Elasticity of Production
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Elasticity of Production (ϵ), and Playing the Percentages

1. Definition

Elasticity of production, *epsilon*, is denoted as "ϵ".

ϵ is the fractional growth in the product, P'/P, from a small fractional increment of input, X'/X:

$$\epsilon = P'/P \div X'/X \qquad (1)$$

To simplify, ϵ is also rewritten this way:

$$\epsilon = P'X/X'P \qquad (2)$$

Expressions with ϵ are simplified by noting that $X'=1$, because X is the independent variable. Then, from (2):

$$\epsilon = X(P'/P) \qquad (3)$$

For those who know calculus, ϵ is the ratio of two log derivatives. That makes it a fairly complex concept. It is two steps beyond a simple derivative - an idea that beginning calculus students find hard enough when taken alone. P' is a derivative (P' is short for dP/dX). P'/P is a log derivative. So is X'/X (which simplifies into $1/X$). The log derivative is simply the derivative divided by the variable itself.

Even so, ϵ is fairly complex (but see #2, below, which makes it simpler). Don't be dismayed if it takes a while to wrap your mind around ϵ. Its basic purpose is not to complicate life, but simplify it. It is a way of uncluttering equations, of organizing fairly complex thoughts in a compact, mnemonic, manipulable package. It helps us see the relationships of things. That will be our persistent goal in using it.

2. ϵ as MP/AP

Rearranging terms in (1) yet again,

$$\epsilon = P'/X' \div P/X = MP/AP \qquad (4)$$

This (4) is a useful, simplifying, and insightful relationship. Suddenly, this mysterious ϵ relates to something we know well, from way back. From this, we can infer values of ϵ

at any point on a traditional graph of a production function. The values are approximate in between critical points, and exact at those critical points.

Location on Graph	Value of ϵ	Comments
Stage I	>1	MP>AP. Increasing average returns to X; (implies negative marginal returns to fixed factor)
Max AP of X	1	Constant returns to X
Stage II	1>ϵ>0	Diminishing returns to X
MP = MC	MC/AP	Optimal combo
Stage II,B	<MC/AP	MP<MC. Incurring marginal losses on each increment to X. Stage of negative marginal rents.
MP = 0	0	Zero marginal returns to X; max AP of fixed input
Stage III	<0	MP<0. Negative returns to X; (implies increasing returns to fixed factor)

We can also infer values of MP from known values of AP:

$$MP = AP \times \epsilon \qquad (5)$$

ϵ is only meaningful and valid at a point, since its value changes from left to right on the graph. It is tolerably stable for small changes, especially where AP is stable. It is not valid over a long reach, and certainly not valid way back to the origin.

IMPORTANT READING INSTRUCTIONS: ***NOW SKIP SECTIONS 3 & 4. GO DIRECTLY TO #5***.[1] Treat #3 and #4 as a technical appendix.

3. Exponents as values of ϵ

A mathematical insight from this technique is that an exponent is an elasticity, when the base is the variable. Likewise, once you know ϵ you can write the production function using ϵ as an exponent of the variable input. Anything that helps you deal with logs and percentages is bound to be useful sometime, so we'll demonstrate it here. To show this, we write

[1] If you pass "GO" collect $200.

the production function in exponential form. See text after Equation (12) for more on this.

$$P = kX^{\alpha} \qquad (6)$$

(where k and α are "parameters"[2])

$$P' = k\alpha X^{\alpha-1} \qquad (7)$$

But from (6), $P/X = kX^{\alpha-1}$, so

$$P' \div P/X = \alpha \qquad (8)$$

Remember, from (3) and (4) that $P' \div P/X = \epsilon$, and $\epsilon = MP/AP$, so:

$$\alpha = \epsilon = MP/AP \qquad (9)$$

That can also be done with log derivatives. First, take the log of both sides of (6):

$$\ln P = \ln k + \alpha(\ln X) \qquad (10)$$

Differentiate both sides w.r.t. X (the log derivative of y is y'/y):

$$P'/P = \alpha\, X'/X \qquad (11)$$

$$\alpha = P'/P \div X'/X = \epsilon \qquad (12)$$

As aforesaid, once we know ϵ and <u>k</u> we can write the production function in the exponential form (6). (6) is good shorthand for expressing something about production functions.

It is a question, though, how close it comes to representing reality on a broader canvas. (6) only applies for small changes of X. Those who depend too narrowly on it run the danger of

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[2]You could also say they are "constants." The word "parameter" means a number that is constant for the time being, but may be changed later, as the facts change. π is a constant that is not a parameter, it never changes.

forgetting that ϵ changes as X changes. They are prone to assuming constant values of the exponents "for simplicity," or "to make the equations more tractable to solving," or "mathematical elegance," or "rigor," or other bad reason. Those are various ways of saying they assume constant values so they can publish an article, or join some clique that mumbles in an exclusionary inside lingo.

The only good reason for using constant exponents would be that it was true; but in general it is not true. See the papers on "The Fallacy of Fixed Proportions," "What is Economic 'Intensity'," "Quantum Leaps and Ecotones," "Contrasts between Large and Medium-sized Firms," "Falling Property Tax Rates and Rising Concentration" (on Reserve), "Complementation, Scale, and the MNP," "The Fallacy of Petty Incrementalism," "Substitution," and "Saving Labor by Wasting Land." Some people very likely use constant elasticities for the very purpose of guiding their readers away from the points brought out in those readings. Not everyone who writes in the idiom of economics is trying to help or edify his readers: quite a few are hired by lobbyists to addle your brains. Don't be victimized.

While we're on exponents as elasticities, a useful application is with the compound interest function. See the paper on "The n% Rule of Thumb."

4. The Cobb-Douglas Function and its limitations

If ϵ remains constant over a considerable reach, then the production function may be written as an exponential function, with ϵ as the exponent. This is the technique used in the "Cobb-Douglas Function," and many derived theorems.

If there are two variable inputs, X and Y, and both may vary, then the Cobb-Douglas function looks like (13):

$$P = kX^{\alpha}Y^{\beta} \qquad (13)$$

When a firm has achieved constant returns to scale, it may be shown (and is intuitively obvious) that $\alpha+\beta = 1$.[3] This is

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[3]If they summed to more than one it would mean you could raise your inputs, in their present ratio, by 1%, and thereby raise output by more than 1% - a prospect so delightful that you'd continue expanding until you got through this stage of increasing returns to scale.

useful in showing that when $\alpha=1$ (constant returns to X, max AP of X), then $\beta=0$ (zero marginal returns to Y). Cf. #2, above; and see the papers, "Wasting Land by Saving Labor," and "Euler's Theorem."

(13) has a feature that many find simplifying, it is "linear in the logs." Take the log of both sides, and you get (14):

$$\ln P = \ln k + \alpha \ln X + \beta \ln Y \qquad (14)$$

I am not sure what is so attractive and useful about that, but here is one possible reason. Holding Y constant, and differentiating (14) w.r.t. X, we get:

$$P'/P = \alpha/X \qquad (15)$$

Solving for P':

$$P' = \alpha \times P/X \qquad (16)$$

We already knew that anyway, from (5). (P' is MP; P/X is AP; and α is ϵ.) Still, it is good when "all paths lead home," because inconsistent results ring alarm bells and force us to drop what we are doing until we find the error. If (14) turns you on, be my guest. However, there are reasons I do not pursue it as much as some other economists do.

Cobb-Douglas is fraught with peril, because ϵ changes its value when one varies the ratio of X to Y. Many analysts use it, nevertheless, on the premise that the exponents may be kept constant by adding X and Y in the same proportions. It is doubtful, though, if firms actually expand that way: every germane data source shows factor proportions changing with scale. See the papers cited in the second paragraph after (6), above.

In addition, even if the values of ϵ were to remain constant over a long reach, the arc-ϵ is different from the value of ϵ at a point.

You should know that Cobb-Douglas exists, and that some theorists love to use it. Some abuse it. For some, it is a another toy for the sandbox - yes, if you haven't noticed already, some academicians get lost playing little useless games. For some, the sand is handy to throw in the eyes of people they are bamboozling. For some, it is a useful tool. If your math is well-practiced, and you can temper mathematical insight with practical judgement, Cobb-Douglas affords some useful insights.

Just don't get carried away and let it substitute for reality. In this course I make little use of Cobb-Douglas, except to illustrate Euler's Theorem. See paper on "Euler's Theorem."

5. Geometric Analogy between finding MP from curve of AP, and finding MR from curve of AR

The geometric relationship of MP to AP is same as that of MR (Marginal Revenue) to AR (Average Revenue or Price) in price theory, a more familiar topic to most students of economics. However, elasticity of demand is defined in a way that is not analogous to elasticity of production, so that the geometric analogy does not translate into an algebraic analogy. The purpose of this section is to show how to fit the algebra of production to the geometry.

As to demand, $MR = AR(1-1/\epsilon)$ (17)

As to production, $MP = AP \times \epsilon$ (5, repeated)

A. The "slice and slab" method of finding MP

i. The geometrical picture

Draw a curve of AP, of negative slope. Select any point on the curve; call it AP_1. It determines a rectangle whose sides are the coordinates of the point. Draw it in. The area of the rectangle is the total product, P. Now select another point, quite close, downwards to the right; call it AP_2. Draw in its rectangle, too. The change in P is the change in the areas of the rectangles. The math convention is to denote such a finite change as "ΔP". (P' is conventionally used as an abbreviation for the derivative, or instantaneous rate of change.)

Compare the rectangles. Most of their areas are in common (overlapping), but #2 contains a vertical slice on the right side, lacking from #1; and #1 contains a horizontal "slab" on top, lacking from #2. ***ΔP is the slice less the slab.*** For a positive ΔP, the slice must be greater than the slab. P is a max when $\Delta P = 0$, and the slice equals the slab.

The value of the slice = AP, where $\Delta X=1$. The value of the slab is $X \times \Delta y$ (where X is on the abscissa, and "y" is a generic indicator for the ordinate). The trick is finding the value of Δy. Three methods follow, the first being a simple numerical example.

ii. A numerical example

You can plant ten fruit trees per acre, and each will yield 400# of salable fruit. Or, you can plant 11 per acre, crowding the trees, and each will yield 370#. What is the marginal product of the 11th stem?

In this case, the slice is 370#, and the slab is 10 x 30 or 300#, so the MP is 70#: the slice less the slab. ***BEWARE!*** The MP is not the 370# yield that is physically produced by the 11th tree. Do not consider the 11th tree just in isolation![4] To identify the MP with the yield of the 11th tree is to miss the whole point of what MP means. Remember, MP is a systems concept. You have to consider the total effect on the system.

Confirm this result by figuring the two total products. With 10 stems @#400, P_1 = #4,000; with 11 stems @#370, P_2 = #4070. P_2 less P_1 = #70.

Notice that in this example, the abstract areas on the graph correspond with physical realities: things we can see, touch, and count. There is no mystery; it's common sense. The idea of Managerial Economics is to convert the abstract principles of theory into real things that are concrete and comprehensible.[5]

In Micro Theory you learned price theory, probably better than you learned production theory. In price theory you learned why Marginal Revenue (MR) is less than Average Revenue (AR) when the demand curve has a negative slope: you sell more, but make less on each item otherwise sold at the original higher price. Marginal Revenue is the new sales less the loss on each old sale. Marginal Product is the same idea, but here it is more trees bearing, and you get less output from each tree.

iii. Reasoning from ϵ

The "slab" must equal the difference of AP and MP:

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[4]In exams, I may trick the unwary by luring them into this error - don't let me get away with it at your expense.

[5]Until you do so, theory is not much use. It may not even be faithworthy. There are lots of fakes and bluffs out there: they manipulate arcane symbols solemnly and intimidatingly, but fail the test of applying their results to anything real. Distrust sham and mumbo-jumbo, even if someone calls it "rigor."

$$AP + X\Delta y = MP \qquad (18)$$

Therefore,

$$X\Delta y = MP - AP = AP(\epsilon-1) \qquad (19)$$

In Stage II, Δy is negative, so the positive numerical value of $X\Delta y$ is:

$$-X\Delta y = |X\Delta y| = AP - MP = AP(1-\epsilon) \qquad (20)$$

When $\epsilon = 1$, the slab = 0, and AP = MP. This is the beginning of Stage II, when AP is maxed. This is also where the AP curve remains flat-topped as X grows, so $\Delta P/P = \Delta X/X$. That is, a 5% rise of X means a 5% rise of P, and so on. This is a reach wherein returns to X are constant. If it continues for any substantial change in X, we could call it a "Stage" of Constant Returns. It would fall between what are conventionally called "Stage I" (increasing returns) and "Stage II" (diminishing returns).

As you move to the right through Stage II, the slab gets wider, and may get deeper too (depending on the slope of the AP curve), and ϵ moves down toward zero. When ϵ finally reaches zero, the slab equals AP, and MP = 0.

iv. Algebraic generalization

Here, to find the change in $\underline{y}$, we fall back on the rules of calculus. (For those without calculus, sorry, but I need this rule!) This means we stop calling the change "Δy", and start calling it "y'". y' is not self-evident because $\underline{y}$ is not self-explanatory, not a single-valued variable. Rather, $y = P/X$, where X is a variable measured on the abscissa.

$$y' = d/dx[P/X] = [XP'-X'P]/X^2 = P'/X - P/X^2 \qquad (21)$$

(remembering that $X' = 1$)

$$Xy' = P' - P/X = MP - AP \qquad (22)$$

Solving (22) for MP,

$$MP = AP + Xy' \qquad (23)$$

AP is the slice, and Xy′ is the slab. So, MP = the slice less the slab (recalling that y′ is negative after Stage I).

Geometrically, this is the same way we find MR from AR in demand analysis. However, (23) is consistent with MP = AP x ϵ, whereas MR = AR [1 - 1/ϵ].[6]

B. Finding MP by the method of fractional changes

Refer to the rectangles you drew in part "A". They represent the product, P. Let "y" represent the ordinate, as before. Then P = Xy. The growth of P is the growth of Xy.

There is a calculus product rule for this:

$$P' = yX' + Xy' \qquad (24)$$

yX′ is the slice; Xy′ is the slab. The calculus rule jibes with the slice and slab approach.

It is simplifying and therefore useful to put (24) into terms of fractional changes. Since y = P/X, and X = P/y, we may substitute those into (24) and rewrite it:

$$P' = PX'/X + Py'/y = P[X'/X + y'/y] \qquad (25)$$

(25) is the same as (18) but with Δ replaced by derivatives. As an exercise, show that the equations are the same - you have to translate AP into P/X, etc., and remember that X′ = 1. We found (18) by a different method, yet it is consistent: good sign!

For a simple, useful, symmetrial and mnemonic result, divide both sides of (25) by P:

$$P'/P = X'/X + y'/y \qquad (26)$$

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[6] ϵ of demand is defined with Q′ on top, where Q is on the abscissa. ϵ of production is defined with P′ on top, where P is the product of ordinate and abscissa. Naturally, the algebra works out differently.

(26) says the fractional growth of P equals the sum of the fractional changes in its two factors (X and y). Remember, though, that in Stage II, where the AP is falling, $y' < 0$. In this case a "sum" means you subtract $|y'/y|$.

(24)-(26) are compatible with (18). That means we could have derived (26) from (18). Let's do it. (18) says that $P' = P/X + Xy'$. (P/X is the slice, and Xy' is the slab.) Show that the right side leads to (25).

$$P/X + Xy' = P[1/X + (X/P)y'] = P[1/X + y'/(P/X)] \quad (27)$$

Recall that $X' = 1$, and $P/X = y$, so:

$$P[1/X + y'/(P/X)] = P[X'/X + y'/y] \quad (28)$$

Q.E.D.

(28) shows that the fractional method and the "slice less slab" method can be viewed as two different ways of seeing the same point, or saying the same thing. Truth has many faces. Now we see it one way, now another, but if what we have is really Truth, the different viewpoints are mutually translatable.

You can fool around with (26) and show that it reduces to what looks like a tautology: $P' = P'$. That does not show it is empty, however. Rather, it shows it is consistent with other ways of making the same point. It is when they are not consistent that there is an error somewhere.

C. Discrete fractional changes (quantum leaps)

Test (26) with some numerical examples. Start with X=100 and y=100, and P=10,000. Does 99 x 101 also = 10,000? Not quite: it's 9,999. Try 95 x 105: that's 9975. Still close enough for practical purposes, but something is not quite right. What's wrong? Remember, we began with a calculus rule. Calculus is a device for showing how such approximations approach absolute precision as the changes in the independent variable approach zero. Otherwise they are a shade off. This is where the slice and slab method is superior: it does not depend on calculus.

A shade of difference is not enough to worry us in the brawl and rush of the management world, with its many rough measures and hunches. Now, however, try 50 x 150. This = 7500. Whoa,

Nelly! Now we have more than a shade of difference. Try 0 times 200 and we have a disaster.

Here is how to handle discrete jumps. Forget the calculus. Take a generic rectangle with B=Base, H=Height, and A=Area.

$$A = BH \qquad (29)$$

Define a as $\Delta A/A$, b as $\Delta B/B$, and h as $\Delta H/H$. Then:

$$A(1+a) = B(1+b) \times H(1+h) \qquad (30)$$

Divide equation (30) by (29), and:

$$1+a = (1+b) \times (1+h) = 1 + b + h + bh$$

$$a = b + h + bh \qquad (31)$$

This is the same as (26), except for the bh. Now if we go from 100 x 100 to 50 x 150, we get:

$$a = -.5 + .5 - .25 = -.25$$

That is a drop of 25%.

If we go to 0 x 200 we get $a = -1 + 1 - 1 = -1$

That is a drop of 100%.

Note that when the changes are equal and opposite, b and h wash out, leaving us with bh.

To maximize A, you want a to be zero. When a=0:

$$b = -h(1+b)$$

$$h = -b/[1+b] \qquad (32)$$

When b is very small, 1+b ---> 1, and you get the simple calculus rule again.

6. Dissipating Rent by Wasting Capital and Work

A. Dissipating rent

See the paper "Saving Labor by Wasting Land." You can also save land by wasting labor. Such "saving," however, may be carried into Stage III, where the MP of work is negative. Think what that means: you actually produce less per acre by applying more work per acre. You're wasting both work and land, at the same time.

This can happen where lands are unfenced, as on an open range. The open range, to be sure, was a transient episode in the 19th Century, and much overblown in horse operas. Open fisheries, however, are quite common up to the present, and are only now being phased out in national waters. Look at each fishing boat and crew as the unit of the variable input applied to a fixed fishery resource. So long as the AP per boat exceeds the AC per boat, and the boats are independent, more boats will come. They will keep coming until the AP is forced down to the AC. This phenomenon ends up destroying the entire rent of the fishery: the cost of fishing expands to equal the gains, with no surplus.

The crowding that results has been given two names, at least: "The Dissipation of Rent"; and "The Tragedy of the Commons." Ask yourself how it got those names, and how they relate to the relationship of the slice and the slab, as discussed above in Section #5,A. I am not going to explain this in detail: you can now figure it out, and should do so. The individual shipowner is motivated solely by the slice, and ignores the slab. A major shipowner with a monopoly of the fishery would consider both. I will not draw you a picture now, but say this: ***YOU SHOULD UNDERSTAND THE FOREGOING SENTENCES, AND BE ABLE TO EXPLAIN THEM ON AN EXAMINATION.*** It's a matter of translating the theory into a practical situation.

A firm that owns the entire fishery, and can exclude interlopers, would manage it differently. It would consider the "slab," not just the slice. Such management is traditionally called "rationalization."

B. Tenure vs. market monopoly

Tenure over resources, with resulting downsizing of the labor force applied, can be justified economically. Now consider an extension of the same idea. Suppose there is just one fishery, and the firm that gets it gets not just tenure, but a market monopoly. In the market monopoly, it controls quantity and price. That means its Marginal Revenue is less than its Average Revenue.

Market monopoly is something like having tenure over a market. Geometrically, this is because Average Revenue is a slice, while Marginal Revenue is a slice less a slab. Economic theory tells us that monopoly is anti-social, and sellers should ideally be motivated just by the slice (price of additional sales), and not by the slab (lower price on existing sales). Thus, orthodox theory says that tenure is socially good, while monopoly is socially bad. As a test of your understanding, explain why that is true. I am not going to do it for you here: you should now be able to do it yourself.

C. A Bias among Economists

The ordinary neo-classical economist these days is a conservative, property-oriented person who loves to make point "A", above. In doing so, most of them overstate it. They favor subjecting all natural resources to private tenure, then leaving everything else to market forces. They shut their eyes to the fact that private owners of resources often underuse them, laying off workers and aiming at maximizing the AP of labor. Thus, they commit the fallacy of Saving Labor by Wasting Land - see paper of the same name. Saving Labor by Wasting Land leads us to the dissipation of rent by another route: underuse. Underuse is as bad as overuse, and in many ways worse. Underuse throws people out of work.

Just why owners given tenure over natural resources tend to underuse them is a big subject, that we deal with elsewhere. Suffice it to say here that the usual treatment by economists these days is unbalanced, in my opinion. I am telling you about the "Dissipation of Rent" not because I think it is our major problem, but because it is a good teaching device to help you understand the meaning of marginal productivity. Our major problem these days is people who overstate the case for privatizing resources. That, however, is a topic too big to pursue here.

7. Finding and Maxing the Bottom Line (Rent)

A. Why the bottom line is rent, not profit

I have mentioned elsewhere that when you equate Marginal Product and Marginal Cost you are maximizing Rent. Rent is the return to the fixed factor. It is not the same as "profit." Profit is a word that belongs in the analysis of personal distribution. It is maximized by owning all the inputs

in sight. In Managerial Economics we are concerned with putting resources to the best use. To do so we impute returns to the resources as such, regardless of who owns them.

B. Maximizing Residual Rent (RR)

$$RR = P - C = X(P/X - C/X) = X(AP - AC) \quad (33)$$

Let $y = AP$

$$R' = X'(y - AC) + Xy' \quad (34)$$

$R = \max$ when $R' = 0$

$$X'(y - AC) = -Xy' \quad (35)$$

Substituting from (22):

$$X'(y - C/X) = P/X - P' \quad (36)$$

Recall that $X' = 1$, and $y = P/X$

$$P/X - AC = P/X - P' \quad (37)$$

$$P' = AC = MC \quad (38)$$

Thus, expectedly, R is a max when $P' = MC$. This point is graphically obvious, but it is good to have some confirming algebra.

As we have seen, there are other methods of approaching this proof. By now, however, we should have confidence in the various methods and their consistency.