he did it, but he'd be happier if he had more money to retire with.

I'll say this for my friend, he had self-confidence: never occurred to him that maybe they were right, and he was wrong. I couldn't talk him out of it. But in fact, (25) is wrong. It overstates the true yield on discount bonds. The reason, in words, is that it credits the bond with a current yield of a/P for every one of n years. But in fact P rises every year, to reach its terminal level of D . So the current yield is less than a/P in every year after the first.

If the last three sentences above make your head swim, that is why mankind invented math, to put such ideas in systematic and irrefutable form, and that is why these notes contain so many equations. They are a lot better than working it out in your head every time. But if you really want to understand what finance is all about -- I mean really understand, as opposed to feeding formulas and grinding out rote solutions -- you have to keep rolling it over in your mind like that, until something comes together inside your own head, in your own way, that makes sense to you.

Most people never have time for that, or never think they have. So they follow trends and authorities and spend their lives as wage-slaves. But if you can make that time, and break through the mental blocks that hold most people down, who knows, maybe you will figure out how to beat the market. Better yet, you may

contribute something to the progress of mankind, which is much more gratifying.

End of sermon.

Getting back to Carter, (25) also understates the YTM on bonds selling "at a premium," i.e. with $P > D$. So it leads you away from good deals with premium bonds, and into bad deals with discount bonds.

It might seem that no real macro-economic issues hang on what kinds of bonds people buy, but they do. Because the kinds of real investments that borrowers make depends a lot on what kinds of bonds they can sell. So if a lot of bond buyers make Carter's mistake it would lead borrowers to finance more things that yield deferred cash, as opposed to fast cash. That slows down the turnover of real capital, as liquid capital turns to slush.

Turning from bonds to direct investments, a lot of direct investments resemble redeemable bonds in this way: they show a mediocre current yield, followed by a "pot of gold at the end of the rainbow," a hoped-for resale to some rich investor after a few years. There is a strong tendency to overvalue that future pot of gold, especially when the herd psychology is running that way, and a lot of bad investments occur as a result. Carter's mistake is just one example of a large genus.

Investments that are bad for the investor are also bad for society, especially if the capital gets frozen or destroyed. As Churchill said, "Capitalism is not a social evil because of its profits, but the lack of them." So this is both a selfish concern and a public one, too.

END OF CARTER'S CUT

Appendix II-4, Managerial, W92

Finding rates of return on investments after taxes

Notes by Mason Gaffney, 7 May 87; rev 15 Sept 91

Originally for use in Economics 100B

1. Simplest case

No redemption, no salvage, no land-value (there is always land, of course, you can't invest in outer space, but some land is so cheap you may treat it as free).

$$RR = a \frac{1 - I^{-n}}{P} \quad (1)$$

Trial-and-error method of solution: "Newton's Method."

$RR = a/P$ when $n = \text{infinity}$. Otherwise, $RR < a/P$

2. Redeemable bond

D = Redemption value

$$RR = a \frac{1 - I^{-n}}{P - DI^{-n}} \quad (2)$$

$RR = a/P$ when $D = P$.

What happens when $D > P$? Why?

What happens when $D < P$? Why?

3. Depreciable capital

(K) is depreciable capital.

RRAT is after income taxes at rate "t." No land-value; no resale.

$$RRAT = \frac{a(1-t)(1-I^{-n}) + t[P/x](1-I^{-x})}{P} \quad (3)$$

This assumes "straight-line" depreciation for tax purposes, meaning the investor can deduct the value of P in equal parts over x years.

4. Resale after n years

Tax is due on "capital gain," at rate "s."

F = Future sale proceeds, at time n. (This value consists of land value plus remaining value, if any, of depreciable capital at time of resale.)

s = effective tax rate on F - L, at time n. For simplicity we assume that all the value of depreciable capital has been written off in the right term of the numerator, i.e. that $x < n$. Note for sticklers. The way the tax law works, actually the same tax rate, t, is applied to capital gains as to ordinary income like "a." But only a fraction of the gain has to be reported as taxable income. The effect is the same as a lower tax rate, and for simplicity I set up the algebra with $s < t$.

q = land share of P. Thus, $Pq = L_0$; and $P(1-q) = K_0$.

$$RRAT = \frac{a(1-t)(1-I^{-n}) + t[P/x](1-q)(1-I^{-x})}{P - [F-s(F-Pq)]I^{-n}} \quad (4)$$

To simplify calculations:

- divide top and bottom by P
- rearrange terms in brackets in denominator

$$RRAT = \frac{a/P(1-t)(1-I^{-n}) + t/x(1-q)(1-I^{-x})}{1 - [F/P(1-s) + qs]I^{-n}} \quad (5)$$

5. Findings

Sensitivity of RRAT

These factors raise RRAT:

- Low x: weak effect
- Low n: strong effect
- Low q: strong effect, when x is short
- High F/P: strong effect, when n is short and q is low
- High a/P: moderate effect, if n and x are long
- Low t: moderate effect, if n and x are long; otherwise very weak
- Low s: strong effect, if n is short.

Appendix II-4, Managerial, W92

Finding rates of return on investments after taxes

Notes by Mason Gaffney, 7 May 87; rev 15 Sept 91

Originally for use in Economics 100B

1. Simplest case

No redemption, no salvage, no land-value (there is always land, of course, you can't invest in outer space, but some land is so cheap you may treat it as free).

$$RR = a \frac{1 - I^{-n}}{P} \quad (1)$$

Trial-and-error method of solution: "Newton's Method."

RR = a/P when n = infinity. Otherwise, RR < a/P

2. Redeemable bond

D = Redemption value

$$RR = a \frac{1 - I^{-n}}{P - DI^{-n}} \quad (2)$$

RR = a/P when D = P.

What happens when D > P? Why?

What happens when D < P? Why?

3. Depreciable capital

(K) is depreciable capital.

RRAT is after income taxes at rate "t." No land-value; no resale.

$$RRAT = \frac{a(1-t)(1-I^{-n}) + t[P/x](1-I^{-x})}{P} \quad (3)$$

This assumes "straight-line" depreciation for tax purposes, meaning the investor can deduct the value of P in equal parts over x years.

4. Resale after n years

Tax is due on "capital gain," at rate "s."

F = Future sale proceeds, at time n. (This value consists of land value plus remaining value, if any, of depreciable capital at time of resale.)

s = effective tax rate on F - L, at time n. For simplicity we assume that all the value of depreciable capital has been written off in the right term of the numerator, i.e. that $x < n$. Note for sticklers. The way the tax law works, actually the same tax rate, t, is applied to capital gains as to ordinary income like "a." But only a fraction of the gain has to be reported as taxable income. The effect is the same as a lower tax rate, and for simplicity I set up the algebra with $s < t$.

q = land share of P. Thus, $Pq = L_0$; and $P(1-q) = K_0$.

$$RRAT = \frac{a(1-t)(1-I^{-n}) + t[P/x](1-q)(1-I^{-x})}{P - [F-s(F-Pq)]I^{-n}} \quad (4)$$

To simplify calculations:

- divide top and bottom by P
- rearrange terms in brackets in denominator

$$RRAT = \frac{a/P(1-t)(1-I^{-n}) + t/x(1-q)(1-I^{-x})}{1 - [F/P(1-s) + qs]I^{-n}} \quad (5)$$

5. Findings

Sensitivity of RRAT

These factors raise RRAT:

- Low x: weak effect
- Low n: strong effect
- Low q: strong effect, when x is short
- High F/P: strong effect, when n is short and q is low
- High a/P: moderate effect, if n and x are long
- Low t: moderate effect, if n and x are long; otherwise very weak
- Low s: strong effect, if n is short.