Rectangle Dimensions
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RECTANGLE DIMENSIONS

A. Optimizing Sides of Rectangle, Constant Area

The first group of problems is based on the simplest of forms, a rectangle, in order to warm us up comfortably on familiar ground, and establish usable procedures. Then we will progress to the less familiar case of a cylinder.

A rectangle has a constant area, A=100. Think of it as a field we are fencing; we want to minimize the length of the fence per unit of area fenced. What is the best ratio of base to height (i.e. depth of the parcel). We already know the answer intuitively, but we also know a lot more that we may not realize we know. The purpose of these problems is not to make a simple problem hard, but to show how many insights of more general value we may derive from the simple problem.

1. Tabulate some values of the Base, B, and the Height, H, that are consistent with A=100. In a third column tabulate the Perimeter, P, and find the minimum value of P for A=100.

Ans: P=40

What is the relationship of B to H when P is minimized?

Ans: B=H=10

2. Plot the values on a graph, with H on the abscissa and B on the ordinate. Use H as the independent variable. It could be either way, but it avoids confusion to stick with one convention. ***IN ALL THAT FOLLOWS, REMEMBER THAT "H" IS THE INDEPENDENT VARIABLE***.

What is the standard mathematical name for the curve showing A=100?

Ans: rectangular hyberbola.

Remember elasticity of demand, as usually taught in introductory micro. There, Revenue = Price x Quantity, and Revenue is constant when $P'/P = -Q'/Q$. Such a condition is called unitary elasticity of demand, and mathematically is the same as what we are doing here.

3. What do you know about the necessary relationship of changes in H to changes in B?

This is different from the functional relationships in "production functions", previously studied, because here we are holding the product constant. The "product" in this case is the area, A, enclosed by the perimeter of the rectangle. With A constant we are looking at substitution of H for B. Scale of output is held constant (although scale of inputs changes).[1]

The ϵ that we have here is called the "Elasticity of Substitution." It is the % change in B required to compensate for a small unit % change in H, in order to hold A constant. Remember our convention that H is the independent variable.

a) With HB = a constant, apply the product rule of calculus in percentage terms. To simplify writing and reading equations, use the convention that $h = H'/H$ and $b = B'/B$. Translate the usual calculus rule ($uv' + vu' = 0$) into our terms.

Ans: $h+b = 0$, therefore $h = -b$

b) Or set the $\ln H + \ln B = \ln 100$ and differentiate. The derivative of the log of a variable is the derivative of the variable divided by the variable. Remember our convention that H is the independent variable.

(Ans: $h=-b$)

c) A quicker method is to remember that exponents are elasticities in simple power functions. Here we write B as a function of H, when A is constant:

$$B=AH^{-1}$$

Then what?

(Ans: it follows immediately that $\epsilon=-1$, from which in turn it follows that $h/b=-1$ and so $h=-b$.)

To repeat, this should remind you of elasticity of demand, as usually taught in introductory micro. There, Revenue = Price

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[1]In real problems, the land is not usually homogenous. Road frontage is worth more than back lots or parcels. Here, to get started simply, we ignore such complications.

x Quantity, and Revenue is constant when $P'/P = -Q'/Q$. Such a condition is called unitary elasticity of demand, and mathematically is the same as what we are doing here.

4. Why bother to do the same thing three different ways? Answers follow.

a. One reason is so you can see that the different ways are mutually consistent and reinforcing. A confirmed mathematician would say they are different ways of saying the same thing.

b. A greater reason, for practical application, is that problems come to your perception in various forms. One time you may recognize it as a maximizing question; another time as a percentage question; again as a log question -- yes, really, there are people out there who customarily talk in logs with great familiarity, just as some people speak in German or Spanish.

Once you solve a problem by one method, usually you can see how to solve it by others. But you never know in which disguise the problem will first present itself to you; or which kind of insight will first put your mind in gear.

5. But hey, wait a minute, what happened to good old-fashioned calculus? Well, sure, let's try that. Find the ϵ of substitution of H for B using standard calculus. Remember our convention that <u>H is the independent variable</u>.

Ans:

$\epsilon = H(B'/B)$, when $BH=k$

$B=k/H$

$B' = -kH^{-2}$

$$B'/B = \frac{-kH^{-2}}{k/H} = -1/H$$

$\epsilon = H(B'/B) = -1$

Longer route, same answer. In cases like this our short cuts save some time. [Use all short cuts with care!]

6. What about big changes in H? Is Berkeley's ghost spooking around? Indeed yes. For finite changes the fractional change in the product of B and H is b+h+bh, and if the product is to remain fixed that must equal zero. Find ϵ. See note.[2]

Answer: $h = -b/(1+b)$

If B doubles (b is 100%) then h = -50%. E.g., 10x10 = 20x5. I still haven't answered the question directly; I'm leaving that to you.

This relationship is called "arc-elasticity" to distinguish it from "point-elasticity." The latter is the calculus concept; it is used for maximizing. Arc-elasticity is often more useful in other kinds of business questions where fine-tuning is impossible and only discrete (gross) changes are considered.[3]

7. What do you think you are supposed to be learning here?

Answer: aside from the useful techniques you are learning that the variable in an equation may be treated as a constant, and vice versa.

Remember the trouble you had, and may still have, when I showed you the MP of variable labor applied to fixed land and then said we can infer from the labor MP curve some information about the land MP? Originally we were varying labor subject to the constraint of fixed land; then I asked you to stop the change in labor at a certain amount, and vary land subject to the constraint of fixed labor. You have to learn to make those mental adjustments. It not only flexes and exercises the mind, it lets you handle lots of variables.

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[2] $H(1+h)B(1+b) = HB$
$(1+h)(1+b) = 1$
$h + b + hb = 0$
$h(1+b) = -b$
$h = -b/(1+b)$

[3] Indeed, point-elasticity can be very misleading if you make the error of assuming that large changes produce results that are in exact proportion to small changes - e.g. that a 40% change has the same result as 40 x a change of 1%. This error is similar to that of assuming that 40 years' growth at interest is equal to 40 x one year's growth, overlooking the compounding effect.

When a constant may have different values it is called a "parameter." Pi is not a parameter, its value never changes. In the rectangle problem here, Area is a parameter because we can change it from 100 to any of various other values.

Qn: When is a perimeter a parameter?

In the rectangle case here we stopped the product at a point where it equals 100 and let the two inputs, B and H, both vary, subject to that constraint.

8. We can use elasticity to find optimal solutions. We have already done so, perhaps without fully realizing it, when we maximized the present value of a growing tree. We maximized the ratio of the harvest value to the compound interest function by setting their % growth rates equal, which is easy because the % growth rate of the base is the interest rate.

Problem: find the minimum value of perimeter/area, (P/A), using $\epsilon = -1$, as shown earlier.

Ans: P is constant when $dB/dH = -1$. Substitute that into $\epsilon = -1$, which in turn means that $B'/B = -1/H$. It follows immediately that B=H.

9. For a check find the minimum value of P as a typical calculus problem, minimizing 2(H+B) subject to HB = k ("k" means any constant).

Ans:

$$P=2(H+B) = 2(H+k/H)$$

$$dP/dH = 2[1-kH^{-2}] = 0$$

$$1 = k/H^2 = HB/H^2 = B/H$$

$$B = H$$

10. Second method: Min P/V subject to HB=k, so that:

$$P/V = \frac{2[H + k/H]}{k}$$

Since the denominator is k, minimizing the ratio P/V boils down to minimizing the numerator, as in the first method.

B. Optimizing Sides of Rectangle, Constant Perimeter

Here we repeat most of questions of Part A (just above), but with a difference. Now we hold P, the perimeter, constant; and let A, the area, vary. Now we are seeking optimality by maximizing A, rather than minimizing P. The perimeter is now the parameter.

1. A rectangle has a constant perimeter, P=40. Tabulate some values of B & H consistent with P=40. In a third column tabulate the Area, A, and find the maximum value consistent with P=40. (Leave alternate rows empty for later use.)

What is the relationship of B to H when A is maximized? How does it compare to the relationship in A,1 when P is minimized?

In Columns 4-7 tabulate h, b, h+b, and b/h. (Remember our usage that lower case letters mean fractional changes in variables.) What is the relationship of b to h when A is max? Enter these values in the rows you left empty before, because these data represent changes between rows. Fractional changes are still changes, like derivatives, or marginal products.

Use fractions rather than decimals. This gives you greater insight as you go along.

2. Plot the values on a graph, with the same coordinates as before, H on the abscissa and B on the ordinate.

What is the standard mathematical name for the curve showing P=40?

What vernacular or architectural term describes the curve showing A?

Cf. this graph with your graph from question A,2. Note how the shapes differ depending on what constraints of constancy you impose.

3-5. What do you know about the necessary relationship now of changes in B to changes in H? (H + B = P/2, a constant).

a. In absolute terms?

b. In fractional or percentage or log terms?

6. How do the relationships in 3-5 change, if at all, when the changes in H and B are large?

7. What are we supposed to learn from doing this problem all over again with fixed P instead of fixed A? Answers follow:

a. More mind-stretching, enabling you to perceive the same problem in different ways.

b. How to approach a trade-off (i.e. substitution) problem and find an optimal solution when your resources are constrained, rather than the scale of output.

c. To realize that "constant scale" is ambiguous -- it could mean constant output, or constant inputs, but not both.[4]

8. Find the maximum value of A, subject to fixed P, using the elasticity value you found in Qns. 3-5.

9. Find the maximum value of A as a typical calculus problem, maximizing HB subject to H + B = k.

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[4]There is a conceit among some theorists that the kind of trade-off analysis beginning on p.213 in our text, using isoquants, shows "pure" substitution, abstracting from economies of scale, and is thus in some sense more true and pure than diminishing returns. The trade-off idea is valid and important, but it does not succeed entirely in abstracting from changes in scale. If output is fixed, inputs change; if inputs are fixed, output changes.

ANSWERS TO PART B

1. Values of H and B when P=40

H	B	A	h	b	h+b	b/h = ϵ
10	10	100	1/10	-1/10	0	-1
11	9	99	1/11	-1/9	-2/99	-11/9
12	8	96	1/12	-1/8	-4/96	-12/8
13	7	91	1/13	-1/7	-6/91	-13/7
*	*	*	*	*	*	*
19	1	19	1/18	-1/2	-16/36	-18/2
20	0	0	1/19	-1/1	-18/19	-19/1

B=H when A is max; same as Part A,1.

b=-h when A is max; same as Part A,1.

2. Curve showing P=40 is a straight line, negative slope. Curve showing A forms an arch.

In Part A, curve showing P was U-shaped; curve showing A was a rectangular hyperbola.

3-5. a. $B'=-H'$

b. $E=b/h=-H/B$ (b) is obvious from the table.

You can also use calculus, as follows.

$\epsilon = H(B'/B)$, subject to $H+B=k$

$B' = 0 - 1 = -1$

$H(B'/B) = -H/B$

6. Neither relationship changes. Note, however, that point-elasticity changes constantly along the straight line.

7. Already answered.

8. A is max when E=-1 when H is substituted for B. ϵ along the straight line is -B/H, which is -1 when B=H.

9. Max BH subject to B+H=k.

$BH = B(k-B) = kB - B^2$

Set derivative equal to zero and solve.

$k - 2B = 0$

$B + H - 2B = H - B = 0$

$H = B$

Alternatively, it is useful to treat this as a binomial.

$H = k + x$

$B = k - x$

$A = HB = (k+x)(k-x) = k^2-x^2$

$dA/dx = 0 - 2x = 0$

$x = 0$