

Pr. Douglas, Replacement Model

Complete Table, Use in Finance.

Ec. 162, Memo. Nov. 10, 1981, by M. Gull/way
Dedicated to Priscilla Douglas, who asked the right question.

This is for more serious students who want really and fully to understand replacement planning, and valuation of durable property, and the basic math. tools needed.

1. An oversimplified model of a typical asset.

We are considering buying or building a structure to generate cash flow on a site we own.

a = annual cash flow

(Cash flow = gross receipts less running costs.)

n = end of life

C = Capital cost, time zero.

G_i = PV of Generation $\#i$

S = Site Value in perpetuity

$$G_i = -C + a \frac{1 - (1+i)^{-n}}{i} \quad (1)$$

$$S = G_i \left[\frac{1}{1 - (1+i)^{-n}} \right]$$

$$= \frac{-C}{1 - (1+i)^{-n}} + \frac{a}{i} \quad (2)$$

2

The trouble with this oversimplified model^{of §} is the same as with G_1 alone, viz., it has no finite maximum w.r.t. n . It just says that if a will run on indefinitely, then the less often we replace, the better.

To correct for this, we modify the model and let cash flow decline, or "decay", with n .

2. A simple but adequate model of a typical asset with decaying cash flow.

d = decay factor, such that

$$\text{cash flow} = a \left[\frac{1}{1+d} + \frac{1}{(1+d)^2} + \dots + \frac{1}{(1+d)^n} \right]$$

$$G_i = -C + a \frac{1 - [(1+i)(1+d)]^{-n}}{d + i(1+d)} \quad (3)$$

$$S = \frac{-C}{1 - (1+i)^{-n}} + \frac{a}{d + i(1+d)} \cdot \frac{1 - [(1+i)(1+d)]^{-n}}{1 - (1+i)^{-n}} \quad (4)$$

A skilled mathematician could inspect (4) carefully, and probably see S rising to a max and then falling, with n . However, let us nail it down firmer.

3. Finding the max of a ratio.

It is intuitive, and you can demonstrate yourself, that a ratio rises in value when the numerator grows by a larger % than the denominator; and stops growing when the two % changes are the same.

$$\frac{u}{v} = \text{a ratio}$$

Ratio constant when:

$$\frac{\frac{du}{dx}}{u} = \frac{\frac{dv}{dx}}{v}$$

(5)

Remember or look up the derivative of a ratio. Usually you find:

$$\frac{d \frac{u}{v}}{dx} = \frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2}$$

(6)

But we may rewrite (6) so it immediately confirms (5).

$$\frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2} = \frac{u}{v} \left[\frac{\frac{du}{dx}}{u} - \frac{\frac{dv}{dx}}{v} \right] \quad (7)$$

This may make you feel better. Our intuitive device is just simplified differential calculus.

4. Maximizing S

To maximize S , solve (5), using (4a)

$$S = \frac{-C + a \frac{1 - [(1+i)(1+d)]^{-n}}{d+i(1+d)}}{1 - (1+i)^{-n}} \quad (4a)$$

$$\begin{aligned} \frac{du}{dn} &= \frac{-a}{d+i(1+d)} \left[-[(1+i)(1+d)]^{-n} [d+i(1+d)] \right] \\ &= a[(1+i)(1+d)]^{-n} \end{aligned} \quad (8)$$

$$\frac{dv}{dn} = i(1+i)^{-n} \quad (9)$$

$$\frac{\frac{du}{dn}}{u} = \frac{\frac{dv}{dn}}{v} \quad \text{when}$$

$$\frac{a [(1+ti)(1+d)]^{-n}}{-C + a \frac{1 - [(1+ti)(1+d)]^{-n}}{d + i(1+d)}} = \frac{i(1+ti)^{-n}}{1 - (1+ti)^{-n}} = \frac{i}{(1+ti)^n - 1}$$

$$\frac{a \frac{(1+ti)^n - 1}{[(1+ti)(1+d)]^n}}{a \frac{1 - [(1+ti)(1+d)]^{-n}}{d + i(1+d)} - C} = i \quad (10)$$

The left side represents $\frac{\frac{du}{dn}}{u}$, and (once the denominator > 0) the derivative is positive so long as the left side is $> i$. It clearly is when $v = 0$, and for a range of n -values greater than that, as it grows slowly. But then it clearly must peak and fall, because:

- a) $v \rightarrow$ an asymptote or constant value
- b) $u \rightarrow \frac{a}{(1+d)^n}$ and keeps getting smaller.

5. a simpler, marginal view of how to max w.r.t. n

The Tangle of (10) can become suddenly clear and evident when we rearrange terms in such a way as to make it clear that the condition says that G_1 should be replaced in the year when the decaying cash flow ceases to earn interest on site value.

The denominator of (10) on the left side is G_1 . Substituting,

$$a \frac{(1+ki)^n - 1}{(1+ki)^n (1+kd)^n} = G_1 \cdot i \quad (11)$$

$$\frac{a}{(1+kd)^n} = i \frac{G_1}{1 - (1+ki)^{-n}} = i \cdot S \quad (12)$$

In (12), the condition for max S is seen as a requirement that the current cash flow, the left side, just cover interest on the site. This makes intuitive sense in its own right.

So what most people do (if they think this

through at all) is to figure they will demolish & replace capital when its cash flow no longer pays interest on the value of S . This is a useful rule of thumb to remember.

6. The ultimate need to max S w.r.t. n

The Trouble is, logically it is circular, because S is unknown, it is what we are determining. Most people just take market value for S , and make out all right, but someone has to do the original thinking for others to follow. This person earns extra income, by being first. He or she can spot bargains, avoid turkeys, etc.

This leads us back to (10). We need to solve it for n . The best way is to set up some standard tables, based on typical parameter values. Let:

$$i = \text{~~0.05~~ } .05 \neq$$

$$C = 100$$

$$a = 20$$

$$d = .08$$

Now we figure left side as function of n .

$\neq$ Using a low or real rate of interest has the same effect as letting a rise annually due to inflation. There are alternative adjustments for inflation.

$$\begin{array}{cccccc} n & 1.05^n & (1.05+1.08)^n & u & v & \frac{u}{v} \end{array}$$

22	2.925	15.90	2.42	39.87	.0607
23	3.072	18.03	2.30	40.98	.0561
24	3.225	20.45	2.18	41.96	.0519
24.5	3.305	21.78	2.12	42.40	.0499

As S is max at $n = 24.5$

$$S = \frac{-100 + 20 \frac{1 - \frac{1}{21.78}}{.134}}{1 - \frac{1}{3.305}} = \frac{-100 + 20 \frac{.95}{.134}}{1 - .303} = \frac{42.4}{.697} = 60.8$$

As a check, recall from #5, above, that at max S ,

$$S \cdot i = \frac{a}{(1+d)^n} \quad (12)$$

Here, $60.8 \cdot .05 = 3.040$
 $20(1.08)^{-24.5} = 3.035$

9

That's pretty close, considering rounding and other approximations used here.

7. Sensitivity analysis: how is optimal replacement time (n) affected by changes in parameters (a , C , i , and d)?

I append some rudimentary sensitivities. I did this by programming (10) on a HP 38E, so I could proceed quickly by trial & error. With practise & experience & better algebra, no doubt we could find an easier way.

✓ The outstanding finding is that n is much more sensitive to a/C than to i or d .

In a few hours, one could work up a set of tables covering all the cases you would meet in a lifetime of making investment & replacement decisions.

Case No.	a/c	i	d	n
1	.20	.05	.08	24 ⁺
2	.25	.05	.10	19 ⁻
3a	.30	.05	.15	15 ⁺
b	.30	.04	.15	15 ⁻
c	.30	.03	.15	14
d	.30	.02	.15	13 ⁺
e	.30	.01	.15	13
4a	.30	.05	.15	15.4 ⁺
			.14	15.3 ⁺
			.13	15.2
			<u>.12</u>	15.2
			.11	15.3
			.10	15.5
			<u>.09</u>	15.8
			.08	16.3
			.07	16.8
			.06	17.5

Sensitivity Analysis

Case

	$\frac{q}{c}$	i	d	n
5a	.30	.05	.15	15.4
	.28			17
	.26			19 ⁺
	.24			22 ⁺
	.22			29
	.21 .21			41
	.20			∞

Conclusion: n is insensitive to $i + d$
but highly sensitive to q/c .

(But when $\frac{q}{c} \rightarrow (i+d)$, n becomes sensitive to $i+d$)

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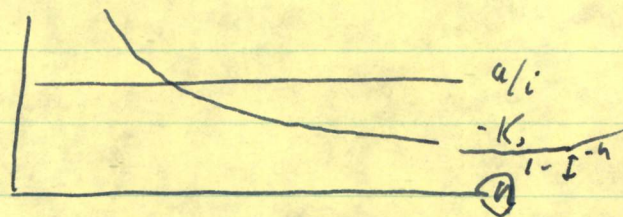
Conclusion: n is insensitive to $i + d$
but highly sensitive to a/c .

(But when $\frac{a}{c} \rightarrow (i+d)$, n becomes sensitive to $i+d$)

1. "Spreading overhead"

$$G_t = -K_0 + a \frac{1 - I^{-n}}{i}$$

$$S = \frac{-K_0}{1 - I^{-n}} + \frac{a}{i}$$



But a remains level, giving no replacement n.

2. Decay.

$$G_t = -K_0 + a \frac{1 - D^{-n}}{i + d} \quad (1)$$

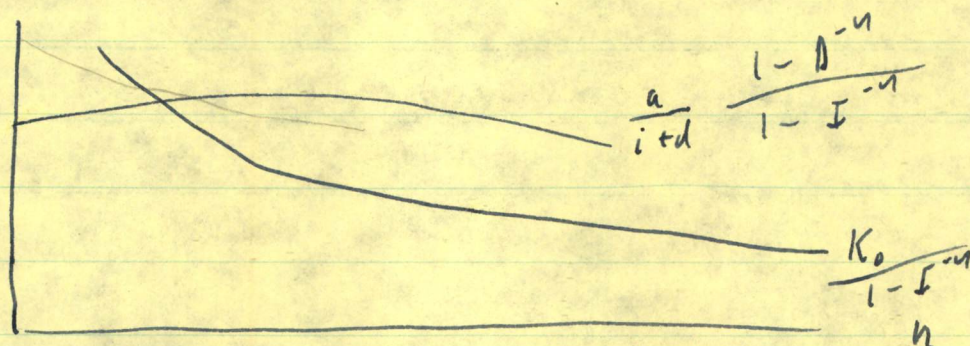
where

$$D \equiv 1 + i + d = I + d$$

(?)

$$C.F. = a(1 + (1-d) + (1-d)^2 + \dots + (1-d)^n)$$

$$S = \frac{-K_0}{1 - I^{-n}} + \frac{a}{i + d} \frac{1 - D^{-n}}{1 - I^{-n}} \quad (2)$$



3. Collect terms

$$S = \frac{-K + \frac{a}{i+d} (1 - D^{-n})}{1 - I^{-n}} \quad (2a)$$

4. Max S when $\frac{u'}{u} = \frac{v'}{v}$

$$u' = \frac{-a}{i+d} [-D^{-n} (i+d)] = a D^{-n} \quad (3)$$

$$\frac{u'}{u} = \frac{a D^{-n}}{-K + \frac{a}{i+d} (1 - D^{-n})} = \frac{D^{-n}}{\frac{1 - D^{-n}}{i+d} - \frac{K}{a}} \quad (4)$$

$$v' = i I^{-n} \quad (5)$$

$$\frac{v'}{v} = \frac{i I^{-n}}{1 - I^{-n}} = \frac{i}{I^n - 1} \quad (6)$$

$$\frac{D^{-n}}{\frac{1 - D^{-n}}{i+d} - \frac{K}{a}} = \frac{i}{I^n - 1} \quad (7)$$

$$\frac{\frac{I^{+n} - 1}{D^n}}{\frac{1 - D^{-n}}{i+d} - \frac{K}{a}} = i \quad (7a)$$

5. To tabulate, & solve by trial & error, may not be any easier from 7a than from (2). Try both ways.

6. Try solving for n .

$$\frac{I^{+n} - 1}{\frac{D^n - 1}{i+d} - \frac{D^n K}{a}} = i$$

$$I^n - 1 = \frac{i D^n}{i+d} - \frac{i}{i+d} - \frac{i D^n K}{a}$$

$$I^n + D^n \left(\frac{Ki}{a} - \frac{i}{i+d} \right) = 1 - \frac{i}{i+d} = \frac{d}{i+d}$$

7. Solve for K/a .

$$\frac{K}{a} i D^n = \frac{i D^n}{i+d} - \frac{i}{i+d} - I^n + 1 = \frac{i D^n}{i+d} + \frac{d}{i+d} - I^n$$

$$\frac{K}{a} = \frac{1}{i+d} + \frac{i}{i+d} - \frac{I^n}{i D^n} + \frac{1}{i D^n} = \frac{1+i}{i+d} + \frac{1-I^n}{i D^n}$$

$$i \rightarrow 00$$

$$d \rightarrow 03$$

$$n \rightarrow 01$$

4

$$\frac{K}{a} = \frac{1}{i+d} + \frac{d}{(i+d)(iD^n)} - \frac{I^n}{iD^n}$$

$$\frac{K/a}{i} \quad \frac{d}{d} \quad \frac{n}{n}$$

$$105 \quad 108$$

$$\frac{1}{i+d}$$

$$i (i+d+1)^n \times i$$

From (7)

$$I^n - 1 = i \left[\frac{D^n - 1}{i+d} - \frac{KD^n}{a} \right] = \frac{i(D^n - 1)}{i+d} - \frac{KiD^n}{a}$$

$$\frac{KiD^n}{a} = \frac{i(D^n - 1)}{i+d} - I^n + 1$$

$$= \frac{iD^n}{i+d} - \frac{i}{i+d} + \frac{i+d}{i+d} - I^n$$

$$= \frac{iD^n}{i+d} + \frac{d}{i+d} - I^n$$

$$\frac{K}{a} = \frac{1}{i+d} + \frac{d}{iD^n(i+d)} - \frac{I^n}{iD^n}$$

Hunt down n, use program "Replace" on key 14 15

<u>q/k</u>	<u>K/a</u>	<u>i</u>	<u>d</u>	<u>n</u>	<u>S/a</u>	$\frac{1}{i(1+d)^n}$
	5	.05	.08	24.5		
.150	2	.05	.08	11	8.88	8.58
.33	3	.05	.08	15	6.67	6.31
.25	4	.05	.08	19.2	4.86	4.56
.20	5			24.5	3.31	3.04
.17	6			31.8	1.947	1.73
.14	7			44.8	0.744	0.67
.125	8			∞		

The program finds i in (7a), for various values of n. Hunt for i = .05, or whatever i is used.

Find S/a from (2a)

$$\frac{S}{a} = \frac{\frac{1 - 0^{-n}}{i + d} - K/a}{1 - I^{-n}}$$

Program on Key 22

<u>i</u>	<u>d</u>	<u>n</u>	<u>$\frac{K}{a}$</u>	<u>s/a</u>
.05	.08	1	-.00...	
		2	.063	17.265
		3.5	.331	14.946
		4	.331	14.910
		8	1.207	11.115
		12	2.245	8.287
		16	3.256	6.178
		20	4.155	4.606
		24	4.914	3.434
		28	5.535	
		32	6.031	
		36	6.421	
		40	6.725	1.060

It is quicker to find K/a . Program on Key 21

<u>i</u>	<u>d</u>	<u>n</u>	<u>K/a</u>	<u>S/a</u>
.01	.08	20	5.200	21.770
.02			4.902	11.044
.04			4.382	
.08			3.577	
.16			2.548	
.24			1.936	
.32			1.537	
.40			1.261	0.822

<u>i</u>	<u>d</u>	<u>K/a</u>	<u>n</u>	<u>s/a</u>
.01	.08	3.577	15	32.34
.02			15.5	15.67
.04			16.5	7.29
.08			20	2.99
.16			35.4	0.59

Morals:

1. ~~Longer~~ Higher $i \rightarrow$ longer life!
2. n not very sensitive to i .
3. s/a varied roughly w. recip. of i , but a little more at low i ; and much more of above .08.