

~~LP 6~~ LP 6: Determining Life, Replacement

PRODUCT OF

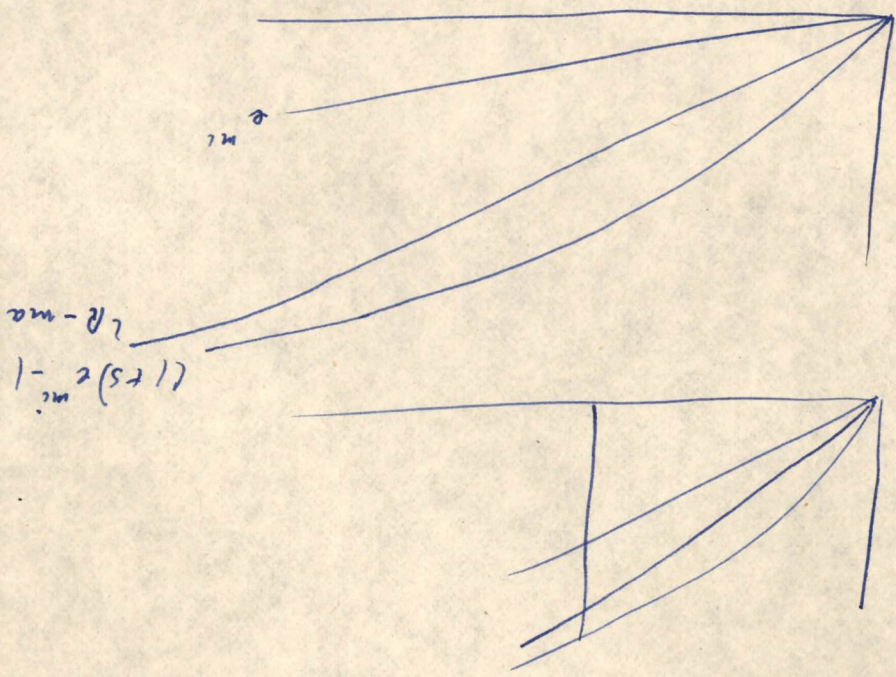
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Inquiries from J. R. Hicks.

Hicks, ch. 17, Interest & the Production Plan.

B-B's error. Took very exceptional examples, wine & trees, observed

a, r is a def. pd. of prod-n

b. Lower i lengthens it.

Generalized from these. But they not typical. - says Hicks

(But I still think lower $i \rightarrow \Delta$ inv. in more durable goods; & this is of utmost importance in cycles, & much in way Hayek said: late boom, lo i , $\rightarrow$ factor payments from inv. in durables not to come on mkt for long time.) (Hicks says as much - lower i changes 'tilt' of production plan, shifts output forward & input backward)

p. 226 Int. too weak to much influence near future; & risk dominates far future. So little role for i . - unless risk not great.

$$e^{mi} = \frac{R(1-t) + Ct + S}{C + S}$$

$$mi = \frac{\ln[R(1-t) + Ct + S]}{m} - \ln(C + S)$$

If $S=0$, ~~and S fixed~~

$$e^{mi} = \frac{R(1-t) + Pt}{P}$$

$$i = \frac{\ln[R(1-t) + Pt]}{m} - \ln(P)$$

~~If~~ If no tax

$$e^{mi} = \frac{R}{P}$$

$$mi = \ln R - \ln P$$

$$r = \frac{\ln R - \ln P}{m}$$

If $P=0$, $\ln P = -\infty$, and $r = \infty$

So let $P=1$. Then $\ln P=0$, and $r = \frac{\ln R}{m}$

If When $R = e^{mi}$, $r = \frac{mi}{m} = i$

Now let there be a yield tax, t .

$$e^{mr} = \frac{R(1-t)}{P}$$

$$mr = \frac{\ln R + \ln(1-t) - \ln P}{m}$$

Let $P=1$, and

$$r = \frac{\ln R + \ln(1-t)}{m}$$

$\ln(1-t) < 0$, so numerator growth is leveraged by it.

Now let $S > 0$

$$e^{mr} = \frac{R + S}{P + S}$$

Levy yield tax

$$e^{mr} = \frac{R(1-t) + S}{P + S}$$

$$mr = \ln[R(1-t) + S] - \ln(P + S)$$

The simplicity is gone.

Approaches to solving for maturity

1. ~~Max r~~
2. ~~$r + a = \text{rev.}$~~
3. ~~Max S~~

1. S fixed, r changes

a. Max r

b. Mgmt sol., $\text{rev}_m = i \cdot V_m + rS$
 - this may be hard when r is vble.

2. i fixed, S changes

a. Max S

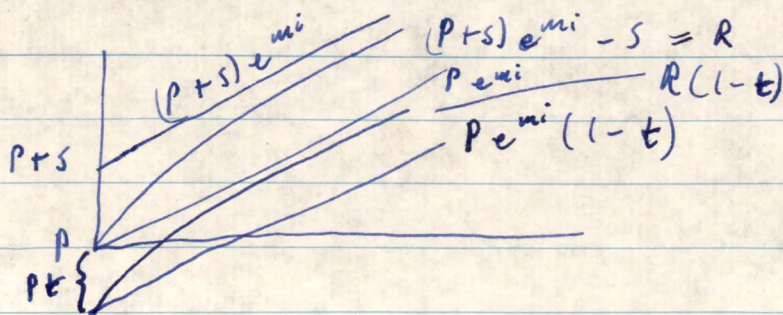
b. Equate $Si + V_m i = \text{Rev.}_m$

$$\text{When } V_m = (P + S)e^{mi} - S$$

And solve for S . But this is probably same as (a)

Task. ~~Max r where~~ Find m to max r where
 $R = (P + S)e^{mi} - S$; and then effect of taxes.

1. When $S = 0$, procedure is to begin w. locus of eq. points, and show that taxes make r & m when i const. with m .



$$(P+s)e^{mr} = R(1-t) + S$$

$$e^{mr} = \frac{R(1-t) + S}{P + S}$$

With sales tax, as above, r is less affected by m than if $s=0$.

That is, $\frac{dr}{dm} \propto \frac{1}{s}$

Working outline

a. Appreciating or ripening capital. P1PO. Bldg. block

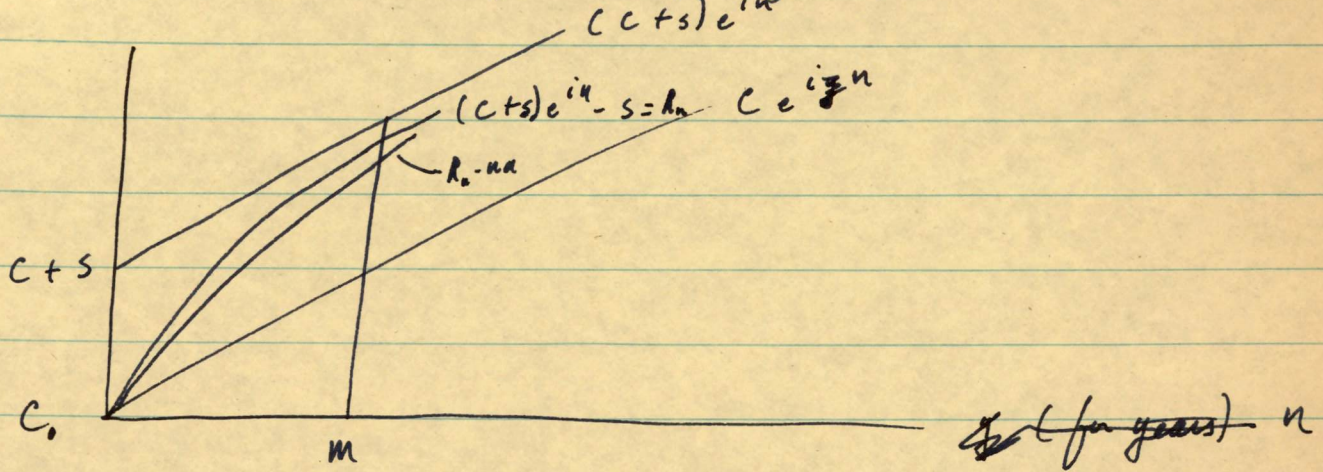


Fig. 1

If $S = 0$, 2 factors, shares are:

Cap. gets $\frac{C e^{im} - C}{C e^{im}} = 1 - e^{-im}$, Δ of m

Labor gets $\frac{C}{C e^{im}} = e^{-im}$, Δ of m .

This is basic.

Its generality has been challenged. ~~Also its~~ I will ~~also~~ generalize, to bldgs & mines.

It omits land. I will include land

For land, use L1LO. Refer to Fig.
~~the int'l. from Davis paper~~

2

if $s > 0$,

Labor gets $\frac{C}{(C+s)e^{im} - s}$, $-d_s F(m)$

Cap. gets & land get:

$$\frac{(C+s)e^{im} - s - C}{(C+s)e^{im} - s}$$

$$= 1 - \frac{C}{(C+s)e^{im} - s}, \text{ or } d_s F(m)$$

$$= \cancel{\frac{(C+s)e^{im} - s - C}{(C+s)e^{im} - s}}$$

Dividing between Cap. & land is tricky

Land gets ms , which is less than $(C+s)e^{im} - s - Ce^{im}$

$(= s(e^{im} - 1))$, bcs. part of what land gets above Ce^{im} is int.

on deferred rent.

So land share is $\frac{mSi}{(C+Si)e^{im} - S}$

are increasing & then
deriv of (m) ,

~~For~~ $\left[\text{Frat} \right]$ reaches max. when:

$$\frac{Si}{mSi} = \frac{(C+Si)ie^{im}}{(C+Si)e^{im} - S}$$

$$(C+Si)e^{im} - S = m(C+Si)ie^{im}$$

$$(C+Si)(e^{im} - mie^{im}) = S$$

$$e^{im}(C+Si)(1-mi) = S$$

$$e^{im}(1-mi) = \frac{S}{C+Si} \quad \left. \vphantom{\frac{S}{C+Si}} \right]$$

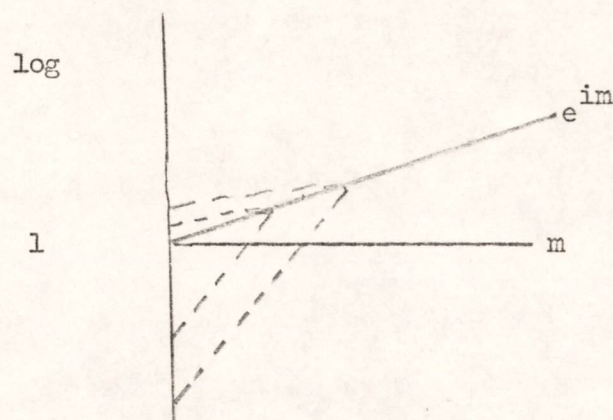
Cap. share is:

$$\frac{(C+Si)e^{im} - S - C - mSi}{(C+Si)e^{im} - S}$$

$$= 1 - \frac{C + mSi}{(C+Si)e^{im} - S}$$

6. Effect of changes in wage and interest rates on durability.

Graphical demonstration.

Figure 1

Let e^{im} be locus of future values having present value of unity at discount rate = i .

At a lower rate, $h < i$, present values from the locus curve are $\frac{e^{im}}{e^{hm}} = e^{(i-h)m}$, an increasing fn. of m ($i - h > 0$), shown by ascending y-intercepts above unity.

At a higher rate, $j > i$, present values from locus curve are $\frac{e^{im}}{e^{jm}} = e^{(i-j)m}$, a decreasing fn. of m ($i - j < 0$), shown by descending y-intercepts below unity.

So if capital is scarce, interest rates rise, causing shorter investments to look better. This speeds capital turnover, substituting labor for capital (cf 2, 3) and vice versa.

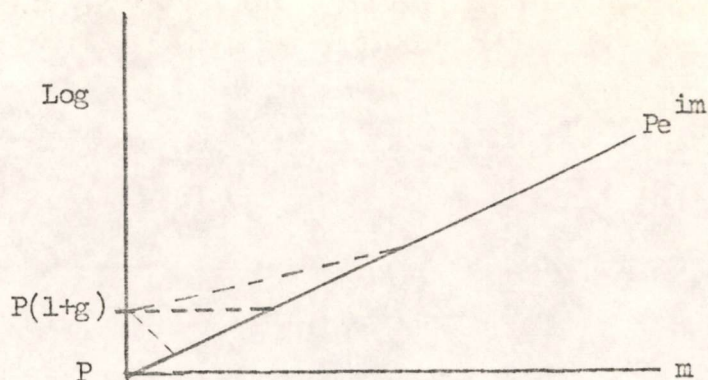


Figure 2

Figure 2 shows rise of wage rate with no change in value of product. Pe^{im} is locus of values yielding interest of i when payroll cost is P . Higher wage rates raise cost to $P(1+g)$. Dash lines show growth of investment at internal rate of return. Higher slopes mean higher rates.

Let $\underline{r}$ = internal rate

$$e^{rm} = \frac{Pe^{im}}{P(1+g)}$$

$$rm = im - \ln(1+g)$$

$$r = i - \frac{\ln(1+g)}{m}, \text{ an increasing fn. of } \underline{m}.$$

So higher wage rates cause longer lives to be preferred, thus substituting capital for labor (cf 2, 3). And of course vice versa.

Thus lower wage rates and higher interest rates lead to full employment, in a free market (if we had one).

But it is the wage and interest rates as perceived and experienced by the entrepreneur that are controlling. Here is where institutional bias intercedes. Labor costs employers much more than labor takes home in pay. But capital and land cost employers less, at least comparatively.

10-29-69, Approach to org. of R.F.F. work

Spatial, temporal, & macro-e. aspects of space networks (& env. enhancement)

I. New Malthusianism; new pessimism, about living standards, inv. outlets, employment.

Raw mtlb. no limit on pop, s.o.l., invt., empl.

As run short, new inv. opp. in exploring, mining, etc.

Env. quality, basis of new pessimism. Conservationists are threatening empl; & v.v.

There is analogous answer.

Want more resources in quality.

Do we have the resources?

We have the labor, by def. of the problem -- too many people. Is answer to fears of ~~too~~ unempl. by automation; & demilitarization.

Do we have the capl.?

Keynesian macro. models on which all nurtured no longer seem relevant. ~~have been misleading.~~ Always a great potential cap. formation.

Now, overcommitment. Hi i. Plenty of inv. opp. Shortage of disposable capl. Basis for current pessimism. (Like Barnett & Morse, capl. is the limit.)

~~But~~ This shortage can be absorbed, adapted to. Ec. has equilibrating thermostat -- will return to that. Problem is many factors clogging the mechanism. Diagnosis goal is to identify public inv. Tax pol.

there; policy goal to clear them away.

As integral part of policy goal, achieve env. quality w. least capt. input & retention. This involves ~~user~~ changes geared to economize. Spatial & temporal aspects. Will return.

II. Equilibrating mechanism.

A. Real terms

1. If labor is 5% & capt. 95%, capt. can't get any more. So investors avoid there when hi i.

These are durable products: houses

If labor gets 95% & capt. 5%, can raise capt.'s share w. min. decline of wage bill. Hi i pulls inv. into there.

These are products of quick payout: hamburgers.

Thus, hi i pull inv. into fast-pay capt; this solves problem of how to meet next payroll.

Low i pushes into slow-pay — solves problem that made Keynesian pessimistic, how to find inv. outlets.

2. A model, after Brems,

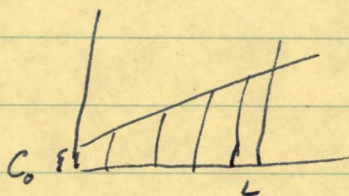
K = Capl. Stock

E = Empl.

W = Wage rate

L = Life

$$K = C_0 \cdot f(L)$$



$$C_0 = E \cdot W$$

$$E = \frac{K}{W \cdot f(L)}$$

To employ labor w. given stock of capl., reduce L .

3. This focuses study on replacement policy -- & includes orig. inv. policy that determines built-in life.

Will return, & expand.

Avoid biases to longevity, i.e. capl.-intensity.

Esply. in interstate sector, ^{over}unbalanced, --

B. Translate macro model to usual terms, centered on spending.

Replmt. freq. is autonomous determinant of gross inv. Keynesian model overlooks, regards r.f. as pliable, determined by spending. Net inv. & prop. to cons. have great leverage.

But r.f. is autonomous. Cons. isn't. Replacement need dominates gross inv.

Capl. owners invest to replace stock as it matures. This gives consumers inc. to buy it. Spending nicely synch. w. output.

Micro & macro integrated. Faster propns. learn on capl. mean faster turnover, replacement. Means more gross inv. More nat. inc. w. given stock of capl.

As we can equilibrate, get along w. cap. stock we have.

B. ~~Spending terms~~

~~Investor in short-lived goods cashes in fast. & spends M to replace. Thus buys the goods as sold.~~

~~Investor in long goods has no need to replace. The money ~~gets~~ he invested gets spent on cons., but it's not real cons., only inflationary.~~

~~Long investor adds in. to spending, but no goods emerge. Infl.~~

~~- P.W. programs ^{can be} ~~are~~ bad offenders. Watch it, when spending to enhance environment.~~

So we can equilibrate. But is this faster turnover compatible w. environmental quality? That's problem.

Identity problem?

- N-5 -

III. Do we have the land?

We can ^{people} use land more intensively. Or stop peopling it less intensively — reverse trends of past decades. Note irony — have feeling of more crowding, but the reverse is occurring, w. rural depopn. and urban sprawl.

This is quite compatible w. turning capl. faster. ~~Capl~~ Long-term capl. ties up land as well as ~~money~~ capl. Labor applied less often.

e.g. renewal of urban sites

e.g. forest.

e.g. replacement of sewers

In terms of service flow, Co of \$/m. designed to be recovered in 10 yrs. reqs. more intensive land use than \$/m. to be recovered in 20; etc.

To people land more intensively, no force required. Hence no reduction in S.O.L. Merely remove biases to sprawl, (l to cap.-intensity & size in agric.) ^{or forced land-consumption.} Is this compatible w. env. quality?

Argument for sprawl is, ~~not~~ reduce density, avoid congestion - pollution. Evaluate this. Is my research job.

This entails working out system of user charges, see how they affect density.

IV. Spatial & Temporal aspects of interstate sector.

Study there, basis for recommending criteria for set of user charges, effluent charges, congestion charges, & assoc. tax policies, to use ~~cap~~ source cap. & land ~~of~~ and enhance environment. (Sewer & transit are central to ques. of wt. polln. & air polln.)

A. Temporal

^{m.f.s. in interstate sector, and land}
These can save great cap. ~~by~~ by replacing indiv. facilities. (& effluent charges encourage that.) But can waste cap. ^{and land} when extended into lean territory prematurely; when left unfinished for years.

The latter has been common. So timing of the exercise.

1. When to initiate

a. Problem of 'ripening,' or progressive obsolescence.

(Like when to open a mine.)

~~b. Prob~~

b. Problem of replacing old i.f.s.

Decid. cost mass facility replacg. Δg cost i.f.

i. If all costs are ~~known~~ given by what.

— If i.f.s. already in

— If i.f.s. not in

In general, contrast of Δg & $-\Delta g$ cost gives enormous advantage to $-\Delta g$ cost or mass facility. Details to be worked out.

~~the~~

ii. Effect of user charges, - on both systems

- On i.f.s. Is main thing making cost rise. Wells, septic tanks, autos. Pvt. costs fall w. scale. Social costs rise.

So user charges ~~down~~ shift balance, encourage replacement of i.f.s. by mass facilities, m.f.s.

Usual use of user charges is badly misdirected. They are typically imposed on m.f.s.: buses, phones, mail, rails, utilities.

- On mass facilities, m.f.s.

$MC < AC$, charges low

But m.f.s. cover wide area, require area differentiation. At fringes, this hurts them.

Also where density is low.

So new criteria won't always & everywhere help m.f.s.

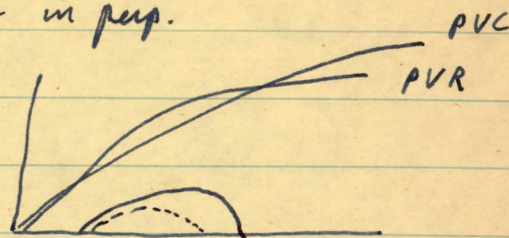
In lean areas, will help i.f.s. Net result: sharper line between city & country. ~~And save~~ Avoid long, half-finished, underused lines. Save capital.

2. How much life to build in?

"Life" ~~add by~~ Pure longevity, as w. choice of intrb. quality.
Size -- a vital element of life.

Trade-offs, max PVR - PVC in perp.

Single horizon case.



Convert to perp. case, mult. by $\frac{1}{1 - (1+i)^{-L}}$

This shortens opt. life. Moves max. to left.

Integrate this w. replacement theory; and ~~the~~ initiation theory (ripening).

In general, argues for shorter life. Argues against deferred initiation, ~~bes. obsolescence~~. Don't know yet.

3. Replacement policy under desig. costs.

This duplicates problem of initiation policy when i.f.s. are consigned.

4. Level ^{& structure} of use changes (temporal)
Peak load
Change over lifetime.

5. ~~Coord.~~ ^{Asynch.} of intersite & on-site inv.
Role of land taxes.
Initiation policy (ripening) again.

6. Evaluation of present policies in terms of criteria established

Biases to low density.

Zoning; bldg. tax; ^{free} premature extensions.

Biases to ^{slow payout} longevity, (capital intensity)

Too much lean territory

Tax policy

Regulatory policy -- bldg. rate base

IV, B.

~~The~~ Spatial aspects.

Does hi density, ^{or intensive land use} lower quality of env.? To some people, these are synonymous.

To the extent that it does, user charges will rifle in on the causes. E.g. cars downtown.

But this is good e.g. of where rifling on specific cause of congr. allows greater density. Reserve streets for mass transit.

The shotgun approach used now may lower density

There are Trade-offs. Density makes some things worse; other things better. Net balance is what? Find criteria. Is probly. l.v. p.s.f. This rises w. density.

At worst, dense areas pull people from others, lower density there.

I think I see, ~~basic tendency~~ and am trying to formulate, basic principle that intensive land use is compatible with, & in some basic sense integral to, high quality of env.

— cost of env. quality is fu. of area (area-sensitivity).

Servicing land: sewers, flood control, pest control, fire control.

The value of the services rises w. the no. of people served.

- labor-saving processes are often land-using, & environment-using
- modern farming
 - subst. of fuel power for muscle power
 - littering

"Intensive" use means using little land & environment per man.

- use better land intensively, spare invasion of ungrd. land.
Positive containment.

Developing ungrd. land wastes both capt. & land.

To study intensive sector. Effects are location, density.
Set of user charges & taxes to optimize.

No need to make work, ~~force~~ ~~use~~ of substn. of labor
for land & capt. But remove biases now forcing the reverse.

That we have biases is clear from corruption of
"productivity" & "efficiency". $\equiv$ AP of labor. Betrays a
whole style of thinking & doing grow's around idea labor
is only cost.

So we min. labor/output. This entails max. cap. & land
/output.

Require hi complement of land & cap./man. If
simul. require full empl., press on limits of res.

15.
2. How to change factor mix, sub. labor for land & cap.

Apply larger inputs of labor; more often.

a. Larger inputs of labor

Higher intensity of land use

This is fairly obvious, & I won't labor it, until get to policy.

~~But~~

Also, there is a counter argument, that requires cap. & model of cap. substn. The counter arg. is, low intensity ~~means~~ creates only a partial scarcity, can be offset by extending p.w., spreading over more land.

To answer this counterargument, need model of substn.

betw. cap. & labor. Bcs. spreading out Δ s

cap./labor, & that is why it doesn't work.

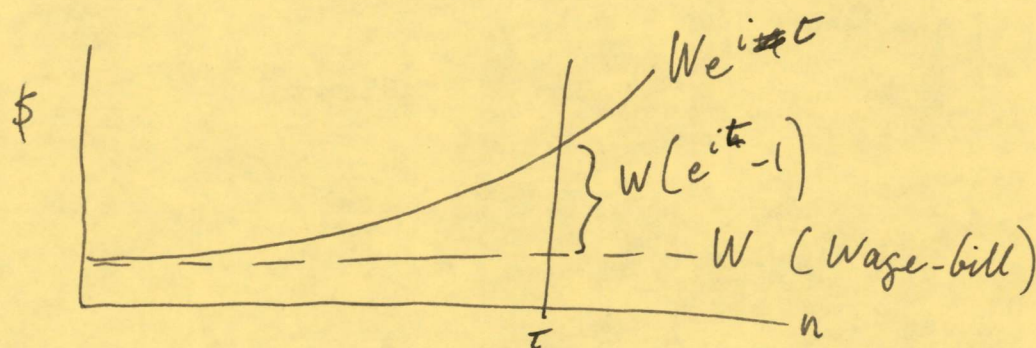
b. More frequent inputs of labor, or how to sub. labor for capt.

To Δ labor share, reduce life of capt.

Why not just replace machines by men; (or superhighways by more driver time?)
 Bes. machines & highways are produced by labor. Share of capt. in value of output is ~~fun~~ of excess of service or cash flow over labor cost, & this is fun. of duration of life.

PIPO model.

Labor share falls w. life.

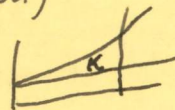


$$\text{Labor share} = \frac{W}{We^{it\tau}} = e^{-it\tau}$$

$$\text{Cap. share} = \frac{W(e^{it\tau} - 1)}{We^{it\tau}} = 1 - e^{-it\tau}$$

Staggering (normalizing, going concern)

Entire sequence exists at once.



cap. intensity indep. of life.

Some think this changes things, ^{cap. intensity indep. of life.} John Robinson; Lutz; many forget.

It doesn't. ^{One of most} fundamental propositions of cap. theory

If normalize, put all on yly. basis:

$$\text{Output} = We^{it}$$

$$\text{Labor} = W$$

$$K = \sum_{n=0}^{\infty} We^{in}$$

To save steps, note this calls for formula for sum of annuity.

To avoid integral, substitute

$$K = W \frac{e^{it} - 1}{i}$$

$$e^{i\Delta t} = (1 + j)$$

$$K = \sum_{n=0}^{\infty} W(1+j)^n = W \frac{1 - (1+j)^t}{1 - (1+j)}$$

$$\text{But } e^{i\Delta t} \approx 1 + j$$

$$K = W \frac{1 - (1+j)^t}{1 - 1 - j} = W \frac{(1+j)^t - 1}{j}$$

To save time, will be sloppy.

$$\therefore K = W \frac{e^{it} - 1}{i} \quad (\text{Formula for annuity})$$

$$\text{Int. bill, yearly} = Ki = W(e^{it} - 1) \quad \text{as before}$$

$$\text{Labor share} = \frac{W}{We^{it}} = e^{-it} \quad \text{as before. (in 1 yr. down to } \frac{1}{2} \text{)}$$

$$\text{Cap. share} = \frac{W(e^{it} - 1)}{We^{it}} = 1 - e^{-it} \quad \text{as before}$$

If add land to model, its share also rises with t . 

But not so much. (This is peculiar to PIPD. With bldgs, land share is more.)

Read nos. from App. I.

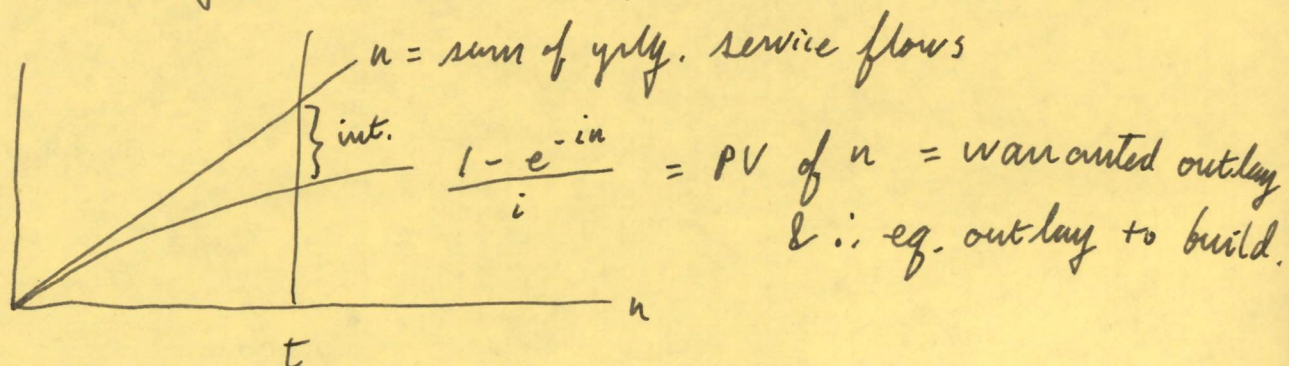
$\therefore$ To sub. labor for cap. & land, shorten life.

Deprec. model

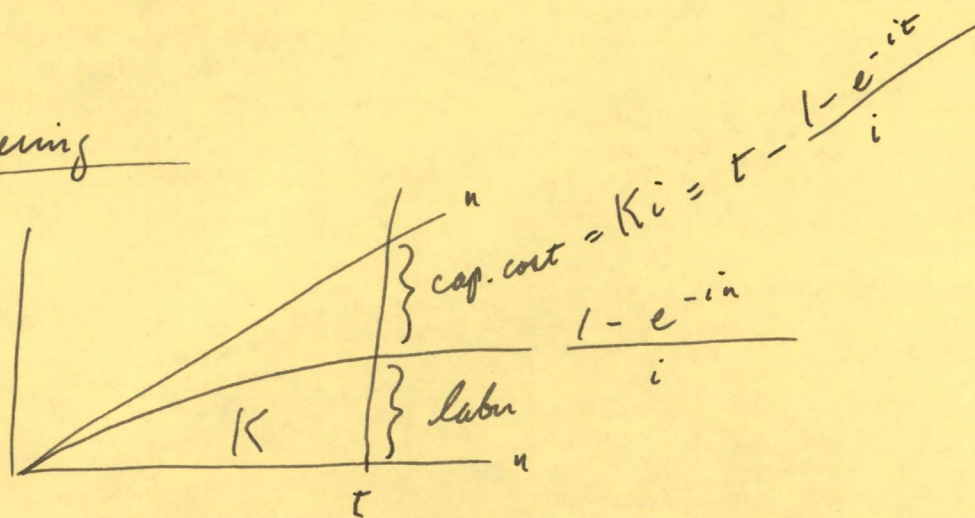
Can we generalize from Trees? Is BR's "agio" universal? It must be, bcs. PIPD is ^{math.} a bldg. block from which all cap. thy. comes.

Single house.

Lasts t years. Yields const. flow of \$1./yr.



Staggering



Annual flow = t

Labor input = $\frac{1-e^{-it}}{i}$ (Build one each year)

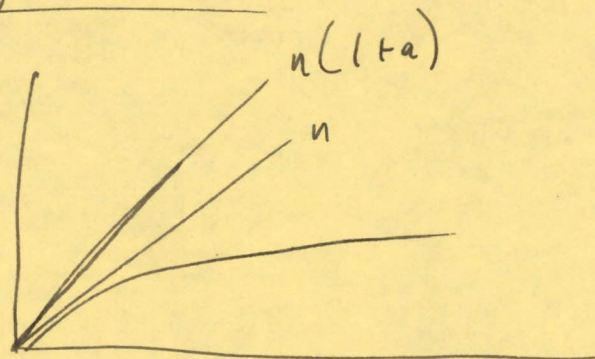
Cap. cost n ^{as residual} = $t - \frac{1-e^{-it}}{i}$

Directly

$$K = \sum_{n=0}^t V_n = \sum_0^t \frac{1-e^{-it}}{i} = \frac{1}{i} \left[t - \sum_0^t e^{-in} \right] = \frac{1}{i} \left[t - \frac{1-e^{-it}}{i} \right]$$

$$Ki = t - \frac{1-e^{-it}}{i}$$

Adding site rent



Here, share of land rises.

Allude to macro. thinking

By substituting labor for cap. & land we speed replacement. This Δ s gross inv. -- demand for labor. Also Δ s gross volume of output from given capt. -- supply of final goods.

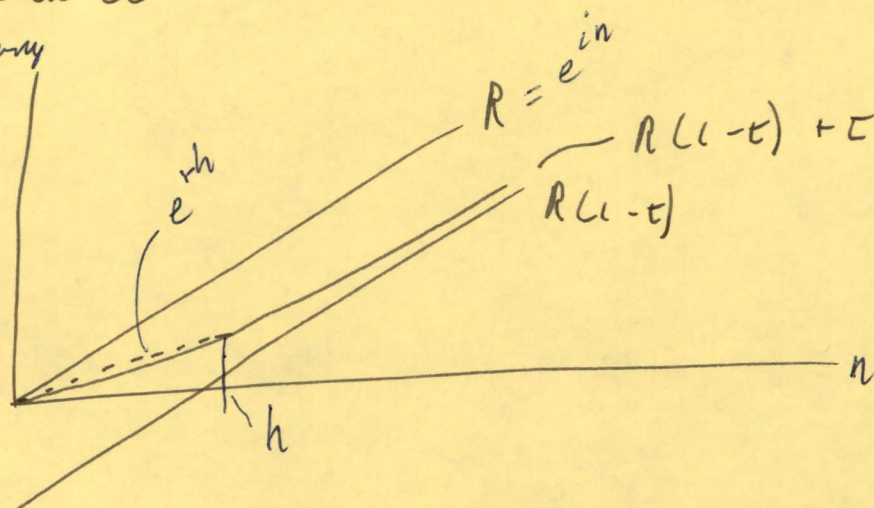
Thus Δ aggr. demand & aggr. supply is balanced Δ s.

B. Effect of Taxes.

P100 case only.

1. ~~A~~ Closed ec.

A. Theory



$$e^{rh} = e^{ih}$$

$$e^{rh} = e^{ih} (1-t) + t$$

Graphical proof.

Examples from tables

General proof on request

PV approach.

$$PVR = e^{in} (1-t) e^{-rn} + t e^{-rn}$$

$$= (1-t) e^{(i-r)n} + t e^{-rn}$$

B. Other factors. See p. 1, Clemson notes.

2. Open ec.

Here we must introduce land.

$$\text{Land value, } S, = \frac{R(h) - We^{ih}}{e^{ih} - 1} \quad (1)$$

Why "-1" ? hear you cry?

$$e^{ih} = \frac{S+R}{S+W} \quad \text{solve for } S. \quad (2)$$

$$S(e^{ih} - 1) = R - We^{ih}$$

Now take tax.

$$\pi \equiv S \text{ after inc. tax.} \quad (3)$$

$$\pi = \frac{R(1-t) - We^{ih} + We}{e^{ih} - 1} = \frac{We^{ih} - tWe^{ih} - We^{ih} + We}{e^{ih} - 1}$$

$$\begin{aligned} \text{But } R &= S(e^{ih} - 1) + We^{ih} && \text{from (2)} \\ &= (S+W)e^{ih} - S \end{aligned}$$

$$\begin{aligned} \pi &= \frac{\cancel{We^{ih}}(1-t) + \cancel{We^{ih}} + We + S(e^{ih} - 1)(1-t)}{e^{ih} - 1} = S(1-t) - We \\ &= S - t(S+W) \end{aligned}$$

$$\frac{S - \pi}{S} = t \frac{S+W}{S}$$

$$\frac{S-\pi}{S} > t \quad \text{and increases with } \frac{W}{S}$$

Marginal land sterilized.

Intensive margin also sterilized.

Rival uses for land reentered: less intensive ones look better

In forestry, this means use less replanting labor.

So either way, inc. tax is biased vs. labor.

So are other ~~taxes~~ aspects of inc. tax.

So are other taxes.

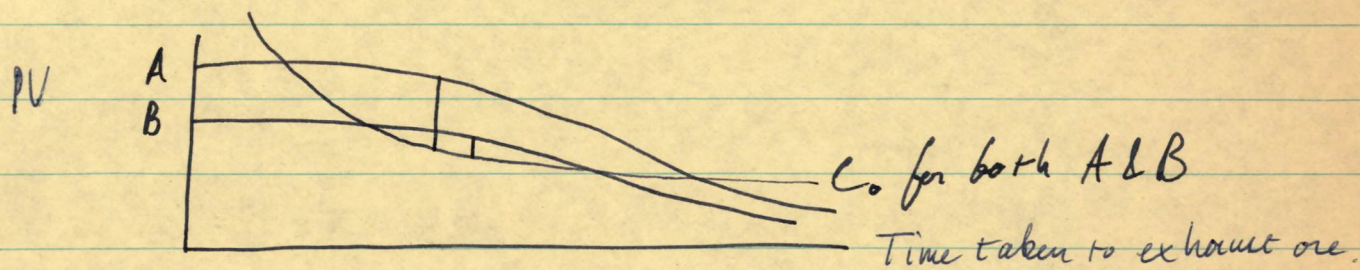
So are non-tax instns.

To achieve full empl., need to change those instns.

Re Mines

1. Division of product
2. Mgt. land ties up capt.
3. Criteria for Timing life.

With mines, capt. used to develop rich mines is recovered faster bcs. the ore's value adds urgency.



A is rich mine; B is ungt.

$a \equiv$ PV of lode A
 $b \equiv$ B

$$a = \frac{A}{L} \frac{1 - e^{-iL}}{i}$$

$$b = \frac{B}{L} \frac{1 - e^{-iL}}{i}$$

(assuming equal lives)

$$A > B \quad ; \quad \text{but} \quad \frac{1 - e^{-iL}}{Li} \rightarrow 0 \text{ as } L \rightarrow \infty$$

so they converge. So opt. life of richer mine is shorter.

In addition, ungt. mine will apprec. by higher % in future. So if capt. input comes in several steps, will allow longer time between steps.

This is akin to the principle that hi l.v.
cause earlier demer., to salvage site.

~~Probably~~ There is an algebraic solution.

Opt. L is one that maximizes:

$$PV = A \frac{1 - e^{-iL}}{iL} - \cancel{\frac{K}{L}} \frac{K}{L} \quad \text{roughly.}$$

$$= \frac{1}{L} \left[A \frac{1 - e^{-iL}}{i} - K \right]$$

(1.) By inspection, the higher is A , the less the leverage of K , and the sooner a max. is reached.

(2.) By differentiation:

$$PV \text{ max when } v \frac{dv}{dL} = u \frac{du}{dL}$$

$$L \frac{A}{i} (0 - -ie^{-iL}) = \left(A \frac{1 - e^{-iL}}{i} - K \right)$$

$$LAe^{-iL} = \frac{A}{i} (1 - e^{-iL}) - K$$

$$LAe^{-iL} + \frac{A}{i}e^{-iL} = \frac{A}{i} - K$$

~~$$Ae^{-iL}(L + \frac{1}{i}) = \frac{A}{i} - K$$~~

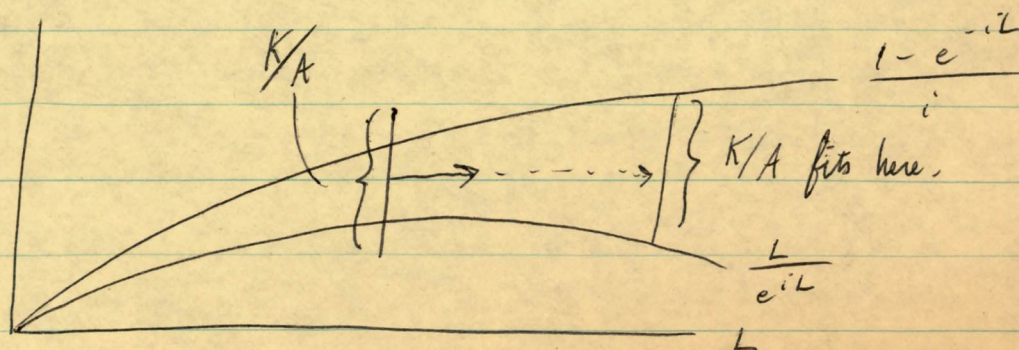
$$Le^{-iL} + \frac{e^{-iL} - 1}{i} = \frac{-K}{A}$$

$$e^{-iL}(L + \frac{1}{i}) - \frac{1}{i} = -\frac{K}{A}$$

$$\frac{L + \frac{1}{i}}{e^{iL}} = \frac{1}{i} - \frac{K}{A}$$

$$\frac{K}{A} = \frac{1}{i} - \frac{L + \frac{1}{i}}{e^{iL}} = \frac{1}{i}(1 - e^{-iL}) - \frac{L}{e^{iL}}$$

$$\rightarrow \frac{K}{A} = \frac{1 - e^{-iL}}{i} - \frac{L}{e^{iL}} \quad \text{Final, operable form.}$$



The smaller is K/A , the earlier it fits.

Opt. life of mine: $\frac{K}{A} = \frac{1 - e^{-iL}}{i} - L e^{-iL}$

Tabulate the rt. side; then find opt. from value of left.

Tabulate the rt. side; then find opt. from value of left.
Rt. side is Δg fn. of L , ^{after a while} so if A is larger, L or K smaller, the sooner rt. side reaches K/A and opt. life.

L	(1) e^{-iL}	(2) $L e^{-iL}$	(3) $\frac{1 - e^{-iL}}{i}$	(4) $(3) - (2)$
5	.71	3.55	4.1	0.55
10	.51	5.1	7.0	1.9
15	.36	5.4	9.1	3.7
20	.26	5.2	10.6	5.4
25				
30	.13	3.9	12.4	8.5
35				
40	.067	2.68	13.3	10.62
45				
50	.034	1.70	13.8	12.10

5
So what are reasonable values for A & K ?

$\frac{K}{A}$ cannot exceed $\frac{1}{i}$ as an upper, upper limit.

K shd. be taken as $15 \times K_{15}$, where K_{15} is the capacity needed to remove all in 15 years. (Bes. when you get to removal in one year, the simple assumptions probably are wrong at the extreme.)

Any value under $\frac{1}{i}$ is thinkable; but values ~~near~~ under one are hardly credible. In between, a wide range is likely. If it's $\frac{K}{A} = 6$, then

opt. $L =$ about 21.

A marg. mine is one where $K/A = \frac{1}{i}$, and $L = \infty$. Then $PV = \frac{A}{L} \frac{1 - e^{-iL}}{i} - \frac{K}{L} = 0$.

That says you don't mine marg. mines at all, which

seems extreme. Here we reach a reductio bcs. of some

assumptions that break down at the extremes. They could

be adjusted. Meantime, main point is clear: you mine rich ones

6
faster. Mgd. ores, no hurry at all.

In practice, a small possibility of higher future values of ore is enough to justify non-use.

On the mgd. mine, you use little $C_0 = \frac{K}{L}$, bcs.

L is so large. But opn. is capl.-mining bcs. you recover

C_0 so slowly. Rich mine, you apply higher C_0 ,

but recover it much faster. L and gets more too.

So rich mines are labor and land-mining. Mgd. mines are cap. mining.

Precisely, how is product divided? I need a mine model to supplement my tree & house models.

Annual basis is possible route.

$$a = \left[\frac{A}{m} \frac{1 - e^{-im}}{i} - \frac{K}{m} \right] \cdot \frac{i}{1 - e^{-im}} = \frac{A}{m} - \frac{K}{m} \frac{i}{1 - e^{-im}}$$

~~Easy~~ Undiscounted $\sum a = ma = A - K \frac{i}{1 - e^{-im}} = \sum \text{all rents}$

$$\text{Wage} = \frac{K}{m}$$

$$\text{Int., as residual} = A - \frac{K}{m} - ma$$

$$= A - \frac{K}{m} - A + K \frac{i}{1 - e^{-im}} = K \left[\frac{i}{1 - e^{-im}} - \frac{1}{m} \right]$$

Recap:

$$\text{Wages} = \frac{K}{m}$$

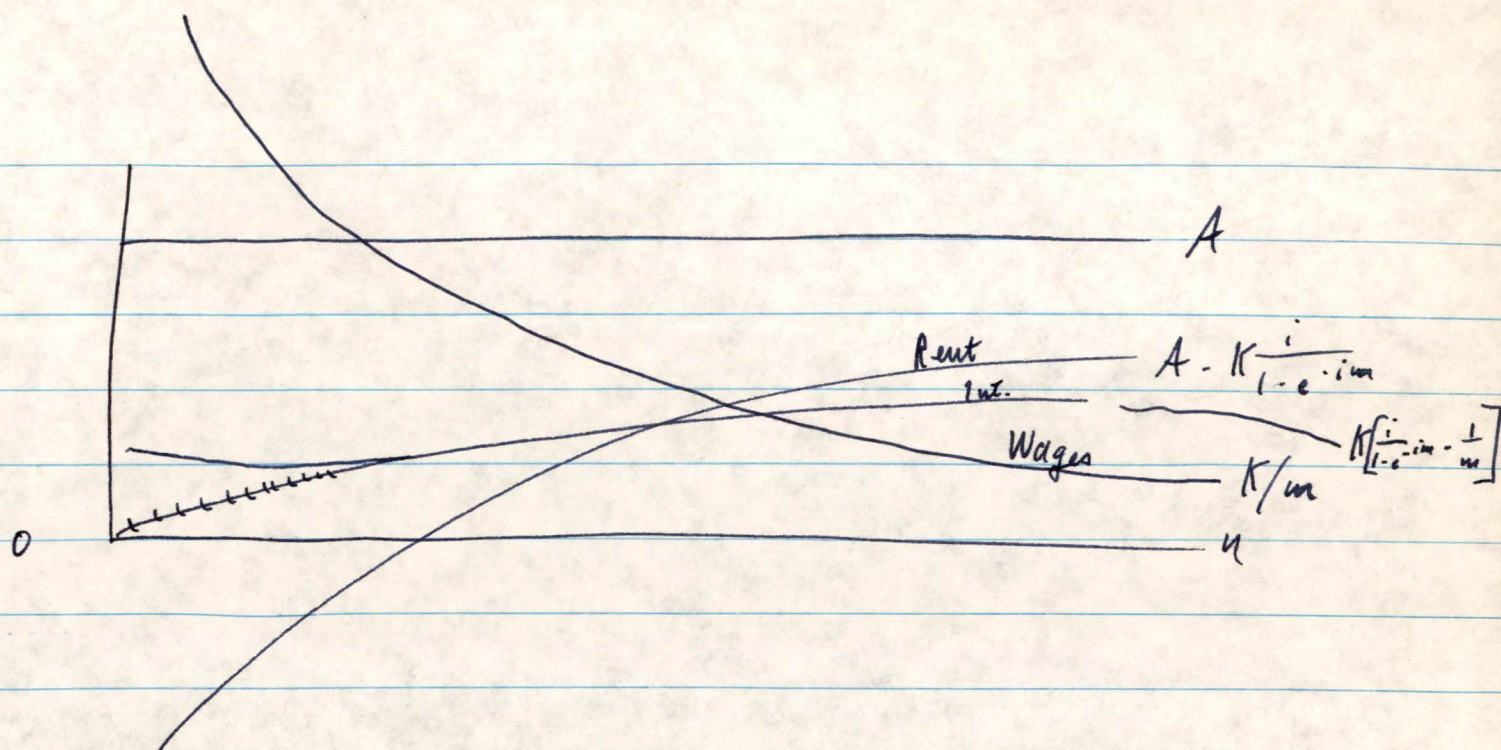
$$\text{Rent} = A - K \frac{i}{1 - e^{-im}}$$

$$\text{Int.} = K \left[\frac{i}{1 - e^{-im}} - \frac{1}{m} \right]$$

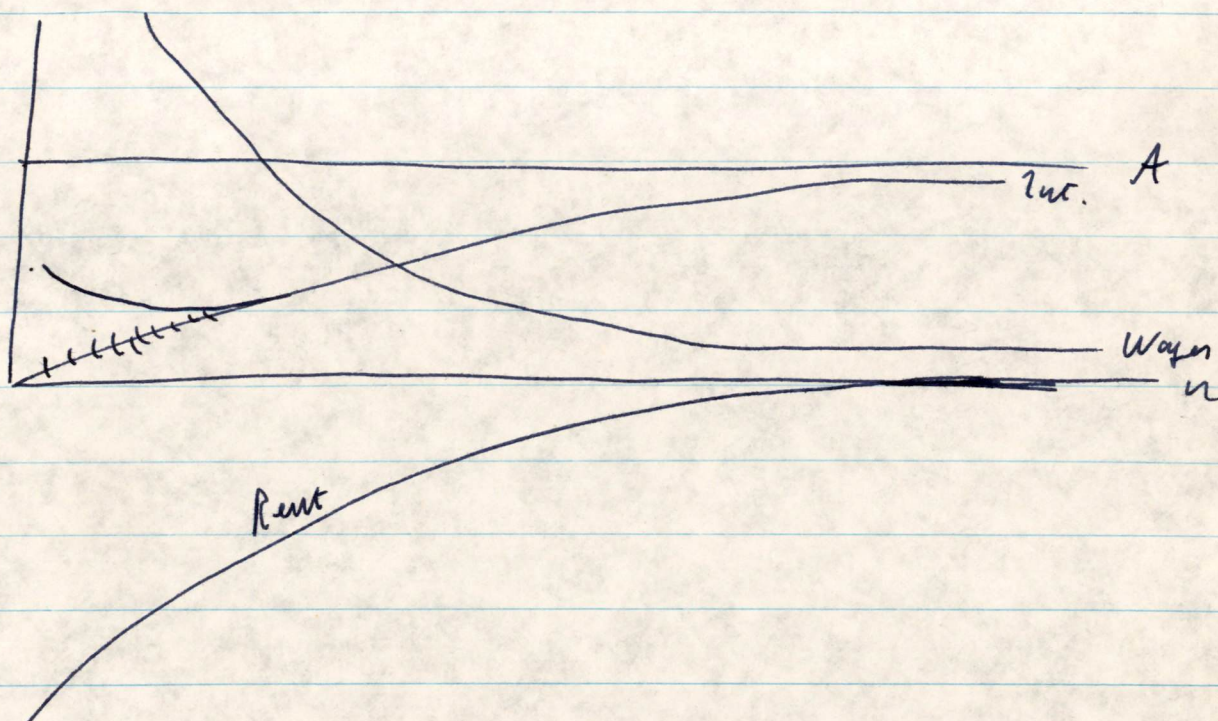
a decr. fu. of m .

an inc. fu. of m

an inc. fu. of m



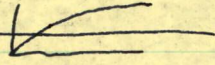
Or, for equal mine:



[The fact that int. is high in earliest years is of no moment, bec. they are hypothetical, since rent < 0 .]

[Prodn. will be timed so rent is max., barring biases.]

Cap. thg.

1. Valuation
2. Staggering, & capt: output
 - a. Set of Trees
 - b. ~~Set of mines~~ 
3. Sensitivity to Δi , comparative
4. Planning life to max. p.v.
 - a. Trees
 - b. Houses
 - c. Mines
5. Replacement
6. Deprecn.
7. Taxn.

Valuation

$$V. \text{ of annuity} = \frac{1 - (1+i)^{-L}}{i}$$

V. of periodic income, or irregular inc., in perpetuity

a. Convert to annuity, $\frac{i}{1 - (1+i)^{-L}} \cdot \sum_1^L R_n (1+i)^{-n}$

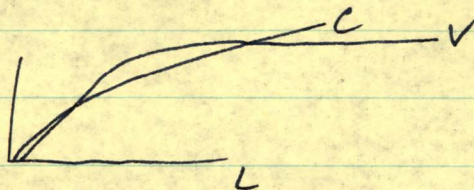
b. Capze annuity in perp., ~~if approp.~~

c. Derive as $S = P [1 + (1+i)^{-n} + (1+i)^{-2n} + \dots]$

$$= \cancel{(1+i)^{-n}} \frac{P}{1 - (1+i)^{-n}}$$

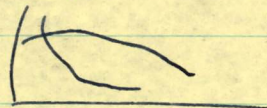
Planning Life to max. V

Use house as e.g.



~~But don't~~ But don't want to max. net value of ~~one~~ site for just one use. So max. in perp.

Convert to annuity -- show spreading cost over time



Max. annuity: capze.

Note how you max. ~~ann~~ by equating % changes of num. & denom.

Note how max. S is same as inqul. condition where $R_n = a$

Case of appreciating land. Get $S = P \frac{1}{1 - \left(\frac{1+r}{1+i}\right)^n}$

Case of mines

1st set

1. It's known that corpors plow back & defer dividends, owner benefits from cap. gains.

The price is paying corp. inc. tax. But why even pay that? Let corpor. invest in assets that appreciate w/ time. Double the benefit.

2. In terms of emplmt, nation's empl. is divided into busy and lazy.

Busy empl. is in small bus.

Lazy large bus. Plow back, hi cap/man.

So, to Δ empl., shift empl. to smaller bus.

If one wanted to look at above in a 'so what' way, he would say the large firm turns empl. slower, but that makes the small turn faster, even out. But using above equinquid. approach, makes it clear it doesn't even out.

3. Bldg. tax reduces durability, but Δ s longevity.

Thus acts w. ~~part~~ double force to Δ land requirements.

(also reduces initial intensity -- triple force.)

4. Nation must attract empl.; and ex. on empl.

5. If only we cd. recover empl. buried in all the p.w. projects undertaken 10 yrs. ago -- at 3% rate of return. How we cd. use it

today, at 10%.

6. Nat. weakness for overcommitment. U.S. strength in work under contract. U.S.? Whole world, using U.S. Capl.

7. Cite specific authors; works: Brems, Benfman, etc.

8. "Free capl." is liquescent.

9. "Stretch out" programs.

10. In simplest tree case, $\text{capl: output} = \frac{(1+i)^L - 1}{i} = \frac{1 - (1+i)^{-L}}{i}$

If people save a lot, some have worried about what happens to capl. later. "Ultimately", it must be consumed.

But capl. is a storehouse of ~~un~~ limitless capacity. We can go on adding to capl. stock forever. As L rises and i falls, $C:0 \rightarrow \infty$.

But today, they shd. be worrying about the opposite problem -- how to get ^{unripe} capl. out of the warehouse!

11. Wickell shows, p. 176, that one year's interest on total stock of staggered wine equals n yrs' int. on C_0 .

This is corollary of fact that ^{value of total staggered} stock at any time equals sum of its parts, and these parts consist of one unit at each age. One yr's int. on Σ is on Σ of staggered components; and n yrs' int. on C_0 is same, only in sequence.

12. Re my my example. The Kernans costing $10\times$ as much as the Acilans, would normally last more than $10\times$ longer, if of same quality.

On other hand, But cap. recovery wd. be slower. Using assum. of const. annual service, proper deprecn. method is sinking fund.

In a staggered model, where the mean age is the half-life, the Kernans at the half-life will hardly be deprecn. at all.

So the ~~net~~ base of int. will be nearer to ΣC_0 than $\Sigma \frac{C_0}{2}$.

With the Acilans, on other hand, it is the latter.

Balancing out the above two, cap/acre does Δ sharply w. longevity, as in the simple model.

13. Mining, & R/O ratio, can be introduced, using concepts from GRT

14. Bruns, Output, Empl, Capl & Growth. pp. 208-9.

Staggering (simple interest model, from BB)

Lower wages mean higher r_f and higher empl.

Bruns' eqn. for showing this is useful, but instead of C_0 he uses wage bill. Since he uses simple interest, I can use comp. int. and my eqn. is different (fn. to him)

Capt. stock for each man is $\frac{\Sigma}{C_0}$

For $C_0 = 1$

So capt. stock: man = $\frac{(1+i)^L - 1}{i}$ ~~man~~ This is for appreej asset.

For appreej. economy, $K \equiv$ appreej. capt. stock
 $P \equiv \frac{\text{Man. hours or Man days.}}{\text{pop. of working men} \times \text{hrs. of work per yr.}}$
 $W \equiv$ Wage rate

Now, $C_0 = PW$

$K = PW \frac{(1+i)^L - 1}{i}$; $P = \frac{K}{W} \frac{i}{(1+i)^L - 1}$. To max. P, min. W & L

If unempl, W falls, Δ empl. 2 ways.

a) $P \propto \frac{1}{W}$

b) Shorter L, and $P \propto \frac{1}{L}$

Once we have the basic model, can adapt it to assets w. different cash flow patterns.

a. Const. annuities.

$V = \frac{1 - (1+i)^{-(L-x)}}{i}$

where L is full life and x is yrs. of life elapsed.

$K = \Sigma = \frac{1 - (1+i)^{-(L-0)}}{i} + \frac{1 - (1+i)^{-(L-1)}}{i} + \dots + \frac{1 - (1+i)^{-(L-L+1)}}{i}$

$= \frac{1}{i} \times L - \frac{1}{i} \left[(1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-L} \right] = \text{(next page)}$

$$\frac{L}{i} - \frac{1}{i} \frac{1 - (1+i)^{-L}}{i} = \frac{1}{i} \left[L - \frac{1 - (1+i)^{-L}}{i} \right]$$

Where there is land rent, above flow to capl., solve by i) Deduct o.c. rent first, treat capl. as residual; ii) Deprec. capl. by sinking fund method, should get same result.

b. Mines --- ? [Here, if yr. capl. is to last longer, you apply less per land. An aggravated case of longevity meaning less intensive use.]

In mine case, what is capl. worth? Maybe deprec. by s.f. method.

c. Full salvage

d. Dim. annuity.

But b, c, and d are optional. (a) is enough to make the point that point case is generalizable.

Bruns ch. 19 purports to generalize to deprec. assets, but is obscure. My (a) is better.

15. Tight inventory control; ~~growth~~ decline of goods in stock, growth of mail order, central warehousing — these show shortage of free capl. Meantime, cap. is heavy in breeding stock, marines, etc.

16. A good follow-up exp. on 14 is to note the contrast of % of total costs which are labor and capl. on long and short invs. Can make this an elasticity -- the % Δ Wage Rate from a % drop of i.v. Make this a fn. of L.

$$\frac{\frac{\Delta W}{W}}{\frac{-\Delta i}{i}}$$

$$W \equiv \text{share of wages} = \frac{i^L}{(1+i)^L} = (1+i)^{-L}$$

$$\Delta W, \text{ as } i \text{ falls from } i \text{ to } j, = (1+j)^{-L} - (1+i)^{-L}$$

$$\frac{\Delta W}{W} = \frac{(1+j)^{-L} - (1+i)^{-L}}{(1+i)^{-L}} = \left(\frac{1+j}{1+i}\right)^{-L} - 1 \approx (1+j-i)^{-L} - 1$$

$$\% \Delta i = \frac{j-i}{i}$$

$$\bar{e}_{\text{last.}} \approx \frac{(1+j-i)^{-L} - 1}{\frac{j-i}{i} - 1} \quad i > j$$

Num. is + fun. of L ; denom. indep. of L .

Give num. e.g. $i = .05$
 $j = .04$
 $L = 1, \text{ ~~50~~ } 70$

$$L = 1, \quad e = \frac{.99^{-1} - 1}{-\frac{1}{5}} = -5 \times .01 = -.05$$

$$L = 50, \quad e = \frac{.99^{-70} - 1}{-\frac{1}{5}} = -5 \times 1 = -5, \text{ or 100 times greater.}$$

The numerical difference is so great, be sure to present it, to make the point forcefully.

17. In equl., capt. on unqul. land turns slower.

18. Artificial price untee. of raw mtrls raises C_0 in later stages, slows turnover. Thus the effect of cheap raw mtrls. on emplmt. is much stronger than ~~is~~ is superficially evident

19. Check ERT.

20. Might handle $\pm \Delta l.v.$ by saying has Δl profound effects, needs to be explored. In equl., destroys savgs; but $\rightarrow \Delta M, \Delta inv. opp.$ Hence inflary, adds to capt. shortage of free capt.

21. Use GNP concept. Can ΔGNP , w/o change $N.C.$, by Δ depress.

Is this a fake gain? Not entirely, bcs. empl. & GNP

22. The longer a bldg. lasts, the greater is the ~~the output ratio~~ share of land rent.

So in a staggered model, the greater is the L.V.: output ratio. Also the L.V.: Capt ratio.

It would seem to follow that capl. longevity makes capl. cover more land, not less as in my sug e.g.

Thus the ~~native~~ economy spreads out

23. Δ capl. intensity of land use, thru longer L , means less labor intensity.

But Δ Co w. const L means greater labor intensity.

Yield of coupon bonds at various present values (V) per \$100 par value.

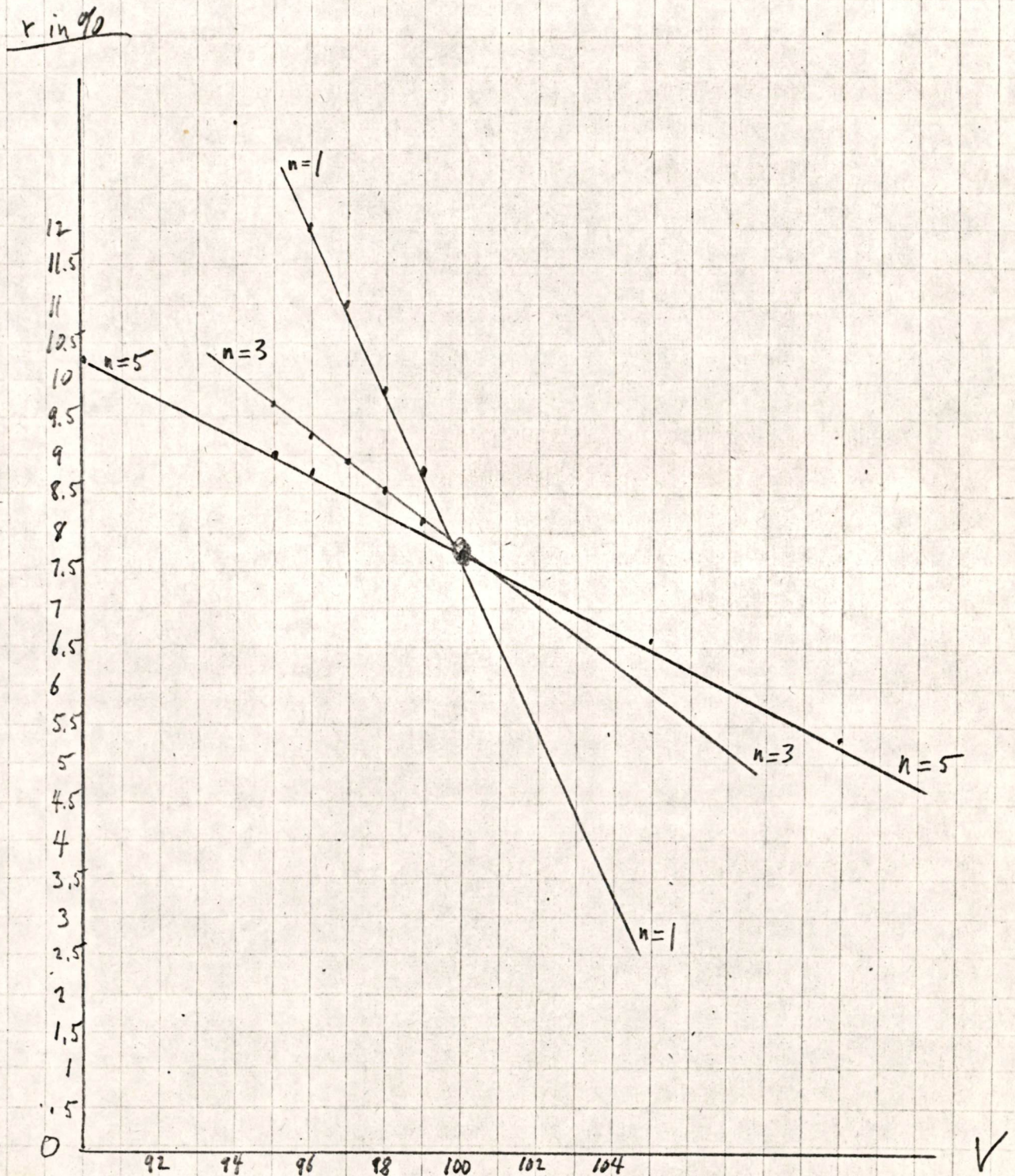
Coupon rate = $7\frac{3}{4}\%$

Source: HP-80

$V \downarrow$ $n \rightarrow$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
90	19.21				11.80				10.36
91									
92									
93									
94									
95	13.25				9.71	9.46			9.01
96	12.12			9.59	9.31	9.11			8.75
97	11.00	9.96	9.43	9.12	8.91	8.76			
98	9.90	9.21	8.86	8.66	8.52	8.42			
99	8.82	8.48			8.13	8.08			
100	7.75				7.75	7.75			7.75
101									
102									
103									
104									
105	2.65				5.91				6.56
106									
107									
108									
109									
110	2.09				4.17				5.44

← This column applies to Brad's bonds on Feb. 15, 1974, after coupon is shipped.

Graph of function ~~for~~ $V = 100 \cdot .0775 \frac{1 - (1+r)^{-n}}{r} + 100(1+r)^{-n}$
for different values of n .



Capital theory, life of bldgs.

BUILDING TAXATION AND SITE VALUE TAXATION--

HOW THEY AFFECT DEVELOPMENT OF LAND

FROM A LOWER TO A HIGHER USE

Neenah Bond

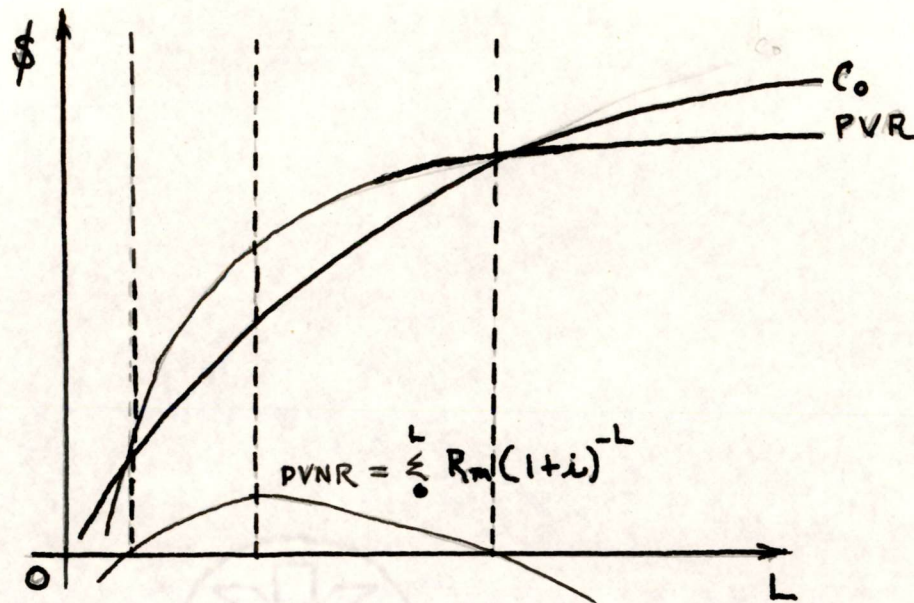
25% COTTON FIBER

Milton Hoffman
Economics 831

BUILDING TAXATION AND SITE VALUE TAXATION--
HOW THEY AFFECT DEVELOPMENT OF
LAND FROM A LOWER TO A HIGHER USE

All economists are concerned about allocation of resources. More specifically, they are concerned about optimal or efficient allocation of resources. It is the effort of this paper to show that present taxes on buildings do not allocate resources efficiently--and more specifically, they cause gross misallocation of one of our major resources, land.

The model that will be used to analyze the effects of the two types of taxes has its origin in Wilksell's famous wine model. But buildings are not like wine, they do not grow at some rate and then be harvested when the rate of growth of wine (trees) is equal to the rate of interest. This approaches Ely's ripening costs theory, but we do not want to maximize the present value, rather equal the present value of n th year to i (assuming for the moment a one cycle only.) The model that must be used is the one that has been developed by Gaffney. In this model it is assumed that costs are made at the initial period, t_0 , and are a function only of durability. As durability increases, costs increase but at a decreasing rate to where eventually additional durability can be had for small incremental costs. The revenues then are considered under the spectrum of choices of durability and then are discounted. The costs do not have to be discounted as they occur in year zero. The graph below illustrates this.



$$PVR^* = a \frac{1 - (1+i)^{-L}}{i} \quad \text{and as } L \text{ approaches infinity } (1+i)^{-L}$$

approaches zero, the limit. The present value of the stream of revenue and costs is $\sum_0^L [R_n(1+i)^{-n}]$. This is the $PVNR^*$ of one generation. But land can be used again. It is non-depreciating while buildings do depreciate. So we must not find only the optimal life of one building but the optimal life of that building when land is to be used again. In the early years, the $R_n(1+i)^{-n}$ may be negative as this includes both costs and revenues.

To find the optimal life of one building when land is to be used again, it is done by the following method. First, find the $PVNR$ of the first generation. The first generation is the one that is built to last to the point where $\sum R_n(1+i)^{-n}$ is a maximum. We then make this into an annuity by $PVNR \frac{i}{1 - (1+i)^{-L}} = a$, and make this into a perpetuity by dividing by i ,

$$\frac{PVNR}{1 - (1+i)^{-L}}.$$

-
- *PVR: Present Value of Revenue
 - *PVNR: Present Value of Net Revenue

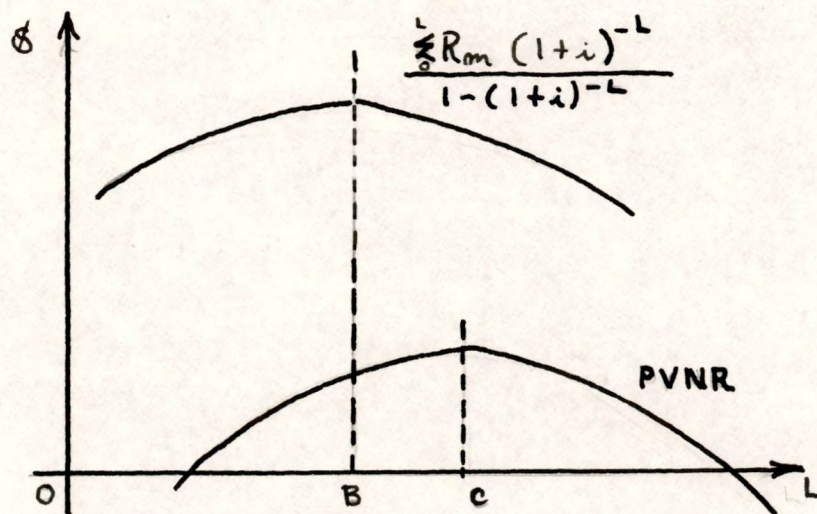
L is the life of one cycle and must assume that all subsequent L 's are of the same length. If there are infinite generations and $L = 10$, $i = 7.2$, the first generation accounts for fifty percent of the value of the perpetuity. In the graph above, the $\sum_0^L R_m (1+i)^{-L}$ is $PVR - C_0$ or the $\sum_0^L R_n (1+i)^{-n}$ for each n where n is any year but L is optimal year. At the maximum of the $\sum R_m (1+i)^{-L}$, the rate of growth is the same as the rate of interest. Then, if there is only one generation of buildings we maximize at top of the PVNR or where $PVR - C_0$ is at a maximum. If there is an infinite number of generations, multiply by the capital recovery factor and divide by i to form a perpetuity,

$$\frac{\sum_0^L R_m (1+i)^{-L}}{1 - (1+i)^{-L}}$$

Or, it is the PVNR discounted. When there are many generations, then maximize the curve again by

$$\frac{\sum R_m (1+i)^{-L}}{1 - (1+i)^{-L}}$$

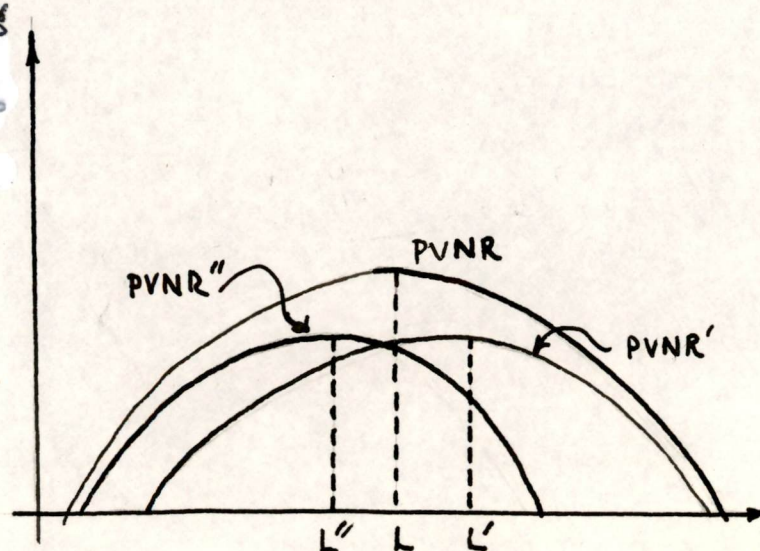
and then the optimal life becomes even shorter.



Then the maximum of the $\frac{\sum_{0}^L R_m (1+i)^{-L}}{1-(1+i)^{-L}}$ for many generations, or the PVNR for one generation, tells us the optimal life of the building. But when do we begin construction of the building? We have now found when to demolish it but when do we build it? This is an important point of our analysis and as we shall see, site value taxation and building taxation will affect the optimal date of construction of the building.

But before it is shown that these two types of taxation will affect the optimal date of construction, we must go a circuitous route--begin at the end by looking at how taxation affects the demolition of buildings. This is not as circuitous as it may seem for the demolition of one building makes way for the construction of a higher and better use building. And by this route we shall not only show how L (life of the building) is affected but also shall determine year zero, the year for construction of the building.

Looking at the PVNR curve, the optimal date of demolition of one building generation land use is at the maximum of the PVNR curve, the $\sum_{0}^L R_m (1+i)^{-L}$. The maximum tells us when to tear down the building.



How do the two types of taxes affect L? The building tax is a larger percentage when the building is new but as the base erodes, the building depreciates. The tax is reduced. Buildings that have long lives pay a smaller percentage of value in taxes as do buildings that have a short life. Buildings of long life reduce their taxes through the building itself depreciating and by depreciating, writing off the building at a faster than normal depreciated rate.

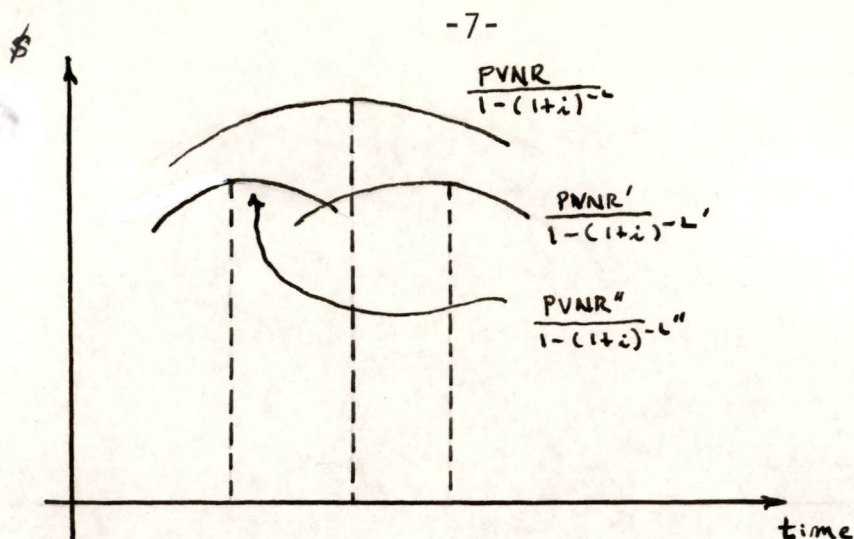
With the building tax, as the building or base erodes, the tax declines and thus the building tax prohibits the highest and best use from being achieved at the time of construction, but by declining in taxation as time passes, it causes the lower use to exist for longer than optimal. It thus causes a double sin--it chooses a building less than the highest and best use prescribes and then attaches the building to the site for a more than optimal time by charging little tax in later years. Thus buildings of short lives cannot avoid later taxes as well as buildings of longer lives. Though taxes in early years, present tax liabilities, are heavier taxes than those in later years, in later years renewal would mean heavy present tax liabilities since we must consider that the site can be reused. Not to renew is to avoid a heavy tax liability and thus not renewing avoids the potential tax liability. And this is not considering that under the income tax laws, buildings may be written off many times and thus completely avoid much of the tax. At any rate, the building tax on buildings of long lives is a smaller percentage of total value than the tax on

buildings of short lives. This affects the PVNR curve by pushing L to L'. It distorts the PVNR curve as illustrated by the PVNR' and pushes L to L'.

The site value tax however does not erode through time but rather as the site moves to higher uses the tax becomes heavier on the present use. Thus, buildings that have long lives pay a heavier penalty for their existence than buildings of short lives. Buildings of short lives can avoid the penalty by giving up their claim to the site that has moved to a higher use. Buildings that refuse to lie down and die are penalized by paying each year heavier taxes as the site moves to a higher use. The site value tax has an opposite effect of that of the building tax. The building tax favors longevity while the site value tax penalizes longevity. Thus, buildings of short lives pay a smaller percentage of value in the form of taxes under the site value tax than do buildings of longer lives. The PVNR curve becomes the PVNR'' curve and L moves to L''.

If the site was going to be used for another building of the same kind, one generation is like the next, and all of the same life, we could multiply the PVNR by the capital recovery factor $\frac{i}{1-(1+i)^{-L}}$ where L is the life (using L for the no tax case, L' for building tax) and then divide by the rate of interest, i, to form a perpetuity, we get

$$\frac{PVNR}{1-(1+i)^{-L}}, \quad \frac{PVNR'}{1-(1+i)^{-L'}}, \quad \text{and} \quad \frac{PVNR''}{1-(1+i)^{-L''}}.$$



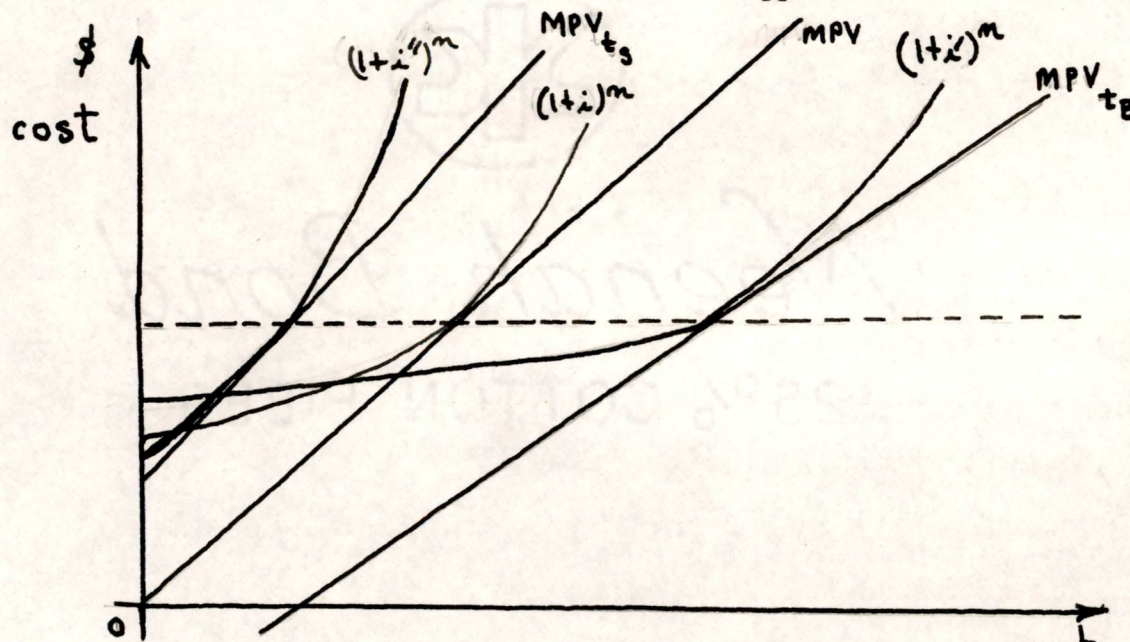
Thus, when the site is to be used again, the optimal life of the building is shortened but the site value tax gives us L'' , no tax gives us L and the building tax gives us L' ; and L'' is sooner than L , and L is sooner than L' .

The assumption that all generations are alike is not a realistic assumption. It is more realistic to assume that the site should not prepare itself for a twin brother of the building it now supports, but for a building of a higher and better use. In other words, the following buildings will not be like the present one. If the present construction is a single family dwelling, we may expect the next site occupier to be a small apartment house and the one after that, a multi-storied office building as the land ripens from residential to urban use.

We must make one assumption about the future generations. Though they are all different in type, use, and construction, we must assume that each lasts the same number of years. In working with annuity tables, we need some common exponent for time, L , L' and L'' . This assumption might not seem too unrealistic when we consider that a single dwelling unit can last 60 years while

multi-storied office buildings are often torn down after 60 years. The Rockefeller Center in New York was constructed on leased land for 99 years. Single dwelling units last for 99 years.

Thus looking to the L , L' and L'' , we must project these through higher and better uses. The sites will not be used for another building of its kind but for a higher use. Taking the maximum points of these curves and projecting them to higher uses, we get moving present value curves, MPV^* . For the no tax case, we get the MPV curve, for the building tax case we get the MPV_{tb} , and for the site value tax we get the MPV_{ts} .



If there is no tax, it is optimal to put the site to the higher and better use then $\frac{\Delta MPV}{MPV} = (1+i)^n$ or the growth of the site value is equal to the growth of the compound interest curve-time 0. If there is a site value tax, the rate of return need is higher, or the tax is capitalized into the site value and reduces

*MPV Moving Present Value

the site value but grows faster. For the same building to be built under the site value tax, it is done when $(1+i'')^n$ is equal to $\frac{\Delta MPV_{ts}}{MPV_{ts}}$ or when the rate of growth of the site under its highest and best use is equal to the rate of growth of the compound interest and tax rate, or time $0''$. For the same building to be constructed under the building tax, the same follows, the building is constructed when $\frac{\Delta MPV_{tb}}{MPV_{tb}}$ is equal to $(1+i')^n$ or time $0'$. Thus, $0''$ is sooner than 0 and 0 is sooner than $0'$.

The implication of the taxes moving the site into higher and better uses is obvious. The site value tax encourages urban renewal. It encourages short lives while the building tax encourages longevity. The site value tax gives us L'' while the building tax gives us L' which comes after L . The building tax makes one hold onto something long after it should be renewed. This is because the base for the tax erodes. As the building depreciates, the tax is reduced. But the building tax commits the second crime, it makes the land move into higher uses later. This is really saying the same thing as if it makes the site cling to an unoptimal use longer, it lets the higher use come later.

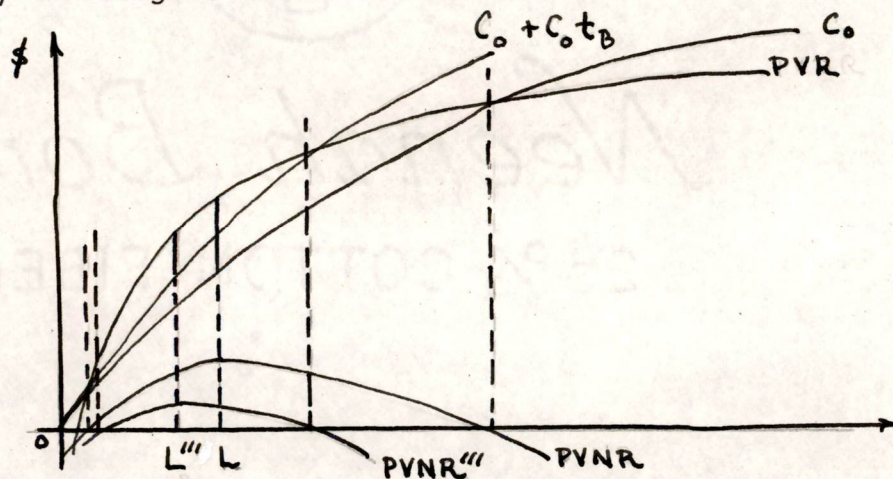
The two taxes also have capital rationing implications. The site value tax favors the disadvantaged in the capital market, the less wealthy. Those unable to accumulate capital at favored rates can still survive since the life of the building is shorter, it satisfies the need of a higher rate, that the capital be returned sooner.

So far, it has been assumed that the two bases allow the same building to be constructed only that the building tax makes the

time for construction come later. It will be shown that the building tax causes the site to be put to a lower use. Thus it makes the site cling to something longer, prevents the land from being used sooner, and when it does permit use, it permits use in a lower form than the highest and best use.

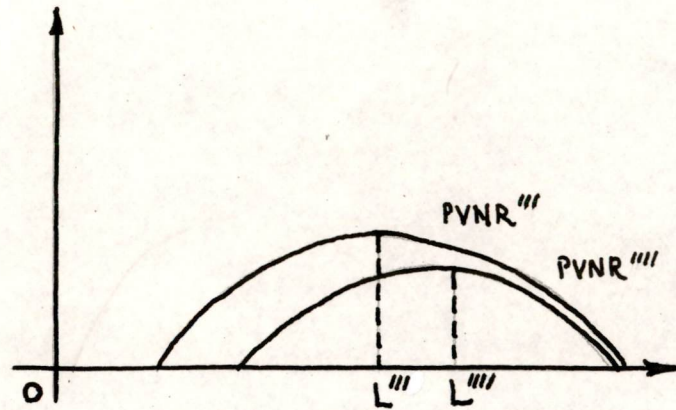
The building tax should really fall on the cost curve, C_o , and not on the PVNR. The site tax is a tax on rent and is correctly placed on the PVNR curve.

By taxing the costs, it prevents the highest use from being constructed as in the one generation curve, as costs increase, durability and higher uses increase.



Since taxing the costs gives us a building that has supposedly less life means it is a lower use than L . But this is only part of the crime. We have now decided what building to construct, a lesser building than the highest and best use building. The tax increases as the new building costs increase but the tax erodes as the building passes through time. Thus, having chosen the building to last L''' years, where L''' is shorter life than L , it makes the building last L'''' where L'''' is after L . The building

tax is a higher percentage in early years but a smaller percentage in later years. Thus, $PVNR''' < PVNR''$



We should then think of the MPV_{tb} curve as an inferior choice to the MPV and MPV_{ts} curves. In reality then, summing up the case against the building tax, it causes a lesser than optimal building to be constructed and it causes the less than optimal building to be held onto too long, preventing the site from moving into its higher use. The verdict weighs heavily against the building tax.

When we consider that under the present tax structure, a building can be depreciated more than once, the end of the unoptimal construction is postponed even more. Another mark against the building tax.

The bias involved in the building tax is shown mathematically by Consigney and Gaffney in Tax-Induced Slow Turnover of Capital. Here R_m is the ratio of revenue in year m to construction costs C_0 -- or the revenue per dollar costs given that $C_0 = \$1$.

Then $(1+r)^m = R_m(1-t)$ where r is the ratio of return after taxes. The tax takes a fixed percentage of gross value (our MPV). The tax is biased in that it taxes capital recovery.

Rewriting the above equation

$$(1+r)^m = (1+i)^m (1-t)$$

$$r = (1+i) (1-t)^{1/m} - 1$$

Since $(1-t) < 1$, the value of r varies from a low when $m = 1$ to a high when $m = \infty$.

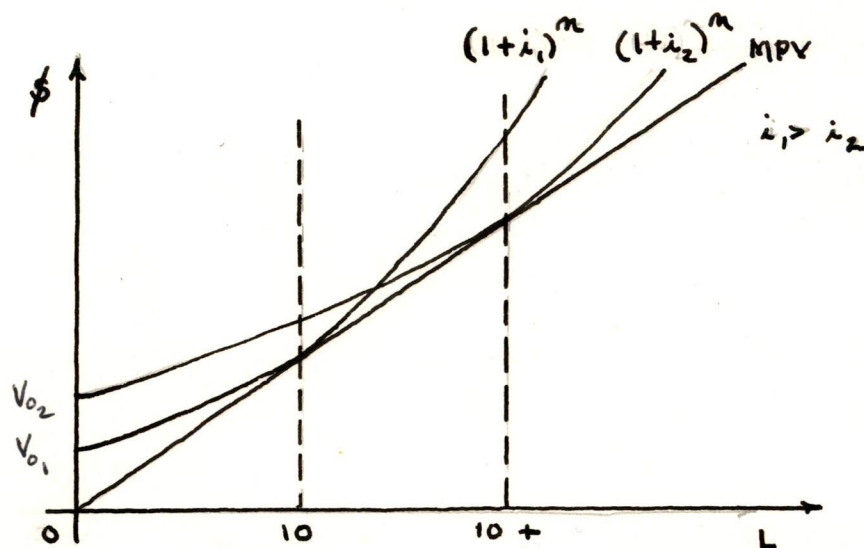
$$\text{If } m = 1, \text{ then } r = (1+i) (1-t) - 1 = i(1-t) - t.$$

$$\text{If } m = \infty, \text{ then } r = (1+i) (1-t)^0 - 1 = i.$$

In the last equation it can be seen that $i = r$. This is not what is being shown in the diagram in this paper. The curves in this paper represent costs that the highest and best use must return before the site can change use.

As long as the model has been developed to its present stage, it is interesting to consider a problem not directly related to taxes, but a factor that does affect timing decisions of moving land into higher uses. Interim use of land affects land's moving into higher use. Consider the case of a vacant lot in the central business district that will be ripe for a high rise in say ¹⁰/years. Does the land stand vacant or useless for 10 years? No. It can be put to some interim use and earn the owner a return. But it is interesting to note that the interim use postpones the time when the highest and best use is constructed, it makes building come after 10 years and since it does that, it causes the land to move into even a higher and ~~best~~ use. How nicely the pieces fall together.

The interim use reduces the rate at which the owner must borrow in the capital market as it reduces his costs. A lower



rate of growth in the compound interest table means that the present value of the site is dearer than a higher rate, $V_{02} > V_{01}$ and this is as it should be since the site is earning an interim rent. And since it reduces the capital cost it postpones the 10 years and by postponing the 10 years the site has ripened into an even higher and best use. We can think of the interim rent as increasing the rate of return to the holder.

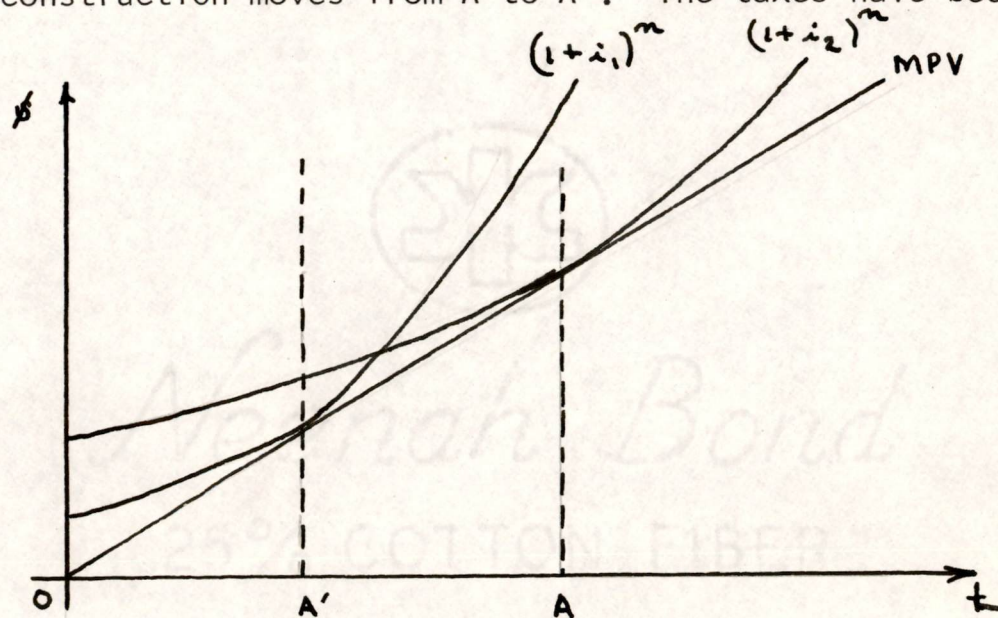
$$(1+i)^n + (1+i)^n r_i = R_m$$

where R_m is the return to owner and r_i is the interim rent.

So far nothing has been said about shifting the tax. A tax can be shifted only by reducing the supply. A neutral tax cannot be shifted since it does not reduce the supply. The site value tax does not reduce the supply, and might even be said to increase the supply of buildings. It is obvious however that anything that causes a reduction in the optimal buildings constructed and postpones the sub-optimal building's construction period, is reducing the supply.

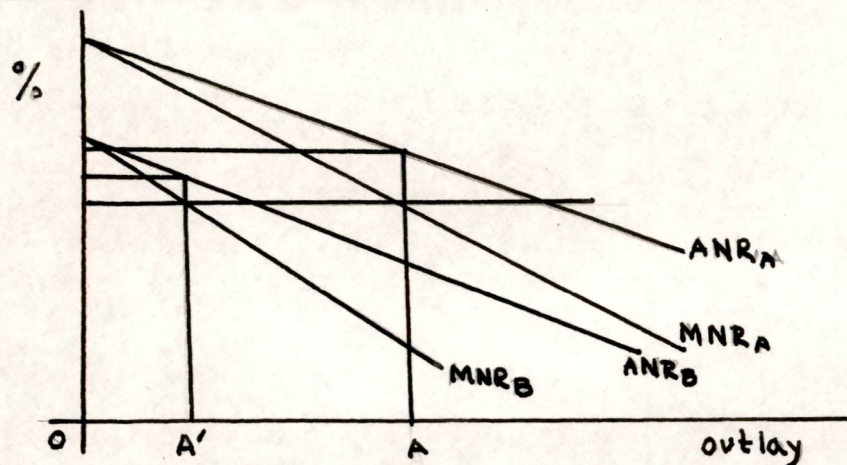
The shifting of the building tax can occur by reducing the supply of buildings. This is created by a shifting away from buildings into other investments until the rate of return on buildings is increased to the rate on other investments. Or there may be a complete shift away from all capital which would bring r , the rate of return after taxes, back up to the old interest rate but cause the interest rate to increase to a new level.

What affect then does shifting have on the model? If there is a shift away from capital, then the rate of interest increases and this causes the optimal time to be sooner. The optimal time of the new construction moves from A to A' . The taxes have been



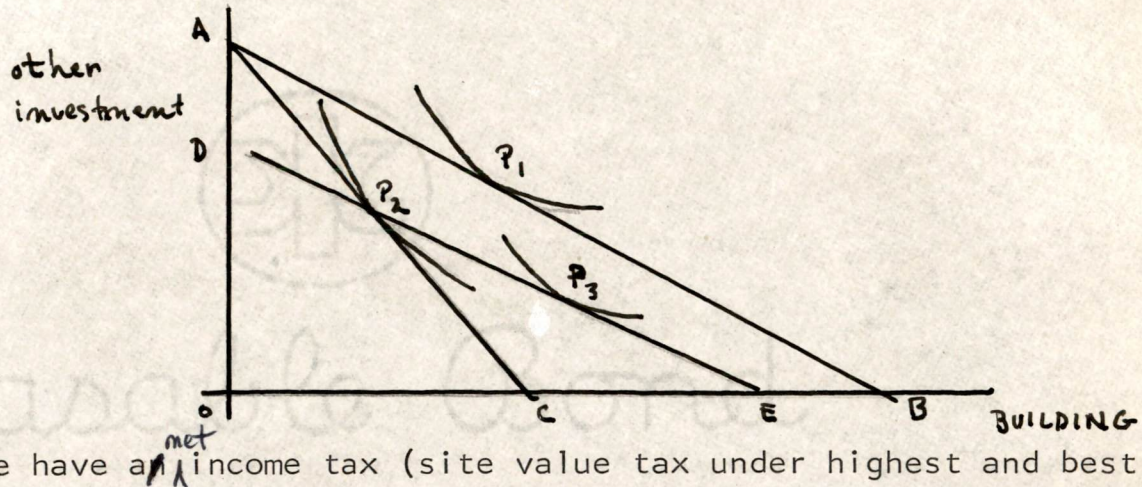
shifted so they do not appear, — that is, if they have been entirely shifted. However, the model assuming that property taxes have not been shifted may not be too unrealistic an assumption when we consider the fact that property taxes are a local tax and the demand for housing in a small jurisdiction must have a high coefficient of elasticity. In that case, the tax shift would not occur but would be shifted in a lower MPV, a lower land rent.

It is interesting to note that the conclusion drawn here about site value taxation can be shown in the model constructed by Turvey in The Economics of Real Property.



The ANR_A curve is before taxation on outlay. If the taxation is a constant percentage it will pull the ANR_A to ANR_B and the corresponding MNR_B interest, the marginal cost or rate of interest and outlay is reduced from A to A' . And, if part of the tax is shifted, or the entire tax is shifted such that ANR_A again becomes the proper curve, the rate of interest will increase and outlay will be reduced. The property tax once again is shown to prohibit the construction of the highest and best use building.

Some Welfare Comments on Building Taxes and Site Value Taxation:



If we have a ^{met} income tax (site value tax under highest and best use) and an excise tax (building tax) to yield the same income, the investor is still better off with the site value tax, assuming that the building is under its highest and best use. When the building tax is levied, the budget line goes through P_2 . If the income is to be the same under the site value tax as under the building tax, the budget line for the site value tax also goes through P_2 but it is DE instead of AC. He can buy the same bundle of goods, P_2 , under either tax so the income must be the same. But with the budget line DE he purchases P_3 and not P_2 . Thus the individual obtains a higher level of satisfaction and from the individual it can be generalized to the whole community.

However, Milton Friedman does not accept this conclusion. He states that we cannot generalize from the individual to the whole. He accepts the reverse situation, that the excise tax is better than the income tax.¹ "It is often concluded on the basis of the individual and generalized to the community as a whole to

¹Friedman, Milton, Price Theory, Aldin, 1962, pp. 56-67.

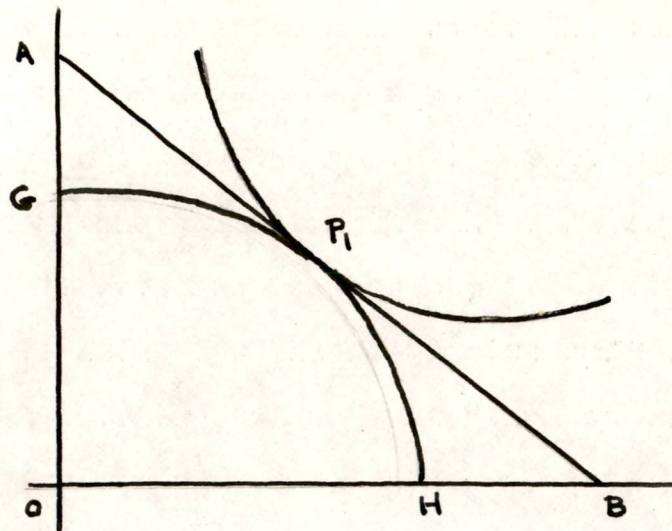
yield the proposition that all members of the community would be better off (on a higher indifference curve) if the excise tax were replaced by an income tax so levied that each member pays the same amount as an income tax that he formerly paid as an excise tax."² If Friedman is correct, all land value tax supporters are fighting for the wrong victory. We should support the present building tax and put "thumbs down" to the ~~the present building tax and put "thumbs down" to the~~ site value tax. Friedman's claim makes our position uneasy.

To remove the uneasiness let's venture to see if Friedman is correct. We have a conflict of views and when the conflict arises from such high levels, we watch for the results with greater anxiety.

Friedman's model assumes that all consumers have identical tastes and preferences and identical kind and quantity of resources owned by each individual. Everyone has the same income. This implies that all can be represented by one.

Given the resources, there is some set of combinations of X and Y that can be produced--the production function.

² Ibid. p. 59.

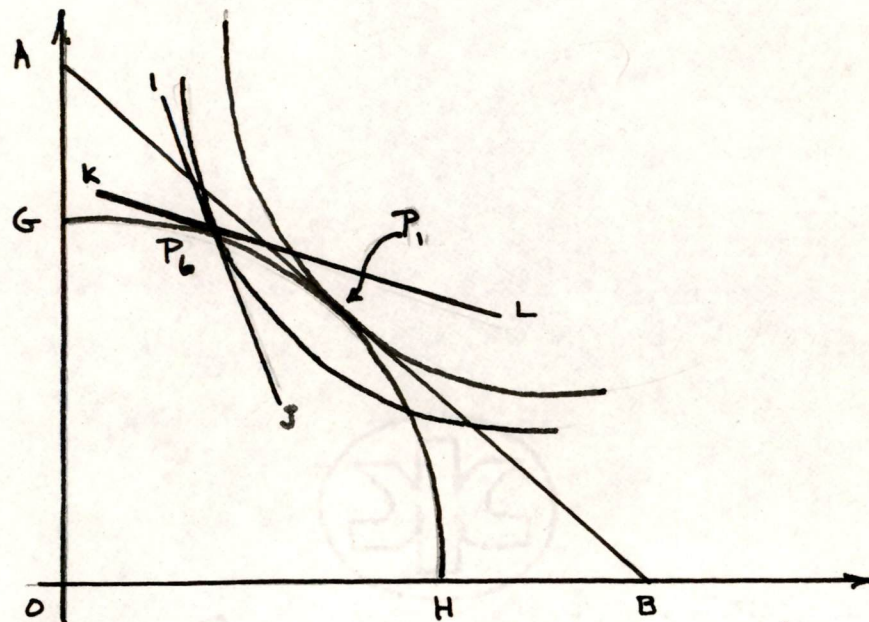


P_1 is an individual equilibrium but since all are alike, the position applies to everyone. Here the rate of substitution in consumption (slope of the indifference curve) is equal to the rate of substitution in purchase (slope of price line) is equal to the rate of substitution in production (slope of production function).

Friedman states that an excise tax does not change the production possibilities curve. "The excise tax does not by itself lessen the physical resources available to the community."³ But surely the building tax, excise tax must affect the production possibility curve. If it prevents land from going to a higher use, then the production function is less than it would be if the site value tax were used. The building tax affects resources in two ways--it affects the amount of construction and the longevity of the building and thus has a double misallocation. But even if we agree with Friedman on this point, that excise tax does not affect the amount of resources available for production, it

³Ibid. p. 60.

is not clear that the remainder of his proof is correct. Yet this is a strong point to our side when site value taxation does increase the availability of resources, land, not horizontally as that is fixed, but vertically.



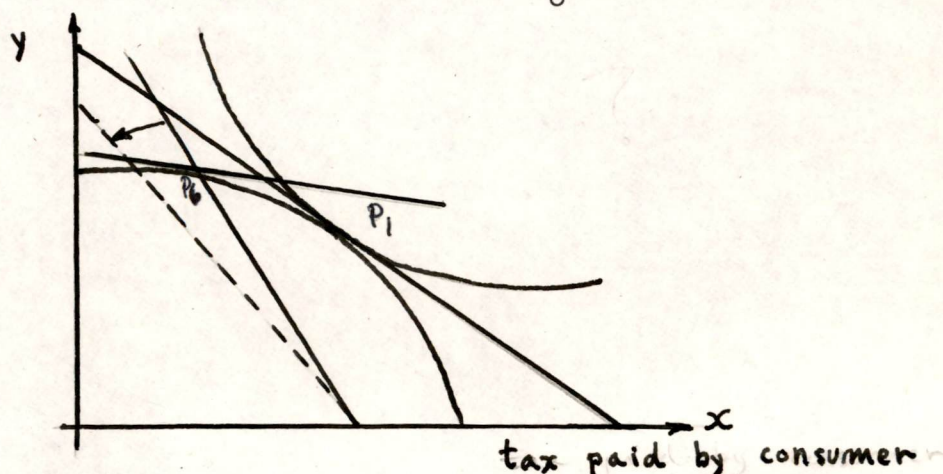
M. FRIEDMAN, PRICE THEORY, p. 64

The excise tax causes the divergence of the price between what the producer receives and the consumer pays. The point where all rates are equal is P_6 and P_6 is clearly inferior to P_1 . LJ is the budget line for the consumer and KL is the constant receipts line for the producer. And the two diverge because of the excise tax on X .

Friedman then argues that suppose P_6 is the initial position due not to taxation but monopoly on the production of X . Now levy the excise tax on Y . Then Friedman says that the excise tax on Y offsets the monopoly on X , and argues that P_1 is the final

equilibrium as the ratios of prices are then equalized by the excise tax on Y and the monopoly in X. But it is unclear how, if P_6 is the initial position due to monopoly and then there is an excise tax levied on Y, how P_1 becomes the equilibrium. It seems to this writer that this is double harm and that equilibrium would come at some point less than P_6 .

Equilibrium would come as a result of monopoly in X and excise tax on Y at some point less than P_6 as illustrated below.



The correct analysis must ask, however, who pays most of the excise tax? If it is the consumer, then we must shift the budget line. If it is the producer then we must shift the receipts line. Both, however, may move. But if either moves because of an excise tax on Y, this writer cannot find an equilibrium at P_1 but gets one somewhere below P_6 . Since our model considers no shifting of the tax, we are concerned about the model that shows the tax being paid by the producer (investor). This agrees with the earlier statement that if the tax is paid by the producer it reduces the production possibility curve. ~~But not only that, it also shifts the consumer to a new lower equilibrium curve even though its budget line is unchanged.~~ P_7 will be the final equilibrium.

⁴ *Ibid.*

above--monopoly in the production of X and excise tax on Y. In both cases agreement with Friedman is not reached. The income tax, the same as the site value tax under highest and best use and clearly the same for a property tax that takes a fixed percentage, t , of net income, is superior to the excise tax, the building tax, and this is ^{as} commonly accepted by economists.

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The Decision to Repair or Scrap a Machine

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A simple algebraic formula for combining repair data with prior experience determines the time when a machine should be replaced in order to minimize the expected cost of equipment purchase and maintenance. A random sample from an exponential distribution represents the cost of each repair, and a time-dependent Poisson process represents the intervals between repairs. Bayes' formula provides the basis for combining data with previous judgement about the characteristics of the equipment. Automobile maintenance records support the basic assumptions of the model, and illustrate the method of deciding when to scrap a given machine.

INTRODUCTION

RISE repair costs frequently provide the incentive for replacing old equipment. As a general rule, a machine should be scrapped when the cost of a necessary repair exceeds the resulting gain in its value. However, a person faced with the problem of deciding whether to repair or scrap an existing machine has difficulty in determining the level of repair cost that is excessive. The problem is particularly acute in the usual case where information on past repairs of similar equipment is limited.

In addition to rising repair costs, many other factors affect the machine replacement decision.¹ These include obsolescence, deterioration of quality of service, rate of return on alternative investments and taxation. While such items can be incorporated into the model described in this paper, they have been excluded to simplify the presentation. Often rising repair costs are the major factor affecting the decision to scrap a machine, in which case the theory to be presented is adequate as it stands.

In early papers on equipment replacement, Clapham² and Eilon *et al.*³ determined the optimal age for replacing any machine in a fleet by studying the average rate at which annual repairs tend to increase with age. Their analysis used linear regression techniques, and consequently assumed that annual repairs are normally distributed with constant variance about a trend line. The methods in these papers do not explicitly consider the random nature of individual repairs to the equipment.

Drinkwater and Hastings developed a method for determining the upper limit for any repair cost on a machine in a fleet of similar equipment.⁴ The limit decreases with the age of the machine. The model consists of two random

variables: the cost of any future repair and the number of repairs in a future period of use. The former is exponentially distributed and the latter is Poisson, both having a mean that depends on the age of the machine. Records of past repairs to a fleet of army trucks support the assumed distributions and provide estimates for their parameters. Subsequent work by Hastings applied dynamic programming to the model to improve the method for determining the repair limits.⁵ Although this method detects a machine having a very high cost for an individual repair, it does not necessarily detect a machine having consistently more frequent or higher repair costs than the average machine of its class. Such differences in quality are a well-known phenomenon, and result from the vagaries of equipment construction and use. In Eilon *et al.* the average variation in repair cost between vehicles is plus or minus 30 per cent.³

The following model establishes a repair limit formula that uses data on past repairs for the machine under study, plus whatever knowledge is available on the general repair characteristic of its class of machine. Records of repairs from a fleet of similar equipment are not required. The theoretical basis for the model is presented in two sections. The first section develops a stochastic model of the random characteristics of maintenance expenses and essentially is a continuous-time version of the Drinkwater-Hastings model. Dynamic programming is avoided, however, by assuming that, after being adjusted for inflation, repair costs do not change with age, only their frequency does. Consequently the resulting non-linear differential equation can be solved algebraically. The second section introduces the Bayesian method for describing uncertainty about the model's parameters for any specific machine and for using data to reduce this uncertainty. It also demonstrates the complete calculation with an example of an equipment replacement decision, and briefly outlines opportunities for future work.

THE MODEL

The characteristic pattern of expense for a machine consists of a large initial purchase price that is followed by low maintenance costs which gradually grow with increasing use of the machine until further repair becomes uneconomic. Table 1 lists such a set of costs for a Canadian automobile. All repairs were made as soon as required to keep the vehicle in good operating condition. The actual costs have been divided by the current value of the Canadian Consumers Price Index⁶ and multiplied by the appropriate figure for 1970 to compensate for changes in the value of money.

The random nature of the costs and the intervals between these costs presents difficulties in deciding whether to repair or replace a machine when a component fails. Not only does randomness introduce uncertainty about the timing and the cost of future expenses, but also it obscures the evaluation of the quality of

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the machine from its past repair record. The problem is greatest with equipment having infrequent, but costly, repairs.

An analysis of the repair costs in Table 1 suggests that it would not be unreasonable to assume that they belong to a random sample from a stable

TABLE 1. PURCHASE AND REPAIR EXPENSES FOR A 1956 FORD SEDAN

Mileage (1000 miles)	Repair	Expense 1970 \$ (Canadian)	Mileage (1000 miles)	Repair	Expense 1970 \$ (Canadian)
0.0	New vehicle	3260	52.0	Ignition	19
4.0	Transmission	99	53.0	Adjusting lights	4
13.0	Ignition	34	55.5	Waterpump	29
21.2	Muffler	20	55.7	Muffler	14
23.5	Ignition	21	55.8	Seat covers	30
24.8	Battery	31	58.0	Wheel bearings	6
26.1	Headlight (left)	4	59.0	Headlight (right)	4
27.7	Pair of tyres (front)	46	59.1	Headlight (left)	4
28.0	Ignition	9	59.1	Licence light	1
28.8	Wheel alignment	21	59.7	Wiper arm	4
30.8	Tailpipe	9	59.8	Tailpipe	15
31.0	Headlight (right)	4	59.8	Ignition	20
32.5	Pinion nut	7	60.3	Wheel balance	12
32.8	Starter	24	60.5	Gear shift	2
32.9	Pair of tyres (rear)	55	66.0	Voltage regulator	10
34.7	Wiper arm	5	66.2	Brakes	2
34.8	Battery	37	68.0	Generator	29
34.9	Ignition	22	68.5	Radiator hose	4
35.0	Wheel balance	5	69.1	Clutch	149
37.5	Pinion	31	69.1	Fuel pump	18
38.0	Headlight (right)	4	70.0	Carburettor	17
39.0	Headlight (left)	3	70.0	Fan belt	2
40.0	Muffler	21	70.3	Window	18
42.0	Wheel balance	4	70.6	Brake cylinder	35
43.0	Radiator	27	74.4	Brake switch	1
43.9	Ignition	10	76.0	Pair of tyres (front)	67
48.4	Tailpipe	10	76.0	Wheel balance	4
49.2	Starter	36	77.5	Muffler	20
49.5	Wheel alignment	21	77.5	Tailpipe	10
49.5	Brakes	108	78.0	Pair of tyres (rear)	60
49.7	Pair of tyres (front)	68	78.5	Pinion	106
49.8	Door handle	4	78.5	Wheel bearings	28
50.3	Pair of tyres (rear)	40	78.7	Heater valve	10
51.0	Window	6	79.0	Ignition	34
52.0	Camshaft	155	79.4	Carburettor	31
52.0	Radiator	19		Total	5099

distribution. The observation that 21 of the first 35 costs exceed the median of \$18 for the group of 70 repairs is equivalent to 21 heads in 35 tosses of a fair coin. The chance of having at least as many heads in 35 tosses is 0.16, a not uncommon event.

When plotted in Figure 1, these costs exhibit the exponential distribution found by Drinkwater and Hastings, which indicates that the components have not been designed to standardize the cost of repair. Each repair cost has a

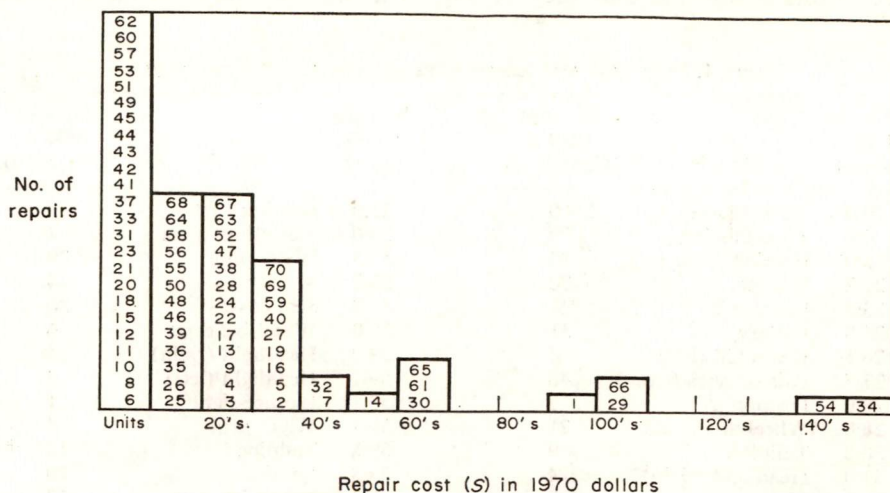


FIG. 1. Distribution of repair costs.

constant chance of exceeding any interval if it is not less than the interval. In the limit as the intervals become very small, any repair bill S is a random sample from an exponential probability density function:

$$\text{Pdf}(S|\gamma) = \gamma \exp(-\gamma S).$$

The parameter γ is a characteristic of the repair costs for any particular machine, and equals the reciprocal of the expected cost of a repair if γ is known precisely:

$$\begin{aligned} \exp(S|\gamma) &= \int_{S=0}^{\infty} S \text{Pdf}(S|\gamma) dS \\ &= \int_{S=0}^{\infty} S \gamma \exp(-\gamma S) dS = 1/\gamma. \end{aligned}$$

Another aspect of the data in Table 1 is the timing of the repairs. (In this case, mileage serves as an appropriate measure of the age of the machine at the time of a failure.) Because there are many components in a machine, the intervals between successive repairs to the machine as a whole are usually the intervals between the failures of different components. Furthermore, the failure of one component in a complex piece of equipment has little effect on the failure of another. Therefore the intervals between each repair are essentially independent of each other and form a Poisson process, as suggested by Drinkwater and

Hastings. The probability of observing N repairs by age A when $F(A)$ repairs are expected is given by the well-known formula:

$$\Pr [N | F(A)] = \frac{[F(A)]^N}{N!} \exp [-F(A)]. \quad (1)$$

If the length of an interval beginning at A_i and ending at A_j is measured in terms of the expected number of repairs during the interval, the probability that the intervals will exceed a given number becomes an exponential distribution:

$$\Pr [0 | F(A_j) - F(A_i)] = \exp [-F(A_j) + F(A_i)]. \quad (2)$$

The expected frequency of repair $f(A)$, which is the derivative of the expected number of repairs $F(A)$, should increase with the age of the machine because the equipment consists of a great variety of components, all of which are new at the beginning and have different expected lives. The probability of a component failing in a given increment of time therefore must increase as the more durable components finally wear out, and the less durable ones undergo second, third and fourth repairs. As a first approximation, the expected frequency of repairs could be assumed to be proportional to age. Therefore $f(A) = \lambda A$, where λ is a characteristic of the repair frequency for any particular machine. The expected number of repairs during an interval between two ages A_j and A_i becomes:

$$F(A_j) - F(A_i) = \lambda(A_j^2 - A_i^2)/2 \equiv \lambda I_{ij}/2. \quad (3)$$

Figure 2 supports the assumption that the occurrence of failures is essentially a Poisson process with an expected number that is proportional to the square of the age of the machine. This histogram of the intervals (I) between the square of the mileages in Table 1 has an exponential distribution. The order of occurrence of the intervals, as given by the numbers in Figure 2, is essentially random. Of the 35 intervals that are shorter than the median of $60 \times (10)^{-6}$ miles², 16 occur in the first 35 repairs. The probability of not having more intervals in a stable distribution is 0.37.

From equations (2) and (3):

$$\text{Pdf}(I | \lambda) = -(\text{d}/\text{d}I) [\exp(-\lambda I/2)] = (\lambda/2) \exp(-\lambda I/2). \quad (4)$$

Both the cost and the frequency components of the model are needed to represent the total repair expenses for the vehicle. The assumption that these two components are statistically independent is supported by the observation that 36 out of 70 pairs of costs and intervals are both above or both below their respective medians. The chance of having 36 or more such pairs with independent components is 0.45.

The basic rule for deciding when to replace a machine is to continue using it until the cost for a prospective repair exceeds the expected net benefits from

future use of the machine if it were repaired. In other words, the limit to the amount that should be spent for a repair is equal to the implicit value of the vehicle after repair, less the value of salvage if it was not repaired. The derivation of the repair limit begins by noting that the average cost per mile, L , for purchase

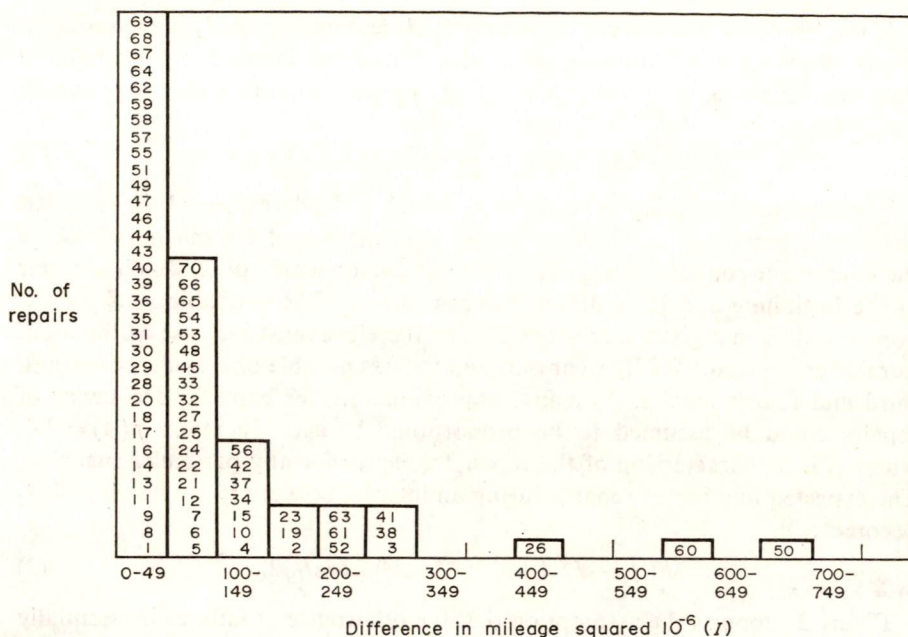


FIG. 2. Distribution of mileage intervals between repairs.

and repair of a series of machines is a constant that equals the current rate of depreciation in the value of any specific machine, $-dV/dA$, plus its expected cost of repair or of loss due to scrapping during the interval ΔA . Because the repair limit W equals the implicit value V less a constant value for salvage, it also decreases at the same rate. In mathematical terms:

$$\begin{aligned}
 L &= -\frac{dV}{dA} + f(A) \left[\int_{S=0}^W S \gamma \exp(-\gamma S) dS + \int_{S=W}^{\infty} W \gamma \exp(-\gamma S) dS \right] \\
 &= -\frac{dW}{dA} + f(A) [1 - \exp(-\gamma W)] / \gamma,
 \end{aligned} \tag{5}$$

Equation (5) is solved via the substitution $Y = W + LA - F(A)/\gamma$ to give the repair limit W as a function of age A :

$$W = (1/\gamma) \ln \left\{ 1 + L\gamma [\exp(F(A) - \gamma LA)] \int_{x=A}^{\infty} \exp(-F(x) + \gamma Lx) dx \right\}. \tag{6}$$

In the special case where the expected frequency of repair is proportional to age, $f(A) = \lambda A$, the formula for the repair limit becomes:

$$W = (1/\gamma) \ln \left\{ 1 + \frac{L\gamma}{\sqrt{\lambda}} [\exp \lambda(A - \gamma L/\lambda)^2/2] \int_{x=\sqrt{(\lambda)(A-\gamma L/\lambda)}}^{\infty} \exp(-x^2/2) dx \right\}. \quad (7)$$

COMBINING DATA WITH PREVIOUS JUDGEMENT

The final step in using the model is to determine the repair characteristics for any particular machine. Although knowledge of parameters like γ and λ traditionally has been described by estimators, a probability distribution provides a more satisfactory description of a person's uncertainty. For example, even before an automobile is used, the owner has some idea of the potential range of repair costs. His uncertainty is broad however because he does not know at this time whether the particular vehicle is a poor one or a good one. In other words, he feels that there is an equal chance that γ or λ is small or large. Nevertheless, as experience accumulates with the use of the automobile, the developing record of repairs reduces his uncertainty about these parameters. Bayes' formula provides the mathematical method for making these adjustments. If the record is good, he begins to realize that the particular vehicle probably has better quality than average. If it is poor, he suspects the worst. He never knows with certainty, because the good record might be the result of a lucky chain of events with an inferior vehicle. Nevertheless, with a suitable analysis of the chance of the various possibilities, a person can make decisions which maximize the expected net benefits from the future. The remainder of this section presents the mathematical basis for performing these calculations.

The algebraic structure of Bayes' formula is simplified if a gamma function is used to describe a person's uncertainty about the cost characteristic γ . This is a flexible class of functions that can have any positive average value and variance, and therefore should be adequate for capturing the essential form of a person's uncertainty. In mathematical terms, the relative chance of any specific value of γ being characteristic of the machine is given by the following probability density function, where $a > -1$, $b > 0$, and $\Gamma(a+1) = a!$ if a is integer:

$$\text{Pdf}(\gamma) = \frac{(b)^{a+1}(\gamma)^a \exp(-b\gamma)}{\Gamma(a+1)}. \quad (8)$$

Two useful statistics of the gamma distribution are the expected value and the standard deviation:

$$\text{Exp}(\gamma) = (a+1)/b, \quad (9)$$

$$\text{Std}(\gamma) = \sqrt{(a+1)/b}. \quad (10)$$

Suitable values for the gamma constants a and b at the time of purchasing the machine should reflect previous experience with comparable equipment. Usually people think in terms of average repair costs, instead of γ . For example, the owner of an automobile might feel that the average cost of a repair probably would be some number between \$15 and \$30 per repair, depending on the quality of the vehicle. In terms of γ , the range would be essentially 0.067–0.033 because the average of a very large number of measurements corresponds to the expected value, which is the reciprocal of γ . This range suggests that the uncertainty about γ has a mean of 0.050 and a standard deviation of 0.017, or thereabouts. The substitution of these values into equations (9) and (10) yields $a = 8$ and $b = 180$.

The accumulation of repair data for a specific machine subsequently revises the initial view of the value of its parameter γ . With N repairs costing a total of D dollars, Bayes' formula converts equation (8) into a new gamma distribution:

$$\text{Pdf}(\gamma|N, D) = \frac{(b-D)^{a+N+1}(\gamma)^{a+N} \exp[-\gamma(b+D)]}{\Gamma(a+N+1)}. \quad (11)$$

When the repair costs of Table 1 are applied to the preceding formula, the record of $N = 70$ repairs costing $D = 1839$ dollars raises the constants of the gamma function from $a = 8$ to $a + N = 78$ and from $b = 180$ to $b + D = 2019$. Thus the record of repairs for this particular vehicle has reduced the uncertainty about its γ from a standard deviation of 0.017 to $\sqrt{(79)/2019} = 0.0044$, while changing the mean from 0.050 to $79/2019 = 0.039$. The expected cost of future repairs therefore will be higher than originally was anticipated. Furthermore, if extensive repair data were available from a number of similar machines, the resulting range of these means would give a good indication of the prior uncertainty about γ for a future machine, as described in the previous paragraph.

A gamma distribution also can describe uncertainty about λ , the parameter for a repair frequency that is proportional to age, $f(A) = \lambda A$. In the automobile example, past experience with similar equipment might suggest that the expected frequency of repairs per 10,000 miles could lie between two and three times the mileage in 10,000-mile units, depending on the quality of the vehicle. In other words, if the automobile is driven 10,000 miles per year, the expected number of repairs per year would lie between two and three times the age of the vehicle. With λ having a mean of $2.5 \times (10)^{-8}$ repairs per mile² and a standard deviation of $0.5 \times (10)^{-8}$, appropriate values for the gamma function would be $d = 24$ and $e = (10)^9$.

The subsequent record of N repairs in U miles revises the gamma function for λ to produce:

$$\text{Pdf}(\lambda|N, U) = \frac{(e + U^2/2)^{d+N+1}(\lambda)^{d+N} \exp[-\lambda(e + U^2/2)]}{\Gamma(d+N+1)}. \quad (12)$$

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For the automobile in Table 1, the record of $N = 70$ repairs in 80,000 miles increases the constants of the gamma function from $d = 24$ to $d + N = 94$, and from $e = (10)^9$ to $e + U^2/2 = 42 \times (10)^8$. Thus the uncertainty about the value of λ for this particular vehicle has been reduced to a mean of $95/42 = 2.26 \times (10)^{-8}$ repairs per mile squared with a standard deviation of $\sqrt{(95)/42} = 0.23 \times (10)^{-8}$. In the more general case where the expected number of repairs is described by $\lambda G(A)$, the substitution of $G(U)$ for $U^2/2$ in equation (12) provides an identical procedure for incorporating data with previous experience on the scale parameter λ .

The foregoing description of the state of knowledge for γ and λ provides the basis for adjusting the repair limit formulas for uncertainty about the parameters. The repair limit formula, equation (7), represents the expected value of the machine by age A if γ and λ are known, while the probability distributions for γ and λ represent the state of knowledge about these parameters at age U . (If the repair record extends to age A , then $U = A$). The unqualified expected value for W is given by the formulas of the previous section when weighted by the probability of various values of their parameters and summed. For example, in the special case where the repair frequency is proportional to age, equation (7) is multiplied by equations (11) and (12) before the product is integrated (numerically) over the ranges of γ and λ :

$$\exp(W|\text{data}) = \int_{\gamma=0}^{\infty} \int_{\lambda=0}^{\infty} (W|\gamma, \lambda) \text{Pdf}(\lambda|\text{data}) \text{Pdf}(\gamma|\text{data}) d\lambda d\gamma. \quad (13)$$

The expected value at the time of purchase should equal the expected net price P of buying another machine. Therefore the substitution of P for $\exp(W|\text{data})$ when $A = D = N = U = 0$ yields an expression for L , the long-term average cost of purchase and repair for many similar machines.

The application of the foregoing theory to repair data on three different vehicles in Table 2 produces repair limits for each at several stages in its life.⁷ At zero mileage, each machine has the same limit of \$3000, the expected net cost of purchase after deducting \$200 for salvage. The substitution of this figure, the preceding values for the parameters (a, b, c, d) and $A = D = N = U = 0$ in equation (13) produces the solution $L = \$0.0529$ per mile after a few iterations. Then as experience accumulates, each vehicle develops a different value for the repair limit which reflects the current assessment of its quality. When $A = D = N = U = 20,000$ miles, the limits have changed to \$2640, \$1460 and \$2130 respectively for the three vehicles. The initial repair record evidently has indicated that the first automobile is better than either of the other two. Subsequent repairs tend to confirm the early evidence because by 80,000 miles the first vehicle has a repair limit of \$770 while the others have dropped to \$180 and \$130 respectively. Eventually a time occurs when the potential cost of a necessary repair exceeds the limit and indicates that the vehicle should be

scrapped. As a matter of record, the latter two automobiles in Table 2 were sold while needing repair for around \$200 soon after 80,000 miles, but the first automobile was still giving satisfactory service after another 20,000 miles.

TABLE 2. RECORD OF REPAIRS AND REPAIR LIMITS FOR THREE FORD SEDANS

Year of mileage	Accumulated mileage (<i>U</i>) (miles)	Accumulated No. of repairs (<i>N</i>) (units)	Accumulated cost of repairs (<i>D</i>) (dollars)	Repair limits Exact (<i>W</i>) (dollars)	Approx. (<i>W</i>) (dollars)
1955	0	0	0	3000	3000
1958	20,000	13	101	2640	2670
1961	40,000	30	517	1010	1020
1964	60,000	47	768	950	970
1967	80,000	68	1152	770	790
1956	0	0	0	3000	3000
1958	20,000	2	133	1460	1430
1961	40,000	22	512	900	890
1964	60,000	47	1170	340	310
1967	80,000	70	1839	180	140
1958	0	0	0	3000	3000
1960	20,000	8	116	2130	2140
1963	40,000	40	798	520	510
1965	60,000	64	1385	220	200
1967	80,000	92	2026	130	100

A convenient approximate solution to equation (13) can be obtained by substituting the expected values for γ and λ into equation (7). This procedure corresponds to the use of estimators for γ and λ which are continuously modified as new repair information becomes available. The expected values for γ and λ are:

$$\bar{\gamma} \equiv \exp(\gamma | \text{data}) = (a + N + 1)/(b + D), \quad (14)$$

$$\bar{\lambda} \equiv \exp(\lambda | \text{data}) = (d + N + 1)/(e + U^2/2). \quad (15)$$

The substitution of equations (14) and (15) into equation (7) gives slightly different results from equation (13). The value for L must be \$0.0541 per mile to make the initial repair limit equal the net purchase price of \$3000. Table 2 shows that by 80,000 miles the limits are \$790, \$140 and \$100 respectively for the three vehicles. These values are sufficiently accurate for practical use because they lie within a few dollars of the exact solution.

When the expected cost of a repair is much lower than the value of the vehicle, equation (7) becomes:

$$\begin{aligned} (W | \gamma, \lambda) = & \lambda A^2/2\gamma - LA + \gamma L^2/2\lambda \\ & + (1/\gamma) \ln \left[\frac{L\gamma}{\sqrt{\lambda}} \int_{x=\sqrt{\lambda}(A-\gamma L/\lambda)}^{\infty} \exp(-x^2/2) dx \right]. \end{aligned} \quad (16)$$

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Substitution of equation (16) into equation (13) consequently only requires numerical integration for the last term. Furthermore, little error is generated if the expected values from equations (14) and (15) are substituted for the γ and λ terms within the logarithm. Therefore equation (13) becomes:

$$\exp(W|\text{data}) = \frac{(b+D)(d+N+1)A^2}{2(a+N)(e+U^2/2)} - LA + \frac{(a+N+1)(e+U^2/2)L^2}{2(b+D)(d+N)} \\ + \frac{(b+D)}{(a+N)} \ln \left[\frac{L\bar{\gamma}}{\sqrt{\lambda}} \int_{x=\sqrt{\lambda}(A-\bar{\gamma}L/\bar{\lambda})}^{\infty} \exp(-x^2/2) dx \right]. \quad (17)$$

Equation (17) gives identical results to those of equation (13) for the examples in Table 2.

The absence of a complete record of repairs raises no difficulty in the analysis. In the case of uncertainty about γ , any N repairs costing a total of D dollars can be used in equation (11) if the decision to record each repair was not based on its cost. The data need not start with a new machine, nor need it cover a continuous period. In a similar manner, U^2 in equation (12) can be replaced by the sum of the differences between the square of the ages bounding any N intervals between repairs. Furthermore, the N associated with d in equation (12) need not be the same as the N associated with a in equation (11). If different, they should be suitably labelled in the repair limit formula of equation (17). Complete records are preferable, of course, because they provide the broadest base for inferring the quality of the equipment.

The use of a proportional function for the expected frequency of repair and a stable exponential distribution for the costs simplifies the calculation, and should be sufficiently accurate for most equipment replacement decisions. Nevertheless, the accumulation of more data may lead to refinements that more faithfully reflect the general characteristics of equipment repairs. No difficulty arises in substituting $G(A)$ for $U^2/2$ in the frequency expression, as has been outlined in previous paragraphs. Modification of the cost function, however, probably necessitates a numerical solution for the differential equation corresponding to equation (5). (The incorporation of obsolescence, rate of return on investments, etc. would have a similar effect.) If repair costs were represented by an age-dependent exponential distribution, for example, the differential equation becomes:

$$L = -\frac{dW}{dA} + \lambda g(A) \{1 - \exp[-\gamma WH(A)]\} \frac{1}{\gamma H(A)}, \quad (18)$$

where

$$\text{Pdf}(S|\gamma, A) = \gamma H(A) \exp[-\gamma SH(A)]. \quad (19)$$

Uncertainty about γ can be represented by a gamma function as before, with the incorporation of cost data substituting $\sum_i^N S_i H(A_i)$ for $D (= \sum_i^N S_i)$ in

equation (11). The introduction of uncertainty about γ and λ into the repair limit formula again proceeds numerically in equation (13), the function $(W|\gamma, \lambda)$ having been obtained in tabular rather than algebraic form by numerical integration from $W \rightarrow 0$ at $A \rightarrow \infty$ back to $A = 0$. The procedure for determining L is unchanged. Finally, the previously described methods for obtaining approximate solutions can be used if a suitably large constant age A is inserted in only the exponential term of equation (18) so that an algebraic solution becomes available.

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