



Net DCF of Generation in Series: Faustmann's Formula and the Land Value Multiplier (LVM)

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Faustmann's Formula and the Land Value Multiplier (LVM)

Our objective here is to find the net PV of a long series of lumpy future values: values that come along every few years or so. This sounds hard, and can be made so. However, using the PIPO formula, we may discover the formula easily. The route will seem indirect at first, but it's a short cut you will learn to appreciate. Following Einstein's advice, the purpose is to "make things as simple as possible, but not moreso."

Notation:

K is capital (from the German Kapital). It has a finite life, through consumption of the corpus, or through depreciation and obsolescence. Think of it as the original cost of planting a tree (or an acre of them, or a forest).

S is Site value, i.e. land. L usually stands for Labor, so we need another letter for land.¹ Land differs from capital in about 20 or more specific qualities, but the one most relevant here is its permanence. Land has infinite life, except for exhaustible resources. (See the paper on exhaustible resources.) That is, after the tree is cut and removed, the Site remains to use again. It is like a catalyst in chemistry: a material that is required to make a process work, but is not used up in the process, not incorporated in the product, not carried away with it. Nevertheless it is "consumed" in the sense that the process uses up a valuable slot of its time; and time on the job is all it has to offer.

Both K and S are measured in value terms. Subscripts refer to dates. Lacking a subscript, time zero is meant.

What we have previously called P, for Present Value, is now divided into two parts, K and S.

¹You may also think of S standing for Soil, or the Spanish Suelo. Other notations commonly used are T, from the French Terre, Spanish Tierra, and Latin Terra; G, for Ground; and B, from the German Boden. I actually do use lower-case b for the annual equivalent of S. b is from the German Bodenrente. Your German prof. would consider it barbaric, since German nouns are all capitalized; but I sacrificed linguistic purity for notational consistency, since my notation uses lower case letters for flow values, and upper case for point values (remember?).

The annualized value corresponding to S is b (from the German *Boden*, soil). $b=Si$.

PIPO cases, with S constant

S is here treated as constant over time, in order to develop the basic math in simplest form. Having done so, you can have fun later playing with various scenarios about rising S . Falling S is not so much fun, but it happens and you'd better consider that, too.

PIPO cases generally deal with growing capital like a tree, or livestock raised for slaughter, which yield their value at one point of future time. " F " is used to denote that Future Value.² But remember that P is now gone, replaced by its two components, S and K . (However, we will see how the valuation technique developed here can be generalized to apply to assets yielding cash flow values, too.)

Compounding

$$F + S = [K + S]I^n \quad (1)$$

You put in S at the start of the cycle, and take it out at the end. It is, as we said, like a catalyst. It is not used up in production. Even so, it is valuable and must be paid for. Like the lawyer, its time is its stock in trade. Each crop of trees occupies a time-slot, and in that important sense "consumes" the site: its time-slots are all it has to give. Its value, S , is a kind of "dead weight" in (1), but it changes the necessary relationships among F , K , I , and n .

Solving for S , if K is given

Solving for S opens up new vistas. Up to now, we have thought of PIPO PV in terms of one cycle of investment and recovery. Actually, the world goes on forever, in sequential cycles,

²Sometimes you will see " H ", for Harvest Value. In forest economics literature you see " S ", for "Stumpage," but we are using S for something else, Site Value. (The foresters use B for site value. It is for Bodenerwartungswerte. Forest economics was developed in Germany, where those long compound words do not faze them.)

generation following generation. Faustmann's Formula deals explicitly with that.

Solving (1) for S:

$$S = \frac{-KI^n + F}{I^n - 1} \quad (2)$$

(2) is named the Faustmann Formula, after a German forester who discovered it in 1849, in the process of appraising forest land. Stringing together nouns in the German way, he called it Bodenerwartungswerte (soil expectation value). I refer to it as Faustmann - that is easier to remember than (2), or the German, or the prolix English equivalent still used in forestry. It also honors a very bright and useful person. Martin Faustmann was a forester, but this formula (with minor adaptations) applies to all chains of repeating cycles.

Faustmann is premised on there being many cycles of investment and recovery after the first one. In fact, that is how Faustmann first derived (2), which we have reached by a different route. For Faustmann's derivation, see the Appendix.

Note that (2) is the same as simple PIPO discounting of the numerator, except for that pesky little "-1" in the base. When $n \rightarrow \infty$, the "-1" is swamped and we have simple PIPO discounting. However, at the other extreme, when $n=1$, the base reduces simply to i, and we have the simple capitalization formula for a perpetual annual flow. In between, we have all the other cases, and need Faustmann to evaluate them numerically.

The PV form of Faustmann

It is generally useful to put Faustmann in the PV form. S is already a PV, and this change won't affect S. What it means is that we discount the numerator to the present, changing the date of the PIPO valuation from time n to time 0. We just multiply both top and base of the right side of Faustmann by the discount factor, I^{-n} , so the value remains the same - it just looks different. This change makes Faustmann more directly comparable with other valuation equations, which are generally based on discounting all future values to time 0.

$$S = \frac{-K + FI^{-n}}{1 - I^{-n}} \quad (3)$$

The top of (3) is the PV of the first cycle of land use. The bottom is a divisor that converts the PV of cycle one into a land value derived from all future cycles (including #1).

The "Land Value Multiplier"

Rather than divide, it is more accommodating to our habits of thought, and therefore more useful, to take the reciprocal, call it a coefficient or multiplier, and multiply. We'll even call it the "Land Value Multiplier" (LVM), which tells you exactly how to operate it. We'll also tabulate it as a factor, so the numerical values are right at hand.

Here are some LVM values, to give you a feel for it, when $i=.05$.

Table 1: Land Value Multipliers when $i=.05$.

n:	1	2	4	8	16	32	64
LVM:	21.0	10.8	5.6	3.1	1.8	1.3	1.05

For example, when $n=16$, cycles following the first one add 80% to the value of cycle one. That 80% is the "postprimary" element in Site Value - the part that comes from generations after generation #1. In general, the postprimary element = $(LVM - 1)$.

You can see why Herr Forstmeister Faustmann didn't want to overlook this factor when valuing forest land. Today, in the southeastern U.S., many stands of Southern Yellow Pine reach harvest age at 16 years. Nominal interest rates are, to be sure, higher than 5%, which would lower the multiplier. However real inflation-adjusted interest rates, the ones that matter, are more like 4%.

When we are valuing a commercial site used for, say, a fast-food stand, the life expectancy is about 8 years. Thus, at a 5% rate of interest, the postprimary values add 210% to the primary value of Generation #1. Otherwise put, S is 310% of the value of Generation #1. That is the same as 3.1 times the value of Generation #1. Be sure you understand why those are all ways of saying the same thing.

As an exercise in heavy thinking, explain to yourself why it makes sense to use real rather than nominal interest rates in figuring the LVM.

As an exercise in applied math, tabulate another string of LVMs, using 4% instead of 5%. They should all be higher than those in Table 1.

Value and Rent

Multiplying S by i we get annual rent. We need a notation for rent:

$$Si = b = \text{annual ground rent}^3 \quad (4)$$

We will soon be learning about the Installment Plan Factor (IPF), which you multiply times the principal of a debt to find the level annual equivalent. Note that in the top of (3) all values are present values. Multiplying by i (as in (4)) the rest of it becomes the IPF, so we have:

$$b = [-K + FI^{-n}] \times \text{IPF} \quad (5)$$

What (5) does is to take the irregular stream of outlays and revenues that we actually get, and convert them to a level annual amount, b , which is the annual value of land, or rent.

This is a yardstick for making the results of all land uses commensurable and comparable, so you can decide which is best. One may have a cycle of 4 years, another of 23 years, but (5) reduces all to a common measure and performance standard.

Rent or Buy?

³ b is from German Bodenrente. Your German prof may shriek, because German nouns begin with Upper Case. I sacrifice linguistic accuracy here to hew to my convention that flows are denoted by lower case letters. Upper Case letters are reserved for point values. I had to choose, and good German lost: I apologize to the foreign language department.

It would be nice to use r , for rent, but r is used elsewhere to mean ROR. One runs out of Roman letters after a while. Some turn to Greek, but it is, well, Greek to most of us today. Besides, we already are using the Greek rho, ρ , for the "force of interest."

You can use (5) to make the rent-or-buy decision. (5) gives you the annual holding cost of any asset lasting n years. The following problem (from an old exam) explains how to do it.

1. You have the choice of renting a house for \$10,000 per year (plus utilities, upkeep, insurance, and other expenses), or buying it for \$100,000, financed at 8%. You expect to live there for 12 years, then sell. To rent or to buy, that is the question. Which is cheaper, given the following resale expectations. Show your work, and give numbers.

- a) No resale value.
- b) Resale value of \$60,000
- c) Resale value of \$100,000
- d) Resale value of \$150,000
- e) At what resale value does the cost of owning just equal the cost of renting?

Answers:

Preparation. You will need these numbers:

$$1.08^{-12} = .3971$$

$$IPF = .08/(1-.3971) = .1327$$

a) $\$100k \cdot IPF = \$13,270.$

b) Starting here, remember the basic 1-2-3 sequence: Discount; Add or Subtract; Annualize.

$$(\$100k - \$60k \cdot .3971) \cdot IPF = \$76,173 \cdot .1327 = \$10,108$$

$$\$10,108 > \$10,000 \text{ (just barely)}$$

Alternative. $\$100k \cdot IPF - \$60k \cdot I^{-n} \cdot IPF = \$13,270 -$
 $\$60k \cdot i/(1.08^{12}-1) = \$13,270 - \$3,162 = \$10,108$

c) $\$100k(1-.3971) \cdot IPF = \$60,290 \cdot .1327 = \$8,000 < \$10k$

d) $(\$100k - \$150k \cdot .3971) \cdot IPF = \$40,435 \cdot IPF = \$5365 < \$10k$

e) Cost of owning = $a = (P - RI^{-n}) \cdot IPF$

Here you are given the cost of owning ($a = \$10k$), and asked to solve for R :

$$R = PI^n - aI^n/IPF = PI^n - a(I^n - 1)/i$$

Plug in known numbers:

$$R = \$100k \cdot 2.5183 - \$10k \cdot 1.5183 / .08 = \$251.83k - \$189.79k = \$62.043k$$

Purposes of exercise:

- 1) Show how to compare costs of ownership with costs of renting or leasing;
- 2) Show how to annualize using the 1-2-3 sequence: discount; add or subtract; annualize;
- 3) Keep you out of the rut of one routine, by asking you to solve for R instead of a.

References:

- 1) Lecture
- 2) Reading #50, on Annualizing with the IPF
- 3) This reading, on Faustmann's Formula, pp.3-5

Optimal life, and replacement timing

A most important by-product of Faustmann is determining the optimal value of n, which is the length of the cycle of investment and recovery. Herr Faustmann discovered he could not determine S without also determining n, so he found them simultaneously. Optimal n is the value of n that maximizes S.

We defer this interesting matter to later, under "Replacement Policy." See the paper on "When to Cut a Tree."

APPENDIX: How Martin Faustmann found his formula

In finding when to cut a tree ("Financial Maturity"), one thinks about the value of the site to grow the next tree. One also thinks about future uses of the capital tied up in the standing timber. A full solution takes both into account. This is a great example of optimizing -- defining a performance standard that combines two or more legitimate considerations. Often they are traded off. In this case they reinforce one another: one adds them together to get to total marginal cost of time. You balance them against current growth, which is the marginal product of time.

To bring the future of the forest site into the picture, a good way is simply to maximize PV in perpetuity. Maximizing PV automatically takes interest into consideration. Looking at a series of future cycles, rather than simply the first, automatically brings the future use of land into the picture. Here is how to do it.

As my students know by now, I use the following definition to unclutter the notation:

$$I = 1+i \quad (1)$$

To keep it uncluttered, and focus on the most interesting points, I assume here two simplifying conditions.

a. All generations after the first are to have the same life (n) as the first.

b. Planting cost = 0. Timber replants itself on this land without human cost. Foresters call it "volunteer regeneration" and often actually let things happen this way, although it is lazy forest management and seldom the best use of land. Better, we can include future planting costs in the cost of harvesting the preceding stand. That just means a lower value for each "F", since "F" is net of harvest costs. Since these outlays come in the same year, the effect on solutions is the same as though we entered replanting cost separately.⁴ We can easily enter planting cost separately later, when we shall have mastered the harder matters. (In the main text, we have already done so.)

⁴The main difference is that we will overstate the value of land that is currently bare, by omitting the first planting cost, K_0 . We can correct for this, when needed.

Faustmann's Solution

The procedure now is to define the present value, or DCF, of land in perpetuity, i.e. from an infinite series of future cycles of planting and cutting. As elsewhere, the symbol for the DCF of land in perpetuity is "S" (Site Value). The symbol for the harvest value of timber is "F", for Future.

$$S = \frac{F}{I^n} + \frac{F}{I^{2n}} + \dots + \frac{F}{I^\infty} \quad (2)$$

(2) is a geometrical progression, infinitely long, whose first term is F/I^n , and whose rate of progression is $1/I^n$. Plug those into the formula for the sum of a geometrical progression and we have:

$$S = F \left[\frac{1}{I^n} \cdot \frac{1}{1 - I^{-n}} \right] = \frac{F}{I^n - 1} \quad (3)$$