



Some Useful, Easily Workable Mathematical Techniques for Optimization, Sensitivity Analysis, and General Managerial Purposes

By Dr. Mason Gaffney

Article citation:

Gaffney, Mason. "Some Useful, Easily Workable Mathematical Techniques for Optimization, Sensitivity Analysis, and General Managerial Purposes." Econ 162A, October 25, 1983. Study guide.

Fair Use Disclaimer

The Robert Schalkenbach Foundation, a non profit private foundation, is publishing this article, at no cost, pursuant to the Fair Use provisions of Section 107 of the Copyright Act which states that under certain circumstances, the fair use of a copyrighted material for purposes of criticism, comment, teaching, scholarship and research is not an infringement of copyright. If you have any concerns about the publication of this article, please contact us at info@schalkenbach.org.

The Robert Schalkenbach Foundation's Digital Collections and Archives is an extension of the Robert Schalkenbach Foundation's Engel Georgist Library. It collects, organizes, preserves, and provides free online access to researchers and the general public of Georgist literature including books, newspapers, pamphlets, scholarly articles and archival materials.

RSF is particularly proud to offer free online access to the works of the late Georgist scholar Mason Gaffney, made possible through the generosity of the Danica Foundation and with the consent of the Mason and Letitia Gaffney Trust.

Robert Schalkenbach Foundation: <https://schalkenbach.org/>

Engel Georgist Library: <https://www.librarycat.org/lib/EngelGeorgistLibrary>

RSF Digital Collections and Archives: <https://schalkenbach.starter1ua.preservica.com/>

Some Useful, Easily Workable Mathematical Techniques for Optimization,
Sensitivity Analysis, and General Managerial Purposes.
Supplement to Ch. 2 of P&B.

I. The Product Rule of Increase (and decrease)

A. Calculus

1. The absolute increase, or "derivative."

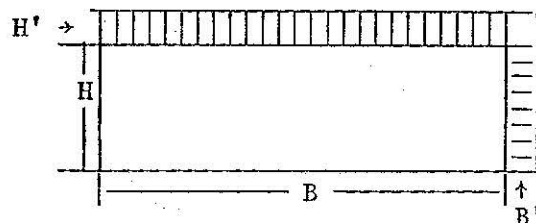
$$\text{Let } A = BH \quad (1)$$

How does A increase with B and/or H? The calculus rule is:

$$A' = BH' + HB' \quad (2)$$

(where prime (') means change as a function of the basic independent variable, "X", of which B and H are functions.

Visually, it's like the area (A) of a rectangle:



BH' is the added strip on top.

HB' is the added strip on the right or growing side.

Add the two strips and get the increase of Area.

"Just a ding-danged minute," you cry. "How about the little white rectangle in the upper right?" "Not to worry" says Matthew Mattox, "Isaac Newton disposed of that. "It is B'H'. But if H' is zero, any B' is wiped out; and if B' is zero, any H' is wiped out--don't you see?" Generations of math students have wondered if they saw, but suppressed all doubt to humor the Professor, and the subject rectangle is so small, anyway, why bother?

The upshot is that calculus rules are accurate for changes so small they may be described as "instantaneous rates of change" along smooth functions--or some approximation thereto that is reasonable in the circumstances. Calculus rules are useful also because they are usually simpler (simpler even than (2) would make you believe, as we will see); and they point you in the right direction. But to predict the results of finite increases, we need to include B'H'.

The Irish Bishop George Berkeley (whose namesake is Berkeley, Cal.), an 18th century critic of Newton, once warned of B'H': "the ghosts of these murdered quantities will return to haunt." For now we will call them "Berkeley's ghosts"; and return to them in I,B.

2. The Percent Increase

The first step is to make (2) simpler. This is done by reasoning in percentages, rather than absolutes. To do so, divide (2) by (1)

$$\frac{A'}{A} = \frac{H'}{H} + \frac{B'}{B} \quad (3)$$

From here on, use lower case letters to mean percent changes of the upper case letters. Then (3) becomes:

$$a = h + b \quad (3.A)$$

That's about as simple as an equation can get. If the height increases by 2% and the base by 3%, the area increases by 5%. (Later we will see A actually increases by slightly more.)

Example: The Riverside public utility pays a fixed share of its revenues into the General Fund each year. If kWh sales rise by 3%, and rates by 10%, the revenues rise by 13%, and so does the General Fund contribution. The Utilities' Board, anticipating rate increases to come will exceed CPI increases (which measure general inflation), and wishing to contribute a lesser share to the General Fund, is proposing to the City Council that future increases be the sum of the percent increases of kWh sales and the CPI. If the CPI grows at a lower rate than the electric "rate" (price per kWh) the Board will succeed in lowering the share of its revenues paid to the General Fund.

3. Log Derivatives

Another approach to (3.A) is via log derivatives, which are fractional changes.[†] That is, the change in the $\ln$ of a variable is the percent change in the variable. (Check that out on your calculator.) By the way this is true of $\ln$ s to base e , not logs to base 10. As to the latter:

$$\log X = \frac{\log X}{\ln X} \cdot \ln X = (\log e)(\ln X) = .4343 \ln X \quad (4)$$

So that the change in the log is .4343X (or 43.43% of) the percent change in the variable. In this course stick with $\ln$ s to base e , I will never ask you about logs to base 10. (4) is given just in case you wondered. The only other base I sometimes use is 2; but not now.

One confusing point of terminology is that people speak of "log derivatives" when they mean derivatives of $\ln$ s, to base e , not logs to base 10. But $\ln$ always means to base e . It is written " $\ln$," but pronounced "log," which is short for "natural log." This seemingly careless talk has developed because mathematicians always assume a logarithm is to base e unless otherwise specified. So do I, and so may you.

The calculus rule is:

$$\frac{d(\ln A)}{dX} = \frac{A'}{A} \quad (5)$$

[†]I.e. the change/base. When multiplied by 100 this is the percent change. I usually refer loosely to fractional change as percent change, just because percent change is a more familiar expression, easier to understand and remember.

From (1), take $\ln$ of both sides:

$$\ln A = \ln B + \ln H \quad (6)$$

The product rule of (3) follows, by differentiating both sides:

$$\frac{A'}{A} = \frac{B'}{B} + \frac{H'}{H} \quad (3)$$

Or, in lower case notation:

$$a = b + h \quad (3.A)$$

Again, we are back to our wonderfully simple and useful (3.A), our refuge and salvation amid strife and confusion. A Captain could remember this while sailing a ship through a hurricane; or a Manager while fighting bankruptcy and competition. We have also learned something about logs and derivatives, and seen that it all meshes together consistently. (Only remember, please, it understates a slightly--see B, below.)

4. Percent Increases and Elasticities (ϵ). [This digresses from the main theme; you may skip for now and return later, if you find that easier.]

B and H are often driven by some independent variable, X; and we often want to know the ϵ of A with respect to X. This is wonderfully simple.

$$\frac{d(\ln X)}{dX} = \frac{1}{X} \quad (7)$$

You could have figured out (7) without calculus. Say, if X is 20 and increases by one, the percent increase is $1/20$.

$$\epsilon \equiv \frac{a}{1/X} = aX \quad (8)$$

This relationship (8) is not limited to the Product Rule, but holds for any functional relationship. Example: what is the ϵ of Present Value with respect to $\underline{n}$?

$$\epsilon = \frac{(1+i)^{-n}}{\frac{1 - (1+i)^{-n}}{i}} n = \frac{ni}{(1+i)^n - 1} \quad (9)$$

Work out some values with your calculator.

(Note how often we run into expressions like the right-hand term in (9), or its reciprocal. So let's name it. It's the "Simple/Compound Ratio" (because it is simple interest divided by compound interest.))

B. The Product Rule with Finite Changes

Let's ask directly, what percent change in A results from given percent changes in B and H? Sticking with our lower case notation for percent changes:

$$A(1 + a) = B(1 + b)H(1 + h) \quad (10)$$

To find a, divide by (1):

$$\begin{aligned} 1 + a &= (1 + b)(1 + h) = 1 + b + h + bh \\ a &= b + h + bh \end{aligned} \quad (11)$$

Great shades of Berkeley! We are back to (3.A) except we have revived Berkeley's ghosts. "bh" tells you how big the ghosts are. You may include them or not, as a matter of judgment, in any given problem.

Example: An avocado grove is afflicted by root rot (phytophthora cinnamomi). Yields drop by 55%. Meantime prices drop by 70%. What happens to revenues? They sure don't drop by 125%, so you had better add the bh of 38.5% for a net drop of 86.5%. (Isn't that cute? Be sure you understand what happened. Go back to the figure on p. 1, and show graphically what has happened.)

C. Maximizing Product Functions

Every calculus student knows you find the maximum (or minimum) of a function by setting the derivative equal to zero.[†] You look up some long formula and grind it out. But did you notice the common sense of it? In (3.A), $a = 0$ when:

$$b = -h \quad (12)$$

The percent changes of b and h must be equal and opposite; then there is no change in the product.

But (11) reminds us that b and h must also be small.

Example: Find the shape of a rectangle to maximize the area enclosed by a fence of fixed perimeter.

Begin with B small and H large. $B + H =$ a constant sum, so $B' = -H'$ as you add to B by taking from H. In order for $B'/B = -H'/H$, it is obvious that B must = H. Or:

$$\frac{B'}{B} = -\frac{H'}{H} \quad (13)$$

$$\text{But } B' = -H', \text{ so:} \quad (14)$$

$$\begin{aligned} \frac{1}{B} &= \frac{1}{H} \\ H &= B \end{aligned} \quad (15)$$

[†]Mathematicians remind us there could also be "cusps," where the max is a point. That does not apply here.

In finding maxima, the calculus rule (3 and 3.A) is easier to work with than (11), which gets rather messy and vague as you approach the maximum. This is because a max is an exact point, which calculus is designed to pinpoint.

(15) tells you a lot about the value of symmetry in economics, if you think about its deeper implications. It tells you why lots and fields and buildings tend to be square; why buildings are expanded vertically (using capital) as well as horizontally (using land), and why there are diminishing returns as you use more of one to less of the other.

A good way to look at symmetry is this. When $A = B \cdot H$, then the marginal product of B is H ; and the marginal product of H is B . That is about as complementary and mutually supportive as two inputs can get! Go back to (2) and be sure you understand how that follows from (2). We will be returning to this.

II. The Quotient Rule of Increase

A. Calculus

A quotient is a Ratio (R), with a Top (T) and a Bottom (B):

$$R = \frac{T}{B} \quad (16)$$

B can also be remembered as Base (as the Base of a percent). Just be sure not to mix it up with B for Base in Section I, where it means Base of a rectangle.

A quotient is also a product:

$$R = TB^{-1} \quad (16.A)$$

and we could stop right here. (3.A) becomes:

$$r = t - b \quad (17)$$

using our convention that lower case letters are percent changes. (See how simple this is going to be when we get done?) All the product rules also apply to quotients. However, that would take some doing, so quotient rules are usually expressed separately. Just note that they must all be consistent with product rules.

If you look up the Quotient Rule in P&B, Ch. 2, or any calculus book, you find an alphabet soup like this:

$$\frac{d\left(\frac{U}{V}\right)}{dX} = \frac{V \frac{dU}{dX} - U \frac{dV}{dX}}{V^2} \quad (18)$$

I needn't tell you to forget it, because you already have. Who can remember all that? But you don't have to, because we can shuffle it around a bit until it is as simple as (17). First, change to our notation:

$$R' = \frac{BT' - TB'}{B^2} = \frac{T'}{B} - \frac{TB'}{B^2} = \frac{T}{B} \left(\frac{T'}{T} - \frac{B'}{B} \right) = R(t - b) \quad (18.A)$$

Then:

$$\frac{R'}{R} = t - b \quad (19)$$

$$r = t - b$$

Now, (19) is so simple you can remember it while kissing your date, or even writing a final.

Examples: Farmer Brown adds 5% to his land and 3% to his harvest. What happens to yields (output/acre)? They drop by 2%.

Payroll Polly has 25 workers and adds Sam. Output rises by 5%. What happens to labor productivity? Up 1%.

But please remember, as before, this rule is only approximate, and for large values of t and b can be far off, unless $t \approx b$. This is covered in II.B. below.

Note that in cases like those, we begin with a Ratio of 1/1, express or implied, and then modify it by adding or subtracting percentage changes. Several people feel at sea because they wonder where is the firm dry land we start from?

Remember how we early on developed the compound interest formula. Begin with \$1, see how it grows. If we can do it for 1, we can do it for any number, by multiplying the result by that number. 1 is the base, and i is a percentage change in the base. Then:

$$F_1 = P_0(1 + i) \quad (20)$$

Now we are carrying this a step further, to cover cases where the Bottom of the ratio also changes. We have already done some of this before, when we found the PV of growing cash flows. Then we said:

$$\begin{aligned} P_0 &= \frac{a_0}{1+i} \left[1 + \frac{1+g}{1+i} + \left(\frac{1+g}{1+i} \right)^2 + \dots + \left(\frac{1+g}{1+i} \right)^{n-1} \right] \\ &= \frac{a_0}{1+i} \left(\frac{1 - \left(\frac{1+g}{1+i} \right)^n}{1 - \frac{1+g}{1+i}} \right) = a_0 \frac{1 - \left(\frac{1+g}{1+i} \right)^n}{i - g} \end{aligned} \quad (21)$$

At that time we were not focussing our consciousness on Ratios as such, but just following the flow of the compounding logic. Now, we are integrating our experience with financial math with the Quotient Rule of differential calculus, to handle Ratios generally.

Let's try some more problems.

The money value of your stamp collection grows at 15%/year, but the value of money drops at 11%/year. What happens to the real value? Grows by 4%/year.

That and the next problem require that you understand how a drop in the value of money is measured. There is a Consumer Price Index (CPI) which begins at 100% in the Base year. When prices rise by $d\%$, the Index goes to $100 + d$.

When, e.g., $d = 5\%$, $\text{CPI} = 105\% (= 1.05)$. Prices are up 5%; a dollar buys 5% less. The value of the dollar, goods/\$, is the reciprocal of the price of goods, i.e. \$/goods. To find the value of goods in constant or base-year \$, we would multiply the current price of a good (\$/good) by the current value of money (goods/\$); or, what is the same thing, divide the current price of a good by the current price of goods in general (\$/goods). If I were king I would have chosen the first way; but the custom now is the second way, divide by an index of \$/goods, called the CPI. We will follow the custom.

The stock of National Natterbox TV rises at 6% annually, while the CPI rises at 11%/year. What happens to the real value after 8 years? Down to $(1 - .05)^8$; (or by $1 - (1 - .05)^8$) of the initial real value.

Old Gramps in Kansas leaves you a lease on his farm, and the remainder to your children. The lease runs 15 years; the cash rent begins at \$10,000 and rises by a yearly increment of 8%. What is the PV of the last year's payment? $10,000 \times (1 + .08 - i)^{15}$.

The "e of production" is the $\% \Delta$ output from a given $\% \Delta$ input. Thus, add half of a back forty to a half-section of Illinois prairie soil--(that's 20 added to 320, for you city folks)--that's a $1/16$ increase of land input. Get a 2.8% increase of output, and:

$$E = \frac{.028}{1/16} = 44.8\% \quad (22)$$

What does (22) mean for yields? ("Yield" means gross output/acre.) They change by $.028 - 1/16 = -.03450$. That is, they fall by 3.45%.

Here's a toughie, you could put on the back burner for a while. If you know e, you can predict the effect on yields of adding acreage. Multiply e by the $\% \Delta$ acres to get the $\% \Delta$ output. Then subtract the $\% \Delta$ acres from the $\% \Delta$ output to get the $\% \Delta$ yields.

$$y = p - i = \epsilon i - i = i(\epsilon - 1) \quad (23)$$

where lower case means percentage change; Y = Yield; P = Product (used in lieu of Output, to avoid confusing "0" with Zero); and I = Input. (We shall return to ϵ --I know you find it bewildering at this point.)

The Quotient Rule of (19) is useful in maximizing present values. The PV of a single or "point" FV is a Ratio whose Bottom grows at a rate i each year. If the Top grows with time, the Ratio is a maximum when $t = b$. (This is precise, because calculus is exactly true when $t = b$; that's why calculus rules are good for maximizing.)

Example. The value of whiskey aging in barrels of the highest grade White Oak, resting quietly in a cool limestone cave in Tennessee, grows at the ff. rates in the series of years beginning with #3: .25, .19, .15, .12, .10, .08, .065. In which year is its PV a maximum? Using a discount rate of 9%, PV is maximized toward the end of year #7, because in year #8 it falls by 1% (.08 - .09). You can also say that PV grows in the sequence .16, .10, .06, .03, .01, -.01, and -.025.

Here's a tricky one for you. White Oak barrels are not cheap. White Oak of "whiskey-barrel grade" takes 50 years or so to age in the forest--a lot longer than whiskey. In Europe, used barrels are recycled several times. Not so here--by law they cannot be. A boon for gardeners to whom the old barrels are retailed, sawed in half, for planters; and much more so for producers of White Oak, of course (for whose benefit the law was passed in our highly political economy). But what is the effect on the date of maximum PV of whiskey? In Europe, add the value of the barrel to the denominator of the numerator--i.e. the Bottom of the Top, if you get my meaning--and reduce its percentage growth rate. Yes, Virginia, adding a constant to a variable dilutes the variable and lowers the percentage growth of the whole.[†] So this factor moves Europeans to age their whiskey a bit less, to salvage the barrels for recycling.

[†] Subtracting a constant, conversely, fortifies the variable and raises the percentage growth--an effect known universally in business as "leverage," because of the multiple effect on percentage growth. The analogy with mechanical leverage is strained, as we will see, and Archimedes would turn over in his grave to hear such usage.

B. The Quotient Rule with Finite Changes

As in (16), let $R = T/B$. Then, using our convention that lower case letters are percentage changes:

$$R(1 + r) = \frac{T(1 + t)}{B(1 + b)} \quad (24)$$

Dividing by (16):

$$1 + r = \frac{1 + t}{1 + b}$$

$$r = \frac{1 + t}{1 + b} - 1 = \frac{1 + t - 1 - b}{1 + b} = \frac{t - b}{1 + b}$$

$$r = \frac{t - b}{1 + b} \quad (25)$$

(25) is the Quotient Rule modified for finite changes. If b is large, and t is not close to b , you'll have to pay attention, obviously. Thus, if input doubles and output triples, you cannot say that yields double.

Rather, $r = \frac{2 - 1}{1 + 1} = \frac{1}{2}$, meaning yields rise by 50%.

Example. Al Avocado is hit with higher water rates so he cuts his irrigating by $1/3$. By irrigating more conservatively, he keeps output constant. What happens to output/acre-foot of water?

Up by $\frac{0 - (-1/3)}{1 + (-1/3)} = \frac{1/3}{2/3} = \frac{1}{2}$. Magic!

To confirm, assume output is 100, while water use drops from 1 unit to $2/3$.

Note that $\frac{100}{2/3} = \frac{300}{2} = 150$, a 50% rise of R caused by a $1/3$ decline in B .

(What a lovely thought: if everyone used $1/3$ less water, we could support 50% more people! If 50% less, 100% more people, etc. No wonder people preach conservation.)

Al's water comes from Crayfish Creek, 15 miles away, through Settler's Ditch, a leaky old canal. 40% of Withdrawals (W) are lost in conveyance, so Deliveries (D) = $.60W$. We can define Transmission Efficiency (E) as D/W . But they neglect cleaning, and Settler's Ditch becomes choked with "phreatophytes"--water-loving plants which transpire and waste water. Line losses rise by 50%--i.e. to 60%, right?--so E falls by $1/3$, to 40%. (Much of that is noise. Hold fast to the signal: E falls by $1/3$.)

A1 has a firm right to a fixed D. By how much must W rise--by 1/3? Not so simple.

$$W = \frac{D}{E} \quad (26)$$

$$w = \frac{0 - - 1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

Efficiency falls by 1/3, so Withdrawals must rise by 1/2. Very interesting. So does inefficiency waste resources? You'd better believe it!

Is it just a coincidence that the 50% rise in required withdrawals equals the 50% rise of line losses? Yes, that's all, a coincidence. But it's interesting to note that Efficiency is the complement of line losses:

$$E = 1 - L \quad (27)$$

where L is the percentage of Input (I) lost in a process.

$$E(1 + e) = 1 - L(1 + \ell)$$

$$1 + e = \frac{1 - L(1 + \ell)}{1 - L} = \frac{1 - L - L\ell}{1 - L} = 1 - \frac{L\ell}{1 - L}$$

$$e = - \ell \frac{L}{1 - L} \quad (28)$$

Thus, if L is already 20%, $\frac{L}{1 - L}$ is 25%, and a 5% ΔL ($\ell = .05$) means a 1-1/4% loss in efficiency.

Another approach is to substitute (27) in (26). Then

$$W = \frac{D}{1 - L} \quad (29)$$

In (29), L appears directly as a percentage decline in the denominator, and we apply (25) directly.

$$w = \frac{0 - - L}{1 - L}$$

Thus, line losses of 75% (as compared with no losses) mean Withdrawals must rise by:

$$w = \frac{3/4}{1/4} = 3 = 300\%.$$

The world is full of things which take a percentage "off the top" like line losses: things like income taxes; share rentals; royalties, commissions; losses from heat, friction and windage; spoilage; vacancies; etc. In all such cases, apply (26), (28), or (29).

Remember that (25) may be applied backwards, like most equations. That is, we may know $\underline{r}$, and be seeking $\underline{t}$ or $\underline{b}$. Then we just solve for $\underline{t}$ or $\underline{b}$ and carry on. It is even easier to solve (16) first for $\underline{T}$ or $\underline{B}$, then apply the product rule for $\underline{T} = \underline{B}\underline{R}$; or the Quotient Rule of (25) for

$\underline{B} = \frac{\underline{T}}{\underline{R}}$, where $\underline{B}$ and $\underline{R}$ have simply been switched. Using (26), here is an example of each.

Water Withdrawals at the head of Pioneer Ditch drop by 10%; transmission Efficiency rises by 20%. What happens to Deliveries? Up 8%. $\underline{D} = \underline{W}\underline{E}$, so use the product rule.

$$\underline{d} = \underline{w} + \underline{e} + \underline{we} = - .10 + .20 - .02 = .08$$

$$(.90 \times 1.20 = 1.08)$$

Another example: Withdrawals rise by 8%; Deliveries rise by 22%. What has happened to Efficiency? Up by 12.96%.

$$\underline{e} = \frac{.22 - .08}{1 + .08} = \frac{.14}{1.08} = .1296$$

Problems can get more interesting as we add more detail. Thus, line losses are always some function of distance. Say Joan has a right to $\underline{D}$ units of water, no matter how far out Pioneer Ditch she locates. How much more $\underline{W}$ does she require by moving from a 2-mile point out to a 6-mile point? Losses = $\underline{W} \times .06\underline{M}^{.8}$ (where $\underline{M}$ = distance in miles).

$$\underline{W} = \frac{\underline{D}}{1 - .06\underline{M}^{.8}}$$

The safest way to handle this one is in 2 stages. At 2-mile point,

$$\underline{w} = \frac{0 - .06 \times 2^{.8}}{1 - .06 \times 2^{.8}} = \frac{.1045}{.8955} = .1167$$

At 6-mile point,

$$\underline{w} = \frac{0 + .06 \times 6^{.8}}{1 - .06 \times 6^{.8}} = \frac{.2516}{.7484} = .3362$$

$$\frac{1.3362}{1.1167} - 1 = .1965$$

Joan requires 19.65% more $\underline{W}$ because of her move.

A truer loss formula would be based on $\underline{D}$ rather than $\underline{W}$; and $\underline{M}$ would be raised to a power greater than one. Let's try that.

$$\underline{W} - .06\underline{M}^{1.2}\underline{D} = \underline{D}$$

$$\underline{W} = \underline{D}(1 + .06\underline{M}^{1.2})$$

This turns out to be simpler.

$$w_2 = .06 \times 2^{1.2} = .1378$$

$$w_6 = .06 \times 6^{1.2} = .5151$$

$$\frac{1.5151}{1.1378} - 1 = .3316$$

Joan requires 33.16% more water because of her move. The higher increase is due to a more realistic formula which bases losses on Deliveries, not W. Too bad it contains no Ratio, but it would if we held W constant and found the effect of M on D. But enough for now; let's move on.

Here's an easy one, to regain your bearings. The equity capital of Gump's Gismos rises by 50%; profits rise by 20%. What happens to return on equity?

$$\text{Changes by } \frac{.20 - .50}{1 + .50} = \frac{-.30}{1.5} = - .20$$

By the way, a profit "rate" is a Ratio, with a Bottom. 90% of the people just see the Top and ignore the Bottom--as though they thought "rate" just meant "per year." The other 10% become potential corporate raiders. They see fat old firms with low real profit rates masquerading as high rates by undervaluing the fat in underused and misused assets.

Example. The Canadian dollar drops 20% against the U. S. dollar, while the yen rises 32%. What happens to the yen in Canadian dollars?

$$\text{Rises by } \frac{.32 + .20}{1 - .20} = \frac{.52}{.80} = 65\%. \quad (\text{To confirm, } \frac{1.32}{.80} = 1.65)$$

What happens to the U. S. dollar in Canadian dollars?

$$\text{Rises by } \frac{0 + .20}{1 - .20} = 25\%. \quad \text{This one has caused a lot of confusion: Theirs}$$

drops 20% against ours, while ours rises 25% against theirs! If the Canadian dollar is worth 80¢, then the U. S. dollar is worth

$$C \$ \frac{1}{.80} = C \$ 1.25.$$

Here we see that reciprocals do not behave as simply as we smoothly assumed when we were talking CPI. You have to watch them when percentage changes are large, and use (25). E.g. the CPI rises by 100%; what happens to the value of the dollar? Down by 100%? Hardly! T is number of dollars, no change; B is CPI, and $b = 1$.

$$r = \frac{0 - 1}{1 + 1} = - 50\%$$

True Story. In August, 1976 the Gaffney's sold their home in Victoria, British Columbia, for \$C 108k. They transferred the funds to U. S. dollars when \$C 1.00 = \$1.03 U. S. One week later the \$C started dropping like a stone. They put the proceeds into a house in Riverside. Within 2 years the \$C dropped to 81¢ U. S., and their house in Riverside rose by 65%. How did the Gaffney's make out in \$C? Their worth rose by

$$\frac{.65 + \frac{.22}{1.03}}{1 - \frac{.22}{1.03}} = \frac{.8636}{.7864} = 109.8\% \uparrow$$

Without lifting a finger! And do they feel unlucky? Old dad reads the foreign exchange quotes whenever he needs an upper, and prefers to measure his assets in \$C. They do feel unlucky for the rising generation of Americans who find housing priced so high, including their own 6 kids. So we are more unlucky than lucky, and so are you. Sad ending--but the point now is how to handle reciprocals.

Here's a tax case. Disposable Income (D) is Gross Income (G) less Tax (T). Let T = tax rate.

$$D = G(1 - T) \text{ because taxes are based on } G.$$

But wouldn't it be better if taxes were based on D? Why should you pay tax on money you never get? Then,

$$D = G - TD$$

$$D(1 - T) = G$$

$$D = \frac{G}{1 + T}$$

For low values of T, machts Nicht. But try T = 100%.

Sounds strange, but the property tax is actually reckoned that way. It is based on market value, which is capitalized from after-tax income. We should be so lucky! Ask me about this one.

† Check. $\frac{1.03 \times 1.65}{.81} - 1 = 109.8\%$. They gained 3% on the exchange and 65% on the price rise, and the U. S. dollar became worth 1/.81 \$ C, or C\$ 1.235.

Accuracy of approximations to quotient rule

We have seen that the change in a Ratio is about $= 1 + t - b$; and this is handy for calculating PV when we subtract the inflation rate from the interest rate to find the real rate, etc. How great is the error? "Who gives a hoot?" you scream. Well, here is motivation. When you're transferring \$108k between countries, for example, believe me, you give a hoot about accuracy to several places. And if you don't the bankers do, and they will clip you plenty.

Again, when you're raising a Ratio to the 60th power, as we do in discounting, a small error can turn into a big one. Try it on your calculator. OK? So carry on, here we go.

Let a Ratio, R_0 , change so that $R_1 = R_0 \frac{1+t}{1+b}$.

$$R_1 = R_0 \frac{1+t}{1+b} \quad (26)$$

What kind of error do we introduce if, to save time, we say "Oh, what the Heck":

$$R_1 \cong R_0(1+t-b) \quad (27)$$

The error as a percentage of the current amount is:

$$\begin{aligned} \text{Error \%} &= \frac{R_0(1+t-b)}{R_0 \frac{1+t}{1+b}} = \frac{1+t-b+b+b(t-b)}{1+t} \\ &= 1 + \frac{b(t-b)}{1+t} \end{aligned} \quad (28)$$

The "Error Factor" is the excess over 1. It is obviously 0 when $b = 0$, and when $b = t$. How bad does it get? If your calculator is programmable, write the program and generate all the answers you want. Here are a few.

Table 1: % error from approximating change in Ratio by calculus rule.
Data in %, to 2 digits.

b \ t	2	4	8	16	32	64
2	0	.04	.11	.24	.45	.76
4	-.08	0	.15	.41	.85	1.5
8	-.47	-.31	0	.55	1.45	2.7
16	-2.2	-1.8	-1.2	0	1.9	4.7
32	-9.4	-8.6	-7.1	-4.4	0	6.2
64	-39.	-37.	-33.	-26.	-16.	0

For large values of $\underline{b}$, the error becomes unacceptably large for most purposes. E.g. when b is 64% and t is 2%, the estimate is 39% below the true Ratio, or 61% of the true Ratio:

$$\frac{1 + .02 - .64}{\frac{1.02}{1.64}} = \frac{.38}{.622} = .61 = 1 - .39$$

On the other hand, when $b = 2\%$ and $t = 4\%$, who's to worry about an error of .04%?

Notice that the error is much more sensitive to $\underline{b}$ than to $\underline{t}$. It would be helpful if we dealt with small values of $\underline{b}$, especially. But in discounting, $\underline{b}$ is normally larger than $\underline{t}$. Therefore we are driven to another easy approximation which is more accurate than (27) when $b > t$. This is the one we have used before, and will continue to use in this course, for discounting:

$$R_1 \cong R_0 \frac{1}{1 + b - t}$$

Table 2 shows why.

Table 2: Ratios calculated precisely, and by 2 methods of estimate

t	b	$\frac{1+t}{1+b}$	$1+t-b$	$\frac{1}{1+b-t}$
.02	.04	.9808	.9800	.9804
.04	.08	.9630	.9600	.9615
.08	.16	.9310	.9200	.9259
.16	.32	.8788	.8400	.8621
.32	.64	.8049	.6800	.7576

Use of the reciprocal form of estimate, the right column, cuts the error by half or more. You can show that (29) is always a better estimate, so long as $b > t$. Just apply the method of (28), and compare results. But I'll leave that to whosoever wishes to explore it, and move on.

How bad is the error in discounting? Take the second row of Table 2 and compare .9630⁵⁰ with .9615⁵⁰. Your discount coefficient becomes 7-1/2% too low, or 92.5% of the true figure. In this imperfect world, that's not bad, not bad at all.

Another benefit of using (29) is that it fits so nicely into the PV formula, where the Bottom turns out to be exactly $b - t$: --only we have called it $i - g$, where $(1 + i)$ is the Bottom of the discount formula (so $\underline{i}$ is $\underline{b}$), and $(1 + g)$ is the Top (so $\underline{g}$ is $\underline{t}$).

How come the Bottom turned out to be $i - g$ instead of $\frac{i - g}{1 + i}$? Because when we derived the formula (19), to keep it simpler, we made the first term $\frac{1}{1 + i}$ instead of 1. That let us multiply the Bottom by $(1 + i)$, and so remove the $(1 + i)$ from under $(i - g)$. This raises the Bottom, and lowers the PV.

It compensates for that bias if we make the Top a little higher. The effect of using $\frac{1}{1+i-g}$ in lieu of $\frac{1+g}{1+i}$ is exactly that. It lowers the second term, which is subtracted from one, and raises the Top.

So it leaves us with a nice simple rule that is pretty accurate, because the errors in it are compensating, as well as small. Before you use the PV formula, subtract any growth rates, g , from i . Add any decay rates. You can also, by the way, add the property tax rate, (T) , (because it is based on value, like interest). Then, just grind out your present value formulas, using $(i - g + d + T)$ in lieu of i , and tout est compris.

So, when and how can we use the calculus Quotient Rule?

1. To know the direction of change in a Ratio--up, down or sideways (if $t = b$, there's no change).
2. When t and b are small.
3. When $t \approx b$.
4. When small errors are tolerable.
5. When you're not going to raise the Ratio to a higher power.
6. When you're rusty or hurried or flustered and trying to get your bearings to get started.
7. When you're marooned on a desert island and your calculator batteries run down.

Complements, and Ratios

When a fixed resource is shared between two parties, their percentage shares add up to one, which means they are complements. Thus a football field is shared between two teams, Red and Blue. When Blue gains 4 yards, or 4% of the turf, Red loses an equal amount, and an equal 4% of the field.

But if Blue gains 4 yards from its 4-yard line, its percentage gain is 100%; while Red has lost $\frac{4}{100-4}$ or 4.17%. In absolute terms this is a "zero-sum game"--what Blue gains, Red loses, and the cynic is right, TANSTAAFL. Again, in terms of percentage of the whole field, Blue gains 4% and Red loses 4%. But in terms of what they had before, Blue gains much more than Red loses, and TITSTAAFL or "There Is Too Such a Thing As A Free Lunch."

Example. 85% of the water used by man in California goes into farming; the rest goes to cities. If 5% of farm use is transferred to urban use, how much do cities gain? $\frac{.05 \times .85}{1 - .85} = 28.33\%$

In general, the Ratio of the percentage gain of the Gainer to the percentage loss of the Donor can be formularized.

Let D = initial % held by Donor

Let X = % of D transferred

$$\frac{\% \text{ gain}}{\% \text{ loss}} = \frac{\frac{XD}{1-D}}{X} = \frac{D}{1-D}$$

Example. In a labor-intensive crop like strawberries, payroll cost is 91% of all costs. Wage rates drop by 2%; what happens to the cash flow of growers? Rises by $.02 \frac{.91}{1-.91} = 20.22\%$. (No wonder they fight Cesar Chavez!)

Example. The proceeds of a tree farm are divided between the Planters and the Investors. The Investor's share of the harvest value is:

$$I = \frac{(1+i)^n - 1}{(1+i)^n} = 1 - (1+i)^{-n}$$

The Planter's share is initially $\frac{1}{4}$; but the amount falls by 20%. How much does the Investor gain? $.20 \frac{.25}{1-.25} = .0667 = 6\frac{2}{3}\%$.

You may skip this. [How does that gain increase his rate of return? This is another whole problem, and we need to know n . Call it 20. Then we know i -- it must be .072, since $(1+i)^{20} = 4$. The new rate we call j .

$$1 - (1+j)^{-20} = 1.0667[1 - \frac{1}{4}] = .80$$

$$- (1+j)^{-20} = .80 - 1 = -.20$$

$$(1+j)^{20} = 5$$

$$j = 5^{1/20} - 1 = .0838$$

$$\frac{j}{i} - 1 = 16.39\%$$

]

Salesman Pete spends 3 times as much time servicing his account in the Bellevue area as he spends on the Fea Vista account, possibly because he enjoys watching the action at Bellevue Jazzercise. Manager Max tells him to shape up or ship out. So Pete increases his time at Fea Vista by 100%. How much less time does he spend at Bellevue? % loss = % gain $\frac{1-D}{D} = 1.00 \frac{1-.75}{.75} = \frac{.25}{.75} = \frac{1}{3}$.

Leverage.

$$\text{Equity} = \text{Assets} - \text{Debt}$$

$$E = A - D$$

$$E(1 + e) = A(1 + a) - D$$

$$1 + e = \frac{A(1 + a) - D}{A - D} = \frac{A - D + aA}{A - D} = 1 + a \frac{A}{A - D}$$

$$e = a \frac{A}{A - D}$$

$$\text{If } A = 2D, \quad \frac{A}{A - D} = \frac{2}{1}$$

$$\text{So } e = 2a$$

Powers and Elasticities.

We know how to differentiate $y = x^2$. The rate of change of y w.r.t. x is $2x$.

We know how to visualize this by looking at a square, anchored to the origin, growing on the top and the right side.

We know how to extend this to higher powers, using first a cube, then a set of cubes (where the number of cubes increases with the base), then a set of sets, etc.

Power functions are useful in engineering and economics because the power is the elasticity of response of y to x . E.g., $y = x^2$ means that a 1% Δx brings a 2% Δy . How do we know?

$$y = x^2$$

$$\ln y = 2 \ln x$$

$$\frac{d(\ln y)}{dx} = 2 \frac{d(\ln x)}{dx}$$

We could go on, but stop right here! Derivatives of lns are percentage changes, remember? So the % $\Delta y = 2(\% \Delta x)$. Or:

$$\epsilon = \frac{\% \Delta y}{\% \Delta x} = 2$$

When x is an input, and y is an output, ϵ is called the ϵ of production.

Now we'll finish the math:

$$\frac{y'}{y} = 2 \frac{x'}{x} = 2 \frac{1}{x} \quad (x' = \frac{dx}{dx} = 1)$$

$$\epsilon \equiv \frac{y'/y}{1/x} = 2$$

For those who find logs mysterious, here is another approach.

$$y = x^2$$

$$y' = 2x$$

$$\frac{y'}{y} = \frac{2}{x}$$

$$\epsilon = \frac{y'/y}{1/x} = 2 \quad (\text{again, } x' = 1)$$

Another useful trick here is to interchange the Bottom of the Top with the Top of the Bottom (that's legal), then:

$$\epsilon = \frac{y'/x'}{y/x} = \frac{\text{Marginal Product}}{\text{Average Product}} = \frac{MP}{AP}$$

Later we will invert ϵ to find MC/AC. But first, let's apply this all to some common geometric forms.

Rates of Change of Some Common Forms, w.r.t. Conventional Geometric Defining Element.

A. Symmetrical Forms.

1. Square: $A = B^2$

$S = 4B$ (S used for both Surface and Perimeter; as appropriate)

$$\frac{dA}{dB} = 2B$$

$$\frac{A'}{A} = \frac{2}{B}$$

$$\frac{A'/A}{1/B} = 2$$

$$\frac{A}{B} = B$$

$$\frac{A}{S} = \frac{B}{4}$$

2. Cube: $V = B^3$

$$S = 6B^2$$

$$\frac{dV}{dB} = 3B^2$$

$$\frac{dV/dB}{V} = \frac{3}{B}$$

$$\frac{V'/V}{1/B} = 3$$

$$\frac{V}{B} = \frac{B^3}{B} = B^2$$

$$\frac{V}{S} = \frac{B^3}{6B^2} = \frac{B}{6}$$

3. Circle: $A = \pi R^2$

$$S = 2\pi R$$

$$\frac{dA}{dR} = 2\pi R$$

$$\frac{dA/dR}{A} = \frac{2\pi R}{\pi R^2} = \frac{2}{R}$$

$$\frac{A'/A}{1/R} = 2$$

$$\frac{A}{R} = \pi R$$

$$\frac{A}{S} = \frac{\pi R^2}{2\pi R} = \frac{R}{2}$$

4. Sphere: $V = \frac{4}{3} \pi R^3$

$$S = 4\pi R^2$$

$$\frac{dV}{dR} = 4\pi R^2$$

$$\frac{dV/dR}{V} = \frac{3}{R}$$

$$\frac{V'/V}{1/R} = 3$$

$$\frac{V}{R} = \frac{4}{3} \pi R^2$$

$$\frac{V}{S} = \frac{R}{3}$$

B. Asymmetric Forms

1. Rectangle: $A = BH$

a) Constant Proportions: Pure Scale Effect

$$A = kB^2 \quad (H = kB)$$

$$S = 2(B + kB) = 2B(1 + k)$$

$$\frac{dA}{dB} = 2kB$$

$$\frac{dA/dB}{A} = \frac{2kB}{kB^2} = \frac{2}{B}$$

$$\frac{A'/A}{1/B} = 2$$

$$\frac{A}{B} = \frac{kB^2}{B} = kB$$

$$\frac{A}{S} = \frac{kB}{2(1 + k)}$$

b) Variable Proportions. H constant, B varies

$$S = 2(B + H)$$

$$\frac{dA}{dB} = H$$

$$\frac{dA/dB}{A} = \frac{H}{BH} = \frac{1}{B}$$

$$\frac{A'/A}{1/B} = 1$$

$$\frac{A}{B} = H$$

$$\frac{A}{S} = \frac{BH}{2(B + H)}$$

2. "Cuboid" or Box:

a) Constant Proportions: Pure Scale Effect

$$V = BHD = k\ell B^3 \quad \begin{array}{l} (H = kB) \\ (D = \ell B) \end{array}$$

$$S = 2(BH + BD + HD)$$

$$= 2B^2(k + \ell + k\ell)$$

$$\frac{dV}{dB} = 3k\ell B^2$$

$$\frac{dV/dB}{V} = \frac{3}{B}$$

$$\frac{V'/V}{1/B} = 3$$

$$\frac{V}{B} = k\ell B^2$$

$$\frac{V}{S} = \frac{k\ell B}{2(k + \ell + k\ell)}$$

b) Variable Proportions: $V = kB^2D$. (D held constant)

$$S = 2(B^2k + DB + BkD)$$

$$= 2B(Bk + D + kD)$$

$$= 2B[Bk + D(1 + k)]$$

$$\frac{dV}{dB} = 2kDB$$

$$\frac{dV/dB}{V} = \frac{2}{B}$$

$$\frac{V'/V}{1/B} = 2$$

$$\frac{V}{B} = kBD$$

$$\frac{V}{S} = \frac{kBD}{2[Bk + D(1 + k)]}$$

3. Cylinder:

a) Constant Proportions: Pure Scale Effect

$$V = \pi R^2 H = \pi R^3 k \quad (H = kR)$$

$$S = 2\pi R^2 + 2\pi RH$$

$$= 2\pi R^2(1 + k)$$

$$\frac{dV}{dR} = 3\pi k R^2$$

$$\frac{dV/dR}{V} = \frac{3}{R}$$

$$\frac{V'/V}{1/R} = 3$$

$$\frac{V}{R} = \pi k R^2$$

$$\frac{V}{S} = \frac{\pi k R^3}{2\pi R^2(1 + k)} = \frac{kR}{2(1 + k)}$$

b) Variable Proportions

$$V = \pi R^2 H$$

$$S = 2\pi R(R + H)$$

1) H held constant; R varies

$$\frac{dV}{dR} = 2\pi RH$$

$$\frac{dV/dR}{V} = \frac{2}{R}$$

$$\frac{V'/V}{1/R} = 2$$

$$\frac{V}{R} = \pi RH$$

$$\frac{V}{S} = \frac{\pi R^2 H}{2\pi R(R + H)} = \frac{RH}{2(R + H)}$$

2) R held constant; H varies

$$\frac{dV}{dH} = \pi R^2$$

$$\frac{dV/dH}{V} = \frac{1}{H}$$

$$\frac{V'/V}{1/H} = 1$$

$$\frac{V}{H} = \pi R^2$$

2. — $\frac{V}{S} = \frac{R}{2(R+H)}$ *Should be $\frac{RH}{2(R+H)}$*

4. "Torus" or circular tube (think inner tube, bagel, doughnut, ring, etc.)

$$V = 2\pi^2 AR^2 \quad A > R$$

$$S = 4\pi^2 AR$$

(R = Radius of tube)

(A = Radius from hub to center of tube)

a) Constant proportions $A = kR$ ($k > 1$)

$$V = 2\pi^2 kR^3$$

$$S = 4\pi^2 kR^2$$

$$\frac{dV}{dR} = 6\pi^2 kR^2$$

$$\frac{dV/dR}{V} = \frac{6\pi^2 kR^2}{2\pi^2 kR^3} = \frac{3}{R}$$

$$\frac{V'/V}{1/R} = 3$$

$$\frac{V}{R} = 2\pi^2 kR^2$$

$$\frac{V}{S} = \frac{2\pi^2 kR^3}{4\pi^2 kR^2} = \frac{R}{2}$$

b) Variable Proportions

1) Hold R constant; A varies

$$\frac{\partial V}{\partial A} = 2\pi^2 R^2$$

$$\frac{\partial V / \partial A}{V} = \frac{2\pi^2 R^2}{2\pi^2 A R^2} = \frac{1}{A}$$

$$\frac{V^1/V}{1/A} = 1$$

$$\frac{V}{R} = 2\pi^2 A R$$

$$\frac{V}{S} = \frac{2\pi^2 A R^2}{4\pi^2 A R} = \frac{R}{2}$$

2) Hold A const.; R varies

$$\frac{\partial V}{\partial R} = 4\pi^2 A R$$

$$\frac{\partial V / \partial R}{V} = \frac{2}{R}$$

$$\frac{V^1/V}{1/R} = 2$$

Conversion of Independent Variable to S, a Proxy for Cost

A. Symmetrical Forms: pure scale effects

1. Square: $A = B^2$

$$S = 4B$$

$$\frac{dA}{dS} = \frac{dA/dB}{dS/dB} = \frac{2B}{4} = \frac{B}{2}$$

$$\frac{dA/dS}{A} = \frac{B/2}{B^2} = \frac{1}{2B}$$

$$\frac{A'/A}{1/S} = \frac{4B}{2B} = 2$$

2. Cube: $V = B^3$

$$S = 6B^2$$

$$\frac{dV}{dS} = \frac{dV/dB}{dS/dB} = \frac{3B^2}{12B} = \frac{B}{4}$$

$$\frac{dV/dS}{V} = \frac{B}{4B^3} = \frac{1}{4B^2}$$

$$\frac{V'/V}{1/S} = \frac{6B^2}{4B^2} = 1.5$$

3. Circle: $A = \pi R^2$

$$S = 2\pi R$$

$$\frac{dA}{dS} = \frac{dA/dR}{dS/dR} = \frac{2\pi R}{2\pi} = R$$

$$\frac{dA/dS}{A} = \frac{R}{\pi R^2} = \frac{1}{\pi R}$$

$$\frac{A'/A}{1/S} = \frac{2\pi R}{\pi R} = 2$$

4. Sphere: $V = \frac{4}{3} \pi R^3$

$$S = 4\pi R^2$$

$$\frac{dV}{dS} = \frac{dV/dR}{dS/dR} = \frac{4\pi R^2}{8\pi R} = \frac{R}{2}$$

$$\frac{dV/dS}{V} = \frac{R/2}{\frac{4}{3} \pi R^3} = \frac{3}{8\pi R^2}$$

$$\frac{V'/V}{1/S} = \frac{3}{2} = 1.5$$

B. Asymmetric Forms

1. With constant proportions: pure scale effects

a) Rectangle: $A = BH = kB^2$ $(H = kB)^\dagger$

$$S = 2B + 2kB = 2B(1 + k)$$

$$\frac{dA}{dS} = \frac{dA/dB}{dS/dB} = \frac{2kB}{2(1+k)} = \frac{kB}{1+k}$$

$$\frac{dA/dS}{A} = \frac{kB}{kB^2(1+k)} = \frac{1}{B(1+k)}$$

$$\frac{A'/A}{1/S} = \frac{2B(1+k)}{B(1+k)} = 2$$

b) "Cuboid" or Box: $V = BHD = k\ell B^3$ $(H = kB)^\dagger$
 $(D = \ell B)$

$$S = 2(BH + BD + HD)$$

$$= 2(B^2k + B^2\ell + k\ell B^2)$$

$$= 2B^2(k + \ell + k\ell)$$

$$\frac{dV}{dS} = \frac{dV/dB}{dS/dB} = \frac{3k\ell B^2}{4B(k + \ell + k\ell)} = \frac{3k\ell B}{4(k + \ell + k\ell)}$$

$$\frac{dV/dS}{V} = \frac{3}{4B^2(k + \ell + k\ell)}$$

$$\frac{V'/S}{1/S} = \frac{3}{2} = 1.5$$

c) Cylinder: $V = \pi R^2 H = \pi k R^3$ $(H = kR)$

$$S = 2\pi R^2(1 + k)$$

$$\frac{dV}{dS} = \frac{dV/dR}{dS/dR} = \frac{3\pi k R^2}{4\pi R(1+k)} = \frac{3kR}{4(1+k)}$$

$$\frac{dV/dS}{V} = \frac{3}{4\pi R^2(1+k)}$$

$$\frac{V'/S}{1/S} = \frac{6}{4} = 1.5$$

[†] As a mnemonic, I will keep $k < 1$ at all times; $H < B$. Likewise, $k > \ell < 1$, $D < H < B$.

d) Torus: $V = 2\pi^2 kR^3$ $A = kR$

$$S = 4\pi^2 kR^2$$

$$\frac{dV/dR}{dS/dR} = \frac{6\pi^2 kR^2}{8\pi^2 kR} = \frac{3R}{4}$$

$$\frac{dV/dS}{V} = \frac{3R}{8\pi^2 kR^3} = \frac{3}{8\pi^2 kR^2}$$

$$\frac{V'/V}{1/S} = \frac{12\pi^2 kR^2}{8\pi^2 kR^2} = 1.5$$

2. Asymmetric forms with variable proportions

a) Rectangle: $A = BH$ (H constant; B varies)

$$S = 2(B + H)$$

$$\frac{dA}{dS} = \frac{dA/dB}{dS/dB} = \frac{H}{2}$$

$$\frac{dA/dS}{A} = \frac{1}{2B}$$

$$\frac{A'/A}{1/S} = \frac{2(B + H)}{2B} = 1 + \frac{H}{B}$$

b) "Cuboid" or Box: $V = BHD$ (H and D constant; B varies)

$$S = 2(BH + DB + HD)$$

$$\frac{dV}{dS} = \frac{dV/dB}{dS/dB} = \frac{HD}{2(H + D)}$$

$$\frac{dV/dS}{V} = \frac{1}{2B(H + D)}$$

$$\frac{V'/V}{1/S} = \frac{B(H + D) + HD}{B(H + D)} = 1 + \frac{HD}{B(H + D)}$$

c) Cylinder: $V = \pi R^2 H$

$$S = 2\pi R(R + H) = 2\pi R^2 + 2\pi RH$$

1) H held constant

$$\frac{dV}{dS} = \frac{dV/dR}{dS/dR} = \frac{2\pi RH}{4\pi R + 2\pi H} = \frac{RH}{2R + H}$$

$$\frac{dV/dS}{V} = \frac{RH}{(2R + H)\pi R^2 H} = \frac{1}{(2R + H)\pi R}$$

$$\frac{V'/V}{1/S} = \frac{2\pi R(R + H)}{(2R + H)\pi R} = \frac{2(R + H)}{2R + H} = \frac{2R + H + H}{2R + H} = 1 + \frac{H}{2R + H}$$

2) R held constant

$$\frac{dV}{dS} = \frac{dV/dH}{dS/dH} = \frac{\pi R^2}{2\pi R} = \frac{R}{2}$$

$$\frac{dV/dS}{V} = \frac{R}{2\pi R^2 H} = \frac{1}{2\pi RH}$$

$$\frac{V'/V}{1/S} = \frac{2\pi R(R + H)}{2\pi RH} = \frac{R + H}{H} = 1 + \frac{R}{H}$$

d) Torus: $V = 2\pi^2 AR^2$

$$S = 4\pi^2 AR$$

1) A held constant; R varies

$$\frac{dV}{dS} = \frac{dV/dR}{dS/dR} = \frac{4\pi^2 AR}{4\pi^2 A} = R$$

$$\frac{dV/dS}{V} = \frac{1}{2\pi^2 AR}$$

$$\frac{V'/V}{1/S} = 2$$

2) R held constant; A varies

$$\frac{dV}{dS} = \frac{dV/dA}{dS/dA} = \frac{2\pi^2 R^2}{4\pi^2 R} = \frac{R}{2}$$

$$\frac{dV/dS}{V} = \frac{1}{4\pi^2 AR}$$

$$\frac{V'/V}{1/S} = 1$$

Summary of Findings from Analysis of Geometric Forms

1. For simple power functions (e.g. $y = x^2$), ϵ always turns out to be the power (as we would expect, but it's good to check and confirm expectations).

2. For less simple power functions (e.g. $y = \frac{f(x^3)}{f(x^2)}$), ϵ turns out to be the ratio of the powers.

3. Because of (2), ϵ is lower (and thus scale economies are weaker) in 3 dimensions (3D) than in 2D. We are deceived and get this backwards if we calculate ϵ w.r.t. Base or Radius, as is customary, because B and R are not proxies for cost.

Thus for the cube and sphere, ϵ w.r.t. B or R is 3; but ϵ w.r.t. S is 1.5. In economics, of course, we are concerned with output w.r.t. cost, so the 1.5 is relevant. When we go from the Base to the Surface as our variable, the ϵ of the square stays at 2. But the ϵ of the cube falls from 3 to 1.5.

Much is made of the "law of the cube" as a source of scale economies in larger buildings, cars, etc. But $\epsilon = 1.5$ is an upper limit on this. So don't get carried away with the economy of 3D. 2D has the highest ϵ . It is generally only to save scarce land, not economies in building, that drives us high into 3D.

4. By extending the reasoning of (3), introducing more elements into an operation reduces ϵ even more. Thus if we build longer airplanes, and wider, and higher, and more of them, flying more often, with smaller seats, flying faster, we increase output to the 7th power, but costs to the 6th power, so $\epsilon = \frac{7}{6} = 1.167$ at best. And since we can't use the added height, and people hate smaller seats, etc., we won't do that well.

But to avoid asymmetry we must increase everything in proportion. Thus, one way or another, we run into limits on economies of scale.

5. Loss of symmetry reduces ϵ . (a) Symmetrical forms cannot lose symmetry, so ϵ never changes (not for geometrical reasons, anyway--there are others). Likewise asymmetric forms maintain their ϵ with scale when we hold the proportions fixed.

(b) But when the proportions vary, ϵ falls.

In all our examples, ϵ never falls below 1. That is because we are looking here only at the sources of economies of scale, not diseconomies. We can say, as to the first, that asymmetric growth destroys them progressively as asymmetry increases.

(c) But look on the bright side. Asymmetry raises ϵ for the dimension that is shorted. Thus for the rectangle, $\epsilon = 1 + \frac{H}{B}$. Think how high ϵ of B is when B is much less than H: 10, 11, 12, ... wow! A 100% increase of output for a 10% increase of cost? Whoever spots an opportunity like that will do well--and a few people have. TITSTAAFL.

To seize opportunities like that calls for an independent and self-confident mind. You have to think B when the herd is stampeding for H. That's why you develop your analytical abilities.

6. Don't be deceived by all these values of $\epsilon > 1$. We have simply omitted the factors that cause diminishing returns. Here is an obvious geometric one. Rooms are built to serve people, not just to enclose space. You are in the middle of your room as it grows larger. Every added cubic foot is farther from you, less accessible, less usable.

If you like geometric models, you can devise one where the product is not simply space, but space discounted by distance from the center of the room. There is diminishing returns.

As to cost, remember the site under the cube. As the building swells, you crowd the lot; or you buy more land. To afford a larger site, you may move to a worse location. Or you may expand vertically, losing symmetry and fighting gravity. All those factors make costs rise by more than by the Surface factor, as in our models, and lower ϵ .

7. When you devise the proposed factor that discounts space by distance from the center, you will find it accentuates the value of symmetry, and the penalties for asymmetry. Thus it doubly lowers the ϵ from any asymmetrical expansion, and forces ϵ below 1: diminishing returns.

8. I will not develop any formal model discounting for distance. We have had a bellyful of models. You can see the point intuitively, and that is enough for now, we have other work to do. But I do expect you to remember points 6 and 7, regardless. Unmodeled does not mean unimportant.

One management principle is that "what gets measured gets done." A teaching principle is that "what gets modeled gets learned." To offset that I'm underlining the reminder, above. Intuitive understanding is also important. Life is too short to model everything.