



Using Semi-Log Scales

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Article citation:

[Gaffney, Mason]. "Using Semi-Log Scales." N.d. Study guide.

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Using semi-log scales

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An excellent way to visualize compounding, or exponential growth,¹ is to plot the growth curves on a semi-log scale, with n on the abscissa and F on the ordinate. The origin of the scale is where $n=0$, but not where $F=0$. Rather, it is where $F=1$.² On such a scale, equal vertical rises represent equal percentage rises. That means every curve of constant percentage growth is a straight line. Compare this with the Bob Hope ski-nose curve you get when you show exponential growth on a common scale.

You can make your own semi-log scale from a common scale. Just label the ordinate origin as 1, and equal rises on the ordinate as 2, 4, 8, 16, 32, 64, etc. When you get up in the clouds of 1024, 2048, etc., you'll get an awesome feel for the power of compounding. Money doubles from 2048 to 4096 just as fast as from 1 to 2 (if you should live that long, and save that much).

The slope of the line is in proportion to the percentage growth rate. Thus, doubling the percentage growth rate doubles the slope. That means you reach any given F -value in half the time if you double the growth rate. Growth curves with different growth rates, starting from the common origin, diverge like rays of a sunburst as n rises.

The vertical distance between them increases with n . This shows that their ratios -- not just their absolute differences -- change with n . A higher ray becomes an ever higher multiple of a lower ray. (For constant multiples, draw parallel lines. They all have a constant ratio: e.g., $.5y$ is parallel to y ; so is $2y$.)

Discounting on the scale is the mirror image of compounding. Instead of growing the curves "decay" exponentially, forming straight lines sloping downwards to the right, starting from the origin. The changes are negative, shown by the negative slope. The absolute changes are much smaller, but the percentage changes are the same, at corresponding points.

¹Remember, "exponential growth" is the growth of an exponential function when the exponent is the independent variable, and the base is constant. It's different from the Cobb-Douglas function, where the exponent is constant and the base grows.

²You get the same effect by plotting the logs of the growth curves on a common scale, but here we just treat the semi-log scale.

Compare these downsloping rays with the discount curve you drew on a common scale. (Don't try to add up the areas under these curves on semi-log paper.)

These decay curves show the effect of interest rates on present values. Higher interest rates selectively downvalue future values in proportion to their futurity. The higher is n , the more sensitive values are to i .

Make yourself a semi-log scale by the doubling method: equal distances on the ordinate are labelled 2, 4, 8, 16, 32, ... etc. Mark off the abscissa in 10s: 10, 20, 30, 40, ... etc. Draw a straight line showing the growth of money at 7.2% - it doubles every 10 years.

Now stretch your mind a bit, and swallow this concept: the curve you just drew is the "locus of all points whose PV = 1, when discounted at 7.2%." Got it? Discounting is just compounding backwards. You started from 1, so discounting takes you back there. Sums on the curve, promised or expected at various dates in the future, all have the same present value.

So what?, you cry. Here is what. Let i fall from 7.2% to 5%. Now each point on the line has a different PV. Take a few, discount them at 5%, and draw a straight line from the point to the ordinate. For example, your first curve equals 4 when $n=20$. At 5%, the PV of 4 is no longer 1, but 1.51. Take $F=8$ when $n=30$. The PV of 8 at 5% is no longer 1, but 1.85.

Draw in these new curves carefully. When you've done it right, as you finally will, the new curves are all parallel to each other, and of lesser slope than the original curve. The future amounts that all had the same PV now have different PVs. The farther future they are, the higher their PVs.

Thus, a change in i reranks investments. A fall of i makes future ones look better. Conversely, of course, a rise of i makes nearer ones look better. Show this by repeating what you just did, but with $i=9\%$. You'll get parallel lines again, but steeper than the original curve. Don't be alarmed if they plunge down pretty deep before hitting the ordinate. Remember, the ordinate below 1 goes down in equal distances to $1/2$, $1/4$, $1/8$, and so on, and never reaches 0.

What you have now should look something like the feathered end of an arrow, so I call this the Arrow Diagram (no relation to the economist Kenneth Arrow). Its point is that the sensitivity

of PVs to i rises with the futurity of the value being discounted.

You can also visualize the effect of changes in the up-front cost of an investment. Suppose that what originally cost you \$1 rises to \$2, but you can't sell it for any more in the future. How does that affect the return on your investment? Go up the ordinate to \$2 and draw straight lines out to points on the original curve. You'll see that you now get negative returns on short investments, but still get positive returns on long ones. As before, the converse is also true. Over to you.

5. The use of semi-log scales, introduction.

~ Semi-log scales are very handy to use in learning to understand and work with compound interest and discount. Here are some of their advantages and characteristics.

~~ a. They keep the small and the large numbers on one sheet of paper, without losing accuracy for the small numbers.

~~ b. They help us analyze sensitivity, our first and present purpose, in percentage terms. "Sensitivity" means how much a function changes in response to changes in its independent variables.

~~ c. The curve of exponential growth becomes a straight line on the semi-log scale. The percentage growth rate is i . The slope of the straight line is proportional to i , so that doubling i also doubles the slope of the line. This is obviously true for $n=1$, and since the lines are straight it is true throughout. Each time n increases by one, y increases by iny (you can also call that 100% of yn , which says it with some excess motion, but yet may be clearer by virtue of long usage.)

~ The slope of the curve is always iny . It is straight because iny/yn remains constant throughout the curve. That is the slope divided by the ordinate, which is the percentage growth rate. It forms a straight line because on the semi-log scale a given vertical distance always represents a given percentage growth from its base.

~~ d. Figure 3, "Growth of Money on a Semi-log Scale", shows two growth paths, at 7.2% and at 3.6%. Half the rate means exactly half the geometric slope, throughout. Half the geometric slope would mean half the ordinate on a common scale, but on the log scale it means a decreasing fraction of the ordinate as n grows. Thus at $n = 20$ the lower curve is $2/4$ or $1/2$ the upper curve. But at $n = 40$ it is $4/16$ or $1/4$ of the upper curve. At $n = 80$ it is $16/256$ or $1/16$ of the upper curve.

~ If you want to see a curve equal to half the upper curve at all points, it will be parallel to the upper curve, and its y -intercept is at $1/2$. It intersects the $1.036n$ curve where $n = 20$. See (6,a,ii) below for more on this.

~~ e. (d) means that cutting i in half doubles the doubling time from 10 to 20 years, as shown on Figure 3. For any future value, F , $ni = k$, a constant. (For those who care, the constant

is $\ln F$ + a very small fudge factor which we needn't get into here.) We have been working with $F = 2$. $\ln 2 = .69$, to which we add a small fudge factor of .03 to get our rule of 72.

~~ f. $1.072n$ is the locus of all values whose present value (PV) is one when discounted at 7.2%. That just says that discounting is backwards compounding. But this is an approach to sensitivity analysis. Discount a number of points on the upper curve back to the present at a lower discount rate, e.g. 3.6%. This means drawing a set of lines parallel to the lower curve, from points on the upper curve, back to the ordinate. They all have different y-intercepts now, which are their present values. The further to the right they are, the higher are the y-intercepts, showing they have been increased by higher percentages. Thus, for $n = 20$, PV rises from 1 to 2, or by 100%. For $n = 40$, PV rises from 1 to 4, or by 300%.

~ So if you expect interest rates to fall in the future, invest today in long maturities and you will really cash in -- if you are right. If you expect rates to rise, stick with the fast payout, or with cash. If you are really cocksure, and a gambler, don't settle for just not buying long maturities, but sell them short. This last move calls for deep conviction, and/or a mood of reckless abandon. We do not speak here of the psychology of it, but only the mathematical framework.

~~ g. Using the semi-log scale gives the same curve as graphing the log of a function on a common scale. This tells you a lot about logs and how they behave, and what they mean.

~ A rule of algebra is that $\ln I^n = n \ln I$. Thus for an exponential function the graph of the log is a straight line, since $\ln I$ is a constant, and n is the abscissa. The straight line starts from the origin, just as on the semi-log scale. ~

~ A very useful and close approximation is that $\ln I = i$. The note explains why, but you may take it on faith if you prefer.* Thus the curve of the log of I^n is simply ni . Every time n increases by one, the curve increases by i . So i is not only proportional to the slope, i is the slope. This makes it clear why the slope increases directly with i .

~~ h. It follows from (g) that $d \ln y / dx = dy / dx / y$. Which is to say that the absolute increase in the log is the fractional increase in the function. That's the way the semi-log scale

works, we have seen; and now we see that the log on the common scale works just the same.

~ That is not just the way the graph works; that is the way that logs themselves work.

~~~ Thus, using a semi-log scale you quickly get a secure base for working with logs themselves. Understanding logs in some fundamental sense will come more slowly, as you work with them. But you can work with them right away, that is the important thing. When in doubt, check what you are doing with the semi-log graph. It will not mislead you.