Logs: Teaching procedure

$y = e^x$

$x = \ln y$

$-y = e^x$

$y = -e^x$

$x \stackrel{?}{=} \ln(-y)$

$8 = 2^3$

$3 = \log_2 8$

$-3 = -(\log_2 8)$

$-8 = (-2)^3$

$3 = \log_{-2} -8$

A neg. no. could only come from a neg. base.
(+ only then if the power is an odd integer)
There is no log of a neg. no., to a + base.

Sensitivity & $\epsilon$ analysis: exponential fns. $A^B = C$ [i.e. are % changes]

Rewrite, using this outline

A: When ~~Rule 1.~~ A is the variable

Rule A-1: B is $\epsilon$, i.e. $\frac{e}{a}$

$(c = aB)$

$(C' = aBC_0)$

Rule A-~~1~~2 $c = BA'$ (% p. 3) $c = BD'$ $(D' = A')$

B: When ~~Rule 2~~: B is the variable

Rule B-1 $\Delta B = k\Delta C/C \equiv kc$

~~$c = BD'$ (p. 3)~~ If $A = e$, $k = 1$, $\Delta B = c$

Rule B-~~2~~3: $\epsilon \equiv B(A-1) = BD$ $(\equiv B \ln A)$

C: When A & B are both variable

Rule C-1: $(1+D)^B \equiv e^{BD}$

Rule C-2: $\Delta(B \cdot D) = c$

Rule C-3: $B \cdot D = \epsilon = \frac{c}{d + b + bd}$

Rule C-4:

[In final write-up, change notation so: $B^N = F$, $B = 1 + D$ ) 2-(

Rule B-1: Given an exp. fn. $e^B = C$ + $B = \ln C$
The absolute $\Delta$ in ln. or exp., B, $\cong$ % $\Delta$ in fn.
$C'/C \cong c = B'$
(Over, Top)

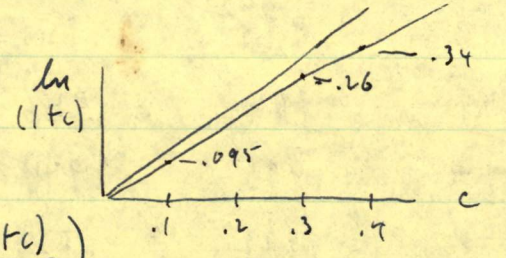
Proof: $e^B = C$
$e^{B+B'} = C(1+c)$
$e^{B'} = 1+c$
$B' = \ln(1+c) \cong c$

→ If you know $B' = \ln(1+c)$, and want to find c exactly,
$c = e^{B'} - 1$

Lemma A. [Memorable alt: $\ln A \cong A - 1$]
~~Rule + holds true~~
$\ln(1+c) \cong c$ in the limit as $c \to 0$
Using calc., $B = \ln C$
$B' = C'/C = c$

When a Rule depends on $x = \ln(1+c)$ it is a calc. rule which is true exactly in the limit as $x \to 0$

For larger c




(Better, graph $\frac{\ln(1+c)}{c}$) — (over)

Lemma B.
Rule holds for ln, not log
$\ln A = \log A \times \ln 10 = \log A \times 2.3026\ldots$

Proof: $x = \log A$ $y = \ln A$
$e^y = 10^x$

$y = x \ln 10 = x \cdot 2.3026$
or $x = y \log e = y \cdot .4342$

In general, $A^B = C$

$A^{B(1+b)} = C(1+c)$

$A^{B'} = 1+c$

$B' \ln A = \ln(1+c) \doteq c$

$(a \equiv c)$

$(b = 2a)$

| $c$ | $\frac{\ln(1+c)}{c}$ | $\frac{1+c}{e^c}$ | $\frac{c}{e^c-1}$ | $\frac{\ln(1+c)}{e^c-1}$ | $\frac{a/b}{\ln(1+a)/\ln(1+b)}$ |
|---|---|---|---|---|---|
| .005 | .998 | | .998 | | .998 |
| .01 | .995 | .999 | .995 | .990 | .995 |
| .02 | .990 | .999 | .990 | .980 | .990 |
| .03 | .985 | .999 | .985 | | .986 |
| .04 | .981 | .999 | .980 | | .981 |
| .05 | .976 | .999 | .975 | .952 | .977 |
| .06 | .971 | .998 | .970 | | .972 |
| .07 | .967 | .998 | .965 | | .968 |
| .08 | .962 | .997 | .961 | | |
| .09 | .958 | .996 | .956 | | |
| .10 | .953 | .995 | .951 | .906 | |
| .11 | .949 | .994 | .946 | | |
| .12 | .944 | .993 | .941 | | |
| .13 | .940 | .992 | .936 | | |
| .14 | .936 | .991 | .932 | | |
| .15 | .932 | .990 | .927 | .864 | .939 |
| .16 | .928 | .988 | .922 | | |
| .17 | .924 | .987 | .917 | | |
| .18 | .920 | .986 | .913 | | |
| .19 | .916 | .984 | .908 | | |
| .20 | .912 | .982 | .903 | .823 | .923 |
| .21 | | .981 | .899 | | |
| .22 | | .979 | .894 | | |

Procedure: To find new insights + oper. rules, subst. ~~D for~~ 1+D for A, (where D = A-1).

$(1+D)^B = C$

$B \ln(1+D) = \ln C$

$BD \cong \ln C$

$e^{BD} \cong C$

~~A B = 1+D~~ $(1+D)^B = C$

But ~~1+D~~ $1+D \cong e^D$ (a very close approx.)

$\therefore (1+D)^B \cong e^{DB}$

Rule 3: An exp. fn. $(1+D)^B \underset{<}{\cong} e^{BD}$ (or $1+D \cong e^D$)

Lemma A.

D + B have nearly identical influence on the fn. value; they are virtually interchangeable.

Lemma B.

The interchangeability depends on $D_{\text{and } B}$ being "small". In practise, $D < .15$ is a good standard; or $DB = 1.5$

E.g.

$1.15^{10} = 4.0456$

$e^{1.5} = 4.4817$ = 11% too high

$e^{.15} = 1.162$, only 1% off. But raising to 10th power magnifies error.

Lemma C.

Can offset bias totally by ff.

$(1+D)^B = e^{B(\ln(1+D)}$ $\ln(1+r) \equiv \rho$ is convention

$1.15^{10} = 4.045558$ This is exact.

Lemma D (over)

Lemma D — This is really an approach to Rule 3

Since $e^{BD} = C$

$BD = \ln C$

But from Rule 1, $\Delta[\ln C] = c$

$\therefore \Delta[BD] = c$

$$\frac{\Delta[BD]}{BD} \equiv \beta$$

$$C = \frac{c}{\beta} = \frac{\Delta[BD]}{\Delta[BD]/BD} = BD$$

$\therefore C = BD$

Rule ~~4~~: The exp. is the $\epsilon$ of the fn., where $A$ is the vble. Use proof from Lemma D, back of prior page.

~~$(A+B)^B = A^B$~~

$$C = A^B$$

$$\frac{dC}{dA} = BA^{B-1} = \frac{B}{A}A^B = \frac{B}{A}C = \frac{C}{A}\cdot B$$

$$B = \frac{c'/c}{'/A}\cdot A = \frac{c'/c}{'/A} = \frac{c'}{c/A} = \frac{\%P}{aP}$$

$\parallel$

$\epsilon$

$$y = aX^{\alpha}$$

$$\frac{dy}{dx} = a\alpha X^{\alpha-1}$$

$$\frac{y'}{y} = \frac{\alpha}{x}$$

$$\frac{y'/y}{'/x} = \alpha = \epsilon$$

W/o Calc. $A^B = C$

$$A(1+a)^B = C(1+c)$$

$$(1+a)^B = 1+c$$

$$B\ln(1+a) = \ln(1+c)$$

$$B = \frac{\ln(1+c)}{\ln(1+a)} \simeq \frac{c}{a} = \epsilon$$

Add p. 1-(

QU/L/LN, asN to $x \doteq y$ key.

Rule A.2

$$A^B = C$$

$$[A(1+a)]^B = C(1+c)$$

$$(1+a)^B = 1+c$$

$$B \ln(1+a) = \ln(1+c)$$

$$B = \frac{\ln(1+c)}{\ln(1+a)} \doteq \frac{c}{a} = \epsilon$$

(a close approx. when either $c$ & $a$ are small, or $B$ is near 1.

Calc. $B A^{B-1} = \frac{dC}{dA}$

$$B A^{-1} = C'/C$$

$$B = \frac{C'/C}{1/A} = \epsilon$$

Thus, it has twice as good a chance of being close as only. approxs. like $\frac{\ln(1+c)}{c} \doteq 1$.)

Lemma A: $c \doteq aB$

Lemma B: $c = (1+a)^B - 1$

Lemma C: ~~Lemma~~ $aB \doteq (1+a)^B - 1$

Lemma D: $aB \doteq e^{aB} - 1$

$\therefore c = e^{aB} - 1$

Rule: When the exponent varies, $\epsilon \doteq B(A-1) = BD$

$$A^B = C$$

Rule B-2 $$A^{B(1+b)} = C(1+c)$$

$$A^{Bb} = 1+c$$

$$Bb \ln A = \ln(1+c)$$

$$B \ln A = \frac{\ln(1+c)}{b} \doteq \epsilon$$

Let $D = A - 1$

Then $B \ln(1+D) \doteq \epsilon$

$$BD \doteq \epsilon$$

Lemma A: $c = C_0^b - 1 \quad (= A^{Bb} - 1)$

Lemma B: $1 + c = C_0^b$

$$C_1 = C_0(1+c) = C_0 C_0^b = C_0^{(1+b)}$$

Lemma C: Alt. route leads to ~~the same answer~~ same answer.

$$B \ln A = \ln C$$

$$B(1+b) \ln A = \ln[C(1+c)] = \ln C + \ln(1+c)$$

$$1 + b = 1 + \ln(1+c)/\ln C$$

(over)

$$b = \ln(1+c) / \ln C$$

$$\frac{\ln(1+c)}{b} = \ln C = B \ln A \doteq B(A-1) = BD$$

$$c \doteq BD$$

$\left(B = \overset{A-1}{\blacksquare}\right)$

$e^{DB} = C_0$ (1)

$e^{D(1+d)B(1+b)} = C_0(1+c)$ (2)

$e^{DB(d+b+bd)} = 1+c$ (3) = (2)/(1)

1. $c = e^{DB(d+b+bd)} = C_0^{(d+b+bd)}$

2. $DB(d+b+bd) = \ln(1+c) \doteq c$

Rule C-3 $\frac{c}{d} \doteq DB\left(1 + \frac{b}{d} + b\right)$ Now let $b = 0$. (partial diff(n.)

✓ Rule $\overset{B-2}{\blacksquare}$: The ~~sum~~ % sensitivity ($\epsilon$) of exp. fn. to ~~either~~ the exponent, ~~or [Base +]~~, is $DB$.

i.e. if $DB = 1$, $c = d$

Rule A-1 Generally, $c = DB \cdot d$

E.g.

| D | B | DB | d | c |
|---|---|---|---|---|
| | | | .20 | |
| .10 | 10 | 1.0 | ■ | .20 |
| .15 | 30 | 4.5 | .067 | .30 |
| .01 | 20 | .2 | 1.00 | .20 |
| .02 | 20 | .4 | .5 | .20 |
| .04 | 20 | .8 | .25 | .20 |
| .08 | 20 | 1.6 | .125 | .20 |
| .16 | 20 | 3.2 | .0625 | .20 |

$\frac{d \ln(1+c)}{dc} = \frac{1}{1+c}$

For any value of B, if you add .01 to D, c is the same!

Res:

$d = D'/D$

$\therefore \quad DB \cdot d = BD'$

and $c = BD'$

(Remember: "Base" is A, in $A^B = C$. $A = 1+D$. $A' = D'$)

Rule 4: The % Δ in fn. = exponent × absolute Δ base;
(or base × absolute Δ exponent?)

(Reconcile w. Rule 1. If base = e, the % Δ in fn. (c) is abs. Δ exp., not that times e.)

$D' = .01$

| D | B | DB | b | c |
|---|---|---|---|---|

$$c = \left[A(1+a)\right]^{Bb}(1+a)^{B} - 1$$

$$c = A^{Bb}(1+a)^{B(1+b)} - 1$$

---

Simpler is to let $A = 1 + D$

$$1.0D^{B} = C$$

$$B\ln(1.0D) = \ln C \quad ; \quad BD \doteq \ln C$$

~~But we already know~~ But we already know $\Delta \ln C = c$

---

Try $e^{DB} = C$

$$DB = \ln C$$

$$\left[\frac{dD}{dx}\cdot B + \frac{dB}{dx} D = \frac{dC}{dx} \Big/ C = c\right]$$

$$D(1+d)\ B(1+b) = \ln[C(1+c)] = \ln C + \ln(1+c)$$

$$(1+d)(1+b) = 1 + \frac{\ln(1+c)}{\ln C}$$

$$1 + d + b + bd - 1 = \frac{\ln(1+c)}{\ln C}$$

$$\ln(1+c) = \ln C(d + b + bd)$$

$$c \doteq \ln C(d+b)$$

| A | a | ~~A~~ D | ~~a~~ d | B | b | $C_0$ | c | $C_1$ | c |
|---|---|---|---|---|---|---|---|---|---|
| 2 | .1 | 1 | .2 | 3 | .05 | 8 → .52 | $= \ln C(d+b)$ | | $= \frac{A_1^{B_1}}{A_0^{B_0}} - 1$ .50 |
| 3 | .2 | 2 | .3 | 4 | .08 | 81 | 1.61 | 253 | 2.1242 |

Not so good.

Let's go one variable at a time.

$$A^B = C$$

$$[A(1+a)]^B = C(1+c)$$

~~$c = \frac{A}{C}$~~ $(1+a)^B = 1+c$

$$c = (1+a)^B - 1$$

$2^3 = 8$

$2(1.1)^3 = 10.648$

$c = .3310$

$1.1^3 - 1 = .3310$

$3^4 = 81$

$3(1.2)^4 = 161.98$

$c = 1.0736$

$1.2^4 - 1 = 1.0736$

OLS

$\frac{c}{a}$ and $\frac{c}{b}$

1. $\frac{c}{a} = \frac{(1+a)^B - 1}{a}$

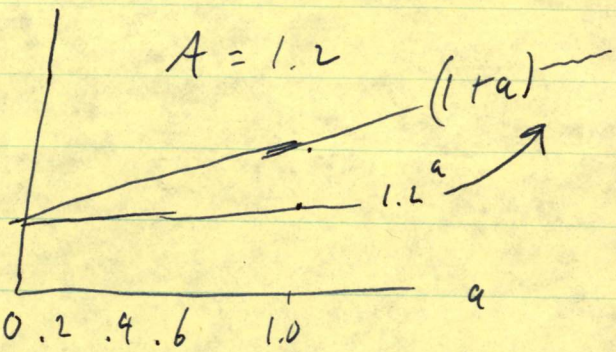
2. $\frac{c}{b} = \frac{A^{Bb} - 1}{b}$

Which is larger? Let $a = b$; $A > 1$

$$\frac{(1+a)^B - 1}{a} \overset{?}{>} \frac{A^{Ba} - 1}{a}$$

$$(1+a)^B \overset{?}{>} A^{Ba}$$

$$1+a \overset{?}{>} A^a$$

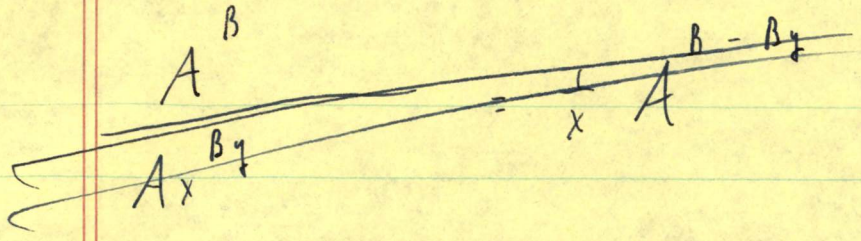



For small + even fairly large a + b, c is more sensitive to a than b. But:

For very small a + b, they are close to the same

For very large a + b, more sensitive to b than a

For $A \overset{\text{L}}{=} 1$, c is more sensitive to a. In fact, the value of $B_1$ has no effect at all on $C_1$.




$$\frac{(Ax)^{By}}{A^{B}} = x^{By} A^{By}$$

$$\frac{A^{By}\, x^{By}}{A^{B}} = A^{B(y-1)}\, x^{By}$$

$c = C_0^{b} - 1$ or $c = C_0^{d} - 1$

If you double D or B, d or b = 1

Then $c = C_0 - 1$ $\left((1+c) = C_0\right)$

If $C_0 = e^{.11 \cdot 10} = e^{1.1} = 3.004$

$c = 2.004$

$C_1 = C_0(1+c) = 3.004 \cdot 3.004 = C_0^{2}$ (!)

$e^{.11 \cdot 10} = e^{1.1} = 3.004$

$e^{.11 \cdot 20} = e^{2.2} = 9.025$

~~$e^{D_0 B_0 (1+x)} = C_0^{(1+x)} = C_0^{(1+x-1)} = C_0^{x}$~~

~~$e^{D_0 B_0} = C_0$~~

~~$e^{(1+x)} = e^{D_0 B_0 x} = C_0^{x}$~~

$C_1 = C_0(1+c)$

But $1+c = C_0^{b}$

$\therefore C_1 = C_0 \cdot C_0^{b} = C_0^{(1+b)}$

Actually, that follows directly from $e^{DB(1+x)} = C_0^{(1+x)}$

Just raise both sides to $(1+x)$ power

$$e^{DB} = C$$

$$DB = \ln C$$

$$DB(1+b) = \ln[C(1+c)] = \ln C + \ln(1+c)$$

$$1+b = 1 + \frac{\ln(1+c)}{\ln C}$$

$$\ln(1+c) = b \ln C$$

$$1+c = C^b$$

$$c = C^b - 1$$

$$e^{.05 \cdot 20} = e$$

$$e^{.05 \cdot 20(1.1)} = 3.004166$$

$$c = .105171$$

$$C^b - 1 = .105171$$

$$e^{.15 \cdot 50} = 1808$$

| b | c |
|---|---|
| .01 | .078 |
| .02 | .162 |
| .03 | .252 |
| .04 | .350 |

The normal range of values w. which I am concerned is

$1.01 < A < 1.6$

$.01 < a < .20$

$.10 < b < 3.0$

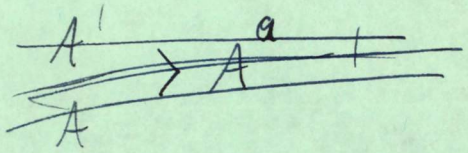
In that range, $A^B = C$ is more elastic to A than to B. and the values of E are:

$$\frac{c}{a} = \frac{(1+a)^B - 1}{a}$$

$$\frac{c}{b} = \frac{A^{Bb} - 1}{b}$$

$1+a > A^a$ when

$a > A^a - 1$




Let $A = 1.2$

$A = 2$

| $a$ | $1.2^a - 1$ | $2^a - 1$ |
|---|---|---|
| .01 | .0018 | |
| .02 | .0036 | |
| .05 | .0092 | |
| .1 | .0184 | |
| .2 | .037 | |
| .4 | .076 | |
| .8 | .157 | .74 |
| 1.0 | .2 | 1.0 |
| 1.6 | .338 | 2.03 |
| 3.2 | .792 | |
| 6.4 | 2.21 | |
| 10.0 | 5.2 | |
| 15.0 | 14.41 | |
| 15.30 | 15.27 | |
| 15.5 | 15.88 | |

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| | |
|---|---|
| .001 | .0002 |
| .0001 | .00002 |

$$e^{\Delta B} = C$$

$$\Delta B = \ln C$$

$$\Delta(1+d)B(1+b) = \ln[C(1+c)] = \ln C + \ln(1+c)$$

$$(1+d)(1+b) = 1 + \frac{\ln(1+c)}{\ln C}$$

$$d + b + bd = \frac{\ln(1+c)}{\ln C}$$

$$c \cong \ln C[d + b + bd]$$

| $\frac{\Delta}{.05}$ | $\frac{B}{20}$ | $\frac{\Delta B}{1.0}$ | $\frac{e^{\Delta B}}{e}$ | $\frac{d}{.20}$ | $\frac{\ln C}{1.0}$ | $\frac{c}{.20}$ |
|---|---|---|---|---|---|---|

$$\ln C[d + b + bd] = \ln(1+c)$$

$$C^{(d+b+bd)} = 1 + c$$

$$c = C_0^{(d+b+bd)} - 1$$

$$\therefore \quad C_0^{(d+b+bd)} = \frac{C_1}{C_0}$$

$$C_0^{(1+d+b+bd)} = C_1$$

e.g. $e^{.12 \cdot .15} = C_0 = 6.0496$

$e^{.12 \cdot .15(1+.09)(1+.14)} = C_1 = 9.362$

$6.0496^{(1.09 \cdot 1.14)} = 9.362$

$D = .11$

$B = 8$

$C_0 = e^{.88} = 2.4109$

$C_n = C_0^{(1+x)}$

$C_n = e^{DB(1+x)}$

| | 1 X | 2 $C_n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | .01 | 2.432 | | | | | | | |
| 2 | .05 | 2.519 | | | | | | | |
| 3 | .20 | 2.875 | | | | | | | |
| 4 | .40 | 3.428 | | | | | | | |
| 5 | .70 | 4.464 | | | | | | | |
| 6 | 1.00 | 5.812 | | | | | | | |
| 7 | 2.00 | 14.013 | | | | | | | |
| 8 | 5.00 | 196.31 | | | | | | | |
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AMPAD EFFICIENCY® LINE No. 2636

$$\frac{\ln(1+c)}{c} \stackrel{?}{=} \frac{(1+c)^n}{e^{cn}} = \frac{1+c}{e^c}$$

$$e^{ck} \stackrel{?}{=} \frac{c(1+c)^k}{\ln(1+c)} = \frac{c}{\ln(1+c)} + \frac{c^2}{\ln(1+c)}.$$

$$\frac{1}{\ln x} \stackrel{?}{=} \ln \frac{1}{x} = \ln(x^{-1}) = -1 \cdot \ln x$$

$$(\ln x)^{-1}$$

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$$\frac{(1+r)^n}{e^{rn}} = \frac{1+r}{e^r} \quad \text{for all } n\text{!?}$$

$$\frac{1.22^1}{\left(1+\frac{.22}{12}\right)^{12}} = \frac{1.2200}{1.2436}$$

$$e^{.22} = 1.246$$

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$$\frac{\ln(1+r)}{r} \stackrel{?}{=} \frac{r}{e^r - 1}$$

$$\ln(1+r) \stackrel{?}{=} e^r - 1$$

If $1 + d \doteq e^d$

Then $\ln(1+d) \doteq d$

If $(1+d) = k e^d$

Then $\ln(1+d) = \ln k + d$

$\ln(1+d) - d = \ln k = K$ No

If $(1+d) = e^d - k$

$e^d - (1+d) = k$ No

# Quick Calculus.

Calc. is ~~quick~~ sensitivity analysis. What happens to solution when elements ("arguments") change?

1. $AB = C$ ~~$a\,b = ab$~~ $\qquad a = \frac{A'}{A}$, etc

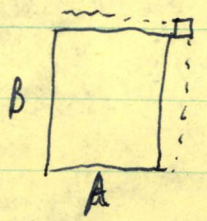
$A(1+a)\,B(1+b) = C(1+c)$

Find $c$

$(1+a)(1+b) = 1+c$

$1 + a + b + ba = 1 + c$

$c = a + b + ba$




Calc. is when $ba$ drops out

E.g. $\ln A + \ln B = \ln C$

~~$dA$~~

$\frac{A'}{A} + \frac{B'}{B} = \frac{C'}{C}$

2. $\frac{A}{B} = C$

$$\frac{A(1+a)}{B(1+b)} = C(1+c)$$

$$\frac{1+a}{1+b} = 1+c$$

$$1+a = 1+c+b+bc$$

$$c = a-b-bc$$

~~(But what's wrong? If $a=b$, $c$ std. $bc=0$)~~

$$c+bc = a-b$$

$$c = \frac{a-b}{1+b}$$

Calc. is when $1+b$ drops out.

$$\ln A - \ln B = \ln C$$

$$a - b = c$$

3. $A^B = C$

$$[A(1+a)]^{[B(1+b)]} = C(1+c)$$

$$\frac{A^{[B(1+b)]}(1+a)^{[B(1+b)]}}{A^B} = \frac{C(1+c)}{C}$$

$$= [A(1+a)]^{Bb}(1+a)^B$$

$$A^{Bb}(1+a)^{B+Bb} = 1+c = (A^B(1+a)^B)$$

$$[A(1+a)]^{Bb}(1+a)^B - 1 = c$$

E.g. $2^3 = 8$

$$2(1.1)^{3(1.05)} = 2.2^{(3.15)} = 11.9848$$

Let $8(1+c) = 11.9848$

$$c = \frac{11.9848}{8} - 1 = .4981$$

$$2(1.1)^{3\times.05} \quad 1.1^3 - 1 = 1.1255 \times 1.331 - 1 = 1.4981 - 1 = .4981$$

$$[B(1+b)]\left[\ln A + \ln(1+a)\right] = \ln C + \ln(1+c)$$

$$B(1+b)(\ln A + a) \cong \ln C + c$$

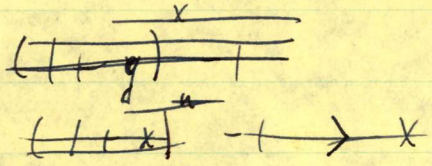
$e^x = y$

$\ln(e^x) = x$

$e^{\ln e^x} = e^x$

$e^x \geq 1 + x$

$e^x - 1 \geq x \geq \ln(1+x)$



$(1+y)^x - 1 \geq xy$

$\frac{(1+y)^x - 1}{y} \geq x$

$(1+y)^x \geq xy + 1$

$x \ln(1+y) \geq \ln(xy+1)$

$\frac{\ln(xy+1)}{x\ln(1+y)} = .967$ for $y = .01$, $x = 8$

not very close

| | |
|---|---|
| $x$ | .20 |
| $e^x - 1$ | .2214 |
| $\ln(1+x)$ | .1823 |
| $\sqrt{(e^x-1)\ln(1+x)}$ | .200913 : .995 |
| $\frac{[(e^x-1)\ln(1+x)]}{x^2}$ | = .991 |

Best approx's.
But simple mean is as good.

$2^3 = 8$

| b | $B'$ | $C_n = 2^{(3+B')}$ | $C_n/8 - 1$ | c | c/b |
|---|---|---|---|---|---|
| .01 | .03 | 8.168 | .021 | | 2.1 |
| .02 | .06 | | | | 2.123 |

No !

$A^B = C$

$[A(1+a)]^B = C(1+c)$

$A(1+a)^B = 1+c$

$c = (1+a)^B - 1$

$\frac{c}{a} = \frac{(1+a)^B - 1}{a} \longrightarrow B$ as $a \longrightarrow 0$

Calc $C = A^B$

$\ln C = B \ln A$

? $\frac{dC}{dA} = B \frac{dA/dA}{A} = \frac{B}{A}$

$\frac{dC}{C}$

$$A^B = C$$

$$A^{B(1+b)} = C(1+c)$$

$$A^{Bb} = 1 + c$$

$$Bb = B'$$

$$c = A^{B'} - 1$$

$$A^{Bb} = C^b \quad , \quad \therefore \quad c = C_0^b - 1$$

$$\therefore \quad C_0^b = \frac{C_1}{C_0} \quad , \text{ or } \quad C_0^{(1+b)} = C_1$$

$$\therefore \quad C_0^{(1+b)} = C_0(1+c)$$

$$\epsilon = \frac{c}{b} = \frac{C_0^b - 1}{b} \quad \overset{?}{=} \quad B \; ?$$

---

Calc. $B \ln A = \ln C$

B is vble; A const.

$$\frac{dC}{dA} = BA^{B-1}$$

$$\frac{C'}{C} = \frac{BA^{B-1}}{A^B} = BA^{-1}$$

$$\frac{C'/C}{'/A} = B$$

$$\frac{dC}{dx} \cdot \frac{1}{C} = \ln A \frac{dB}{dx} \quad \therefore \quad \frac{C'/C}{B'/B} = \ln A \cdot B$$

7. Transition to Cobb-Douglas technique of analyzing ecs. of scale.

Elast. of prodn. $\dfrac{\dfrac{\Delta P}{P}}{\dfrac{\Delta I}{I}} = \dfrac{\dfrac{\Delta P}{\Delta I} \cdot}{\dfrac{P}{I}} = \dfrac{MP}{AP}$

Is technique for analyzing ecs. of scale.

- If input is just one of several, and product is net
- If input is all inputs. I prefer the first method, because it lets proportions vary, which is essence of scale.

This is opt. where $MNP = ANP$ of indep. vble. input.

8. In Cobb-D. technique, you state relationship as equ., where elast. of prodn. is ~~parameter~~ unknown, i.e. dependent variable.

$$y = k a^{\alpha} b^{\beta}$$

Show that $\alpha$ & $\beta$ are elasts. of prodn.

a. Numerical example.

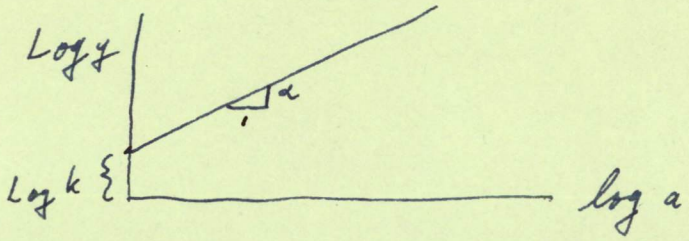
$$y = x^2$$

If x doubles, y quadruples

i.e., $\dfrac{\%\,\Delta y}{\%\,\Delta x} = 2$

b. Graphical

$y = k \cdot a^{\alpha}$ ; so $\log y = \log k + \alpha \log a$




$$\frac{\Delta \log y}{\Delta \log a} = \alpha$$

But the increase in a log is the % $\Delta$ of the vble.

That is, a const. $\Delta$ of a log is a const. % $\Delta$ of a vble.

$$\text{So} \quad \alpha = \frac{\%\,\Delta y}{\%\,\Delta a}$$

c. Differentiation

$$\frac{d \log y}{da} = 0 + \alpha \frac{d \log a}{da}$$

$$\frac{\frac{dy}{da}}{y} = \alpha \frac{\frac{da}{da}}{a}$$

$$\alpha = \frac{\%\,\Delta y}{\%\,\Delta a}$$

9. The basic idea is, if you can $\Delta \frac{\text{output}}{\text{input}}$ by $\Delta$ input, you're still realizing ecs. of scale.

You can set this up any way you want.

Problem is getting data.

General bias of empirical studies, overstate ecs. of scale by understating output of small firm (psychic income) and understating inputs of large (undervalued land.)

10. Opt. scale,

where sum of $\alpha$ & $\beta$ = 1.

$$\text{So } \frac{\%\,\Delta y}{\%\,\Delta a} + \frac{\%\,\Delta y}{\%\,\Delta b} = 1$$

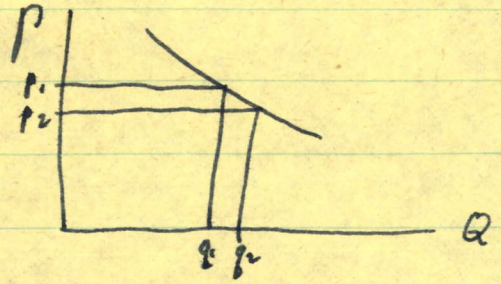
Then a 1% $\Delta$ of a and b $\rightarrow$ 1% $\Delta y$

---

Using MNP approach, opt. is where $\frac{MNP}{ANI} = 1$.

The $\Delta$ of comply. inputs has been "paid off" by netting from MNP.

# Elasticity of Production and Marginal Product




$$e = \frac{\Delta Q/Q}{\Delta P/P} = \frac{P \cdot \Delta Q}{Q \cdot \Delta P}$$

$$y = P_2 \cdot \Delta Q$$

$$x = Q_1 \cdot \Delta P$$

$$MR = y - x = P \cdot \Delta Q - Q \cdot \Delta P$$

For unit basis, divide by $\Delta Q$. Must explain this. $\frac{\Delta P}{\Delta Q}$ means the $\Delta P$ that corresponds with $\Delta Q = 1$.

$$\frac{MR}{\Delta Q} = \frac{y-x}{\Delta Q} = P - \frac{Q \cdot \Delta P}{\Delta Q} = P - P\,\frac{Q \cdot \Delta P}{P \cdot \Delta Q} = P - \frac{P}{e}$$

---

E of production, on other hand, is reciprocal of E of demand.

$$e_p = \frac{Q \cdot \Delta P}{P \cdot \Delta Q}$$

$$\frac{y-x}{\Delta Q} = P - \frac{Q \cdot \Delta P}{\Delta Q} = P - P\,\frac{Q \cdot \Delta P}{P \cdot \Delta Q} = P - Pe$$

so,

$$MP = aP - aP\frac{MP}{aP} = aP - MP$$

$$2MP = aP$$

This makes no sense!

Re e of demand,

$$\frac{\Delta Q/Q}{\Delta P/P} = \frac{\Delta Q/\Delta P}{Q/P} =$$

$Q/P$ is, say, $\frac{100 \text{ bushels}}{\$2.00 \text{ per bushel}} = 50 \frac{\text{bu.}^2}{\$}$ (?)

Unit problems are confusing!

This is not wrong, but confusing, because unit is $\Delta P$ instead of $\Delta Q$.

$$MR \text{ also} = \frac{dR}{dP}$$

$$R = P \cdot Q$$

$$\frac{dR}{dP} = \frac{d(P \cdot Q)}{dP} = \frac{dP}{dP} \cdot Q + \frac{dQ}{dP} \cdot P = Q + \frac{dQ}{dP} \cdot P$$

---

$$MR \text{ also} = \frac{dR}{dQ}$$

OK. $$\frac{dR}{dQ} = \frac{d(P \cdot Q)}{dQ} = P\frac{dQ}{dQ} + Q\frac{dP}{dQ} = P + Q\frac{\Delta P}{\Delta Q}$$

Now, let $P$ be <u>Ave. Product</u>; and $Q$ be no. of workers.
$TP$ is Total Product.

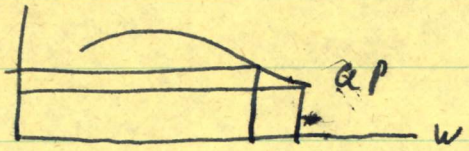
$$MP = \frac{d(TP)}{dQ} = \frac{d(P \cdot Q)}{dQ} = P\frac{dQ}{dQ} + Q\frac{dP}{dQ} = P + Q\frac{dP}{dQ}$$

$$= P + P\frac{Q \cdot dP}{P \cdot dQ} = P + P\frac{dP/P}{dQ/Q} = P + P\frac{dP/dQ}{P/Q}$$

NB. $dP/dQ$ is <u>not</u> $\frac{d(TP)}{dQ}$

and $P/Q = \frac{TP/Q}{Q} = \frac{TP}{Q^2}$ ... strange!

# Elast. of Prodn., Ave. Prod. & Mgnl. Prod.




$P$ = total product

$$MP = \frac{dP}{dw}$$

$$P = w \cdot \frac{P}{w}$$

$$\frac{dP}{dw} = w\left(\frac{\frac{dP}{dw}\cdot w - P}{w^2}\right) + \frac{P}{w} = \frac{dP}{dw} - \frac{P}{w} + \frac{P}{w} = \frac{dP}{dw}$$

---

$$\frac{dP/dw}{P/w} = \frac{dP/P}{dw/w} = \text{elast. of prodn.}$$

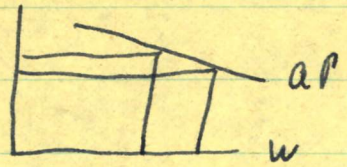
To prove:

$$dP/dw = \frac{P}{w} - \frac{P}{w}\,\frac{dP/dw}{P/w} = \frac{P}{w} - dP/dw$$

$$2\,\frac{dP}{dw} = \frac{P}{w}$$

So, it's not true! Except when $2\,\frac{dP}{dw} = \frac{P}{w}$

MP & aP




$\Delta MP = y - x = \Delta w \cdot aP + W \cdot \Delta aP \qquad (\Delta aP < 0)$

$\frac{\Delta P}{\Delta w} \equiv MP = aP + W \frac{\Delta aP}{\Delta w} \qquad (\Delta aP < 0)$

$\frac{\Delta aP}{\Delta w} = \frac{\Delta (P/w)}{\Delta w} = \frac{\frac{\Delta P}{\Delta w} \cdot w - P}{w^2} = \frac{\Delta P/\Delta w}{w} - \frac{P}{w}$

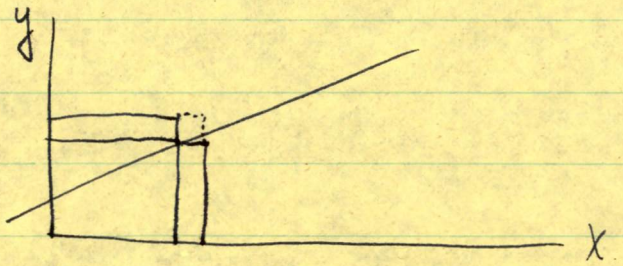
$\frac{\Delta P}{\Delta w} \equiv MP = \frac{P}{w} + \frac{\Delta P}{\Delta w} - \frac{P}{w} = \frac{\Delta P}{\Delta w}$

---

$e = \frac{MP}{aP}$

so, $MP = e \cdot aP$

$$\frac{\Delta y/y}{\Delta x/x}$$



Oct. 17
1917

If $a^{\alpha} \cdot b^{\beta} = P$

Then $dP/da = b^{\beta}\left(\alpha a^{(\alpha-1)}\right)$

and

$$aP_a = \frac{P}{a} = a^{\alpha-1} b^{\beta}$$

$$\frac{dP/da}{P/a} = \frac{b^{\beta} \alpha a^{\alpha-1}}{a^{\alpha-1} b^{\beta}} = \alpha$$

So, $\alpha$ = elast. of prodn, when a increases.

---

Also,

$$\alpha \ln a + \beta \ln b = \ln P$$

$$\frac{\frac{dP}{da}}{\frac{P}{a}} = \alpha \frac{da/da}{a} = \frac{\alpha}{a}$$

$$\frac{dP/da}{P/a} = \alpha$$

The morals ~~is~~ are that: ~~the~~

1. The exponential form is a compact way of expressing an otherwise complex relation.

You would, or someone would first establish by observation or experiment the facts of production response: MP, for given values of AP.

Then, having MP/AP, you know $\alpha$ (for that range of a/b values only, however.) You or they then record the findings in exponential form.

Our purpose here is so you can interpret what they are telling you, because this is a common way of talking.

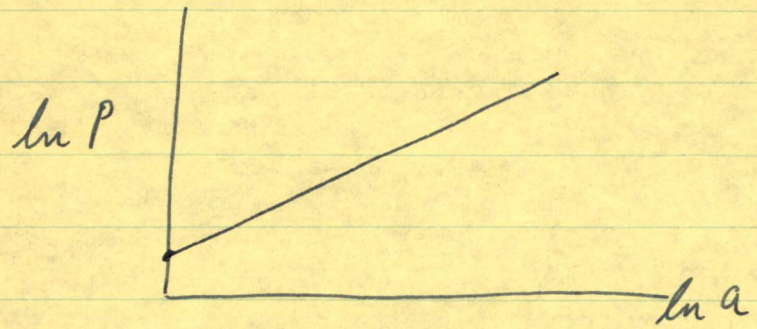
2. Knowing 2 values, you now can find the 3d.
$MP = \alpha AP$

3. If $\alpha > 1$, add more a. Ag returns.

4. In eg., $\alpha < 1$. No cause for alarm, normal dim. rets.

5. In eg. $\alpha + \beta = 1$. Const. returns to scale.
a 1% $\Delta$ of both a & b means a 1% $\Delta P$

6. Linear in the logs. (const. elast.)




$$\frac{\Delta \ln P}{\Delta \ln a} = \alpha$$

This facilitates manipulation. The ~~logs~~ lns. are linear, even tho mgnl. returns are diminishing.

7. Tunes your mind for thinking in terms of % changes, also useful re
- demand analysis
- comp. interest
- decreasing costs, .6 rule.

8. ~~You may~~ Even if you never use these tools positively, need them definitively to understand what others are saying — & often, that they are snowing you.

9. Cautions

a. $\alpha$ & $\beta$ are not const. w. scale or propn.

b. Assumes inputs are complements

$$\frac{dP}{da} = b^{\beta}(\alpha a^{(\alpha-1)})$$

If they are substitutes, this is misleading.

$i = .10$

| n | F | PV | a | a/i |
|---|---|---|---|---|
| 5 | 300 | 186.28 | 49.14 | 491 |
| 10 | 500 | 192.77 | 31.37 | 314 |

$$PV\,G_1 = \frac{F_n(1+i)^{-n}}{} = \frac{F_n}{(1+i)^n}$$

$$PV \Sigma G = \frac{F_n}{(1+i)^n - 1}$$

| n | F | PV | a | a/i |
|---|---|---|---|---|
| 1 | 100 | 90.91 | 100 | 1000 |
| 2 | 150 | 129.97 | 71.43 | 714 |

Now subtract $C_0$

$$PV\,G_1 = \frac{F - C_0(1+i)^n}{(1+i)^n} = \frac{F}{(1+i)^n} - C_0$$

$$PV \Sigma G = \frac{F - C_0(1+i)^n}{(1+i)^n - 1}$$

$$\frac{dG_1}{dn} = \frac{F' - C_0 i (1+i)^n}{\cancel{i(1+i)^n}}$$

Cf. % growth rates of denominators wrt $n$

$G_1$: $\%\Delta = \dfrac{i(1+i)^n}{(1+i)^n} = i$

so PUG$_1$ grows as long as $\dfrac{F'}{F} > i$

$$\Sigma G \quad \%\Delta = \frac{i(1+i)^n}{(1+i)^n - 1} = \frac{i}{1-(1+i)^{-n}}$$

a much higher % growth rate, so $F'/F$ must be much higher.

$$\frac{d\frac{u}{v}}{dx} = \frac{-u\frac{dv}{dx} + v\frac{du}{dx}}{v^2} = \frac{du/dx}{v} - \frac{u}{v}\frac{dv}{dx}/v$$

$$= \frac{u}{v}\left[\frac{du/dx}{u} - \frac{dv/dx}{v}\right]$$

For Glutz

$i = 10\%$

| n | F | PV | a |
|---|---|---|---|
| 5 | 100 | 62 | 16.38 |
| 6 | 111 | 63 | 14.39 |

Now cf. 5 yr. taken twice vs. 10

| | | | |
|---|---|---|---|
| 5 | 100, ~~100~~ | 62 | |
| 10 | 100 | 39 | |
| | | 101 | 16.38 |
| 10 | ~~117~~ | ~~45~~ | |
| | 265 | 102 | 16.63 |

$$PV = \frac{F_n}{(1+i)^n} + \frac{F_{2n}}{(1+i)^{2n}}$$

$$G_1 = \frac{F_1}{(1+i)^n}$$

$$S = \frac{F_1}{(1+i)^n - 1}$$

$$\frac{G_1}{S} = \frac{(1+i)^n - 1}{(1+i)^n} = 1 - (1+i)^{-n}$$

$$G_2 = \frac{F_1}{(1+i)^{2n}}$$

$$G_1 + G_2 = \frac{F}{(1+i)^n} + \frac{F}{(1+i)^{2n}} = \frac{F}{(1+i)^n}\left[1 + (1+i)^{-n}\right]$$

$$\frac{G_1 + G_2}{S} = \frac{\frac{F}{(\ )^n}\left[1+(1+i)^{-n}\right]}{\frac{F}{\left[(1+i)^n - 1\right]}} = \frac{\left[1+(1+i)^{-n}\right]\left[1-(1+i)^{-n}\right]}{\cancel{1-(1+i)^{-n}}}$$

$$= 1 + (1+i)^{-n} - (1+i)^{-n} - (1+i)^{-2n}$$

$$= 1 - (1+i)^{-2n}$$

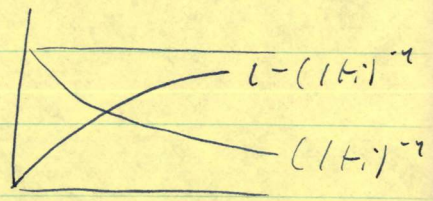
So, if $n_1 = \frac{1}{2}$ of $n_2$, then 2 G of $n_1$ gives same % of S as one G of $n_2$.

---

$$PV = \frac{F}{(\ )}$$

$$S = \frac{F}{(\ ) - 1}$$

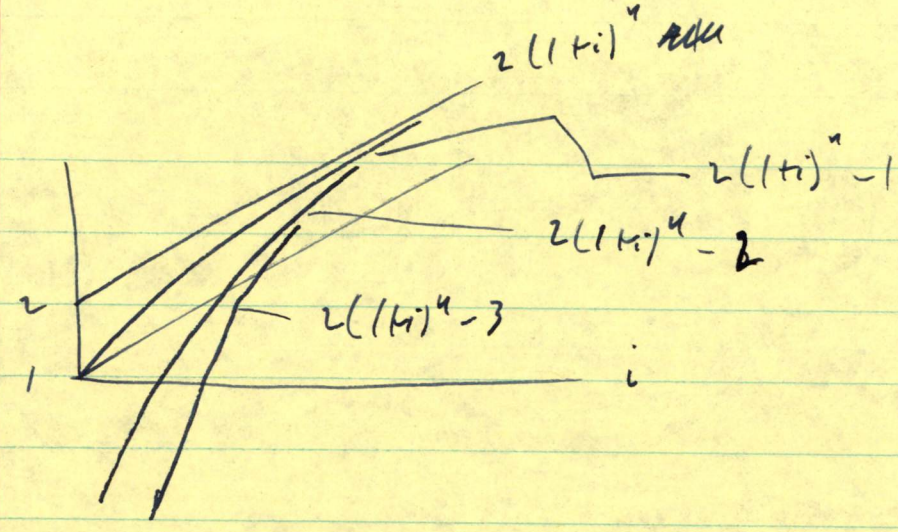
$$\frac{PV}{S} = \frac{(\ ) - 1}{(\ )} = 1 - (1+i)^{-n}$$



Items. 11/2, Time & Lat.

1. ~~Replmt. Pol.~~ (max's quotients)
2. ~~Log scales~~
3. Problems
4. ~~Infln: Read i.r.~~
5. Sensitivity of periodic pnts to $\oint C_0, i, + u$
6. ~~Lur. Criteria~~

Subject for article.




$$(1+\hat{i})^n - (1+i)^n = \left[1 - \left(\frac{1+i}{1+\hat{i}}\right)^n\right](1+\hat{i})^n$$

$$= \left[\left(\frac{1+\hat{i}}{1+i}\right)^n - 1\right](1+i)^n$$

Equity share.

$g$ = gross growth rate

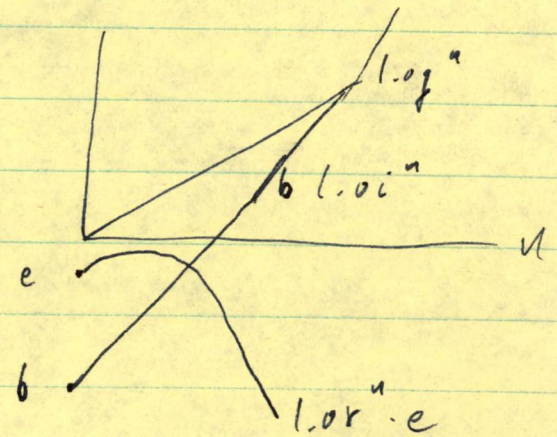
$i$ = i.r.

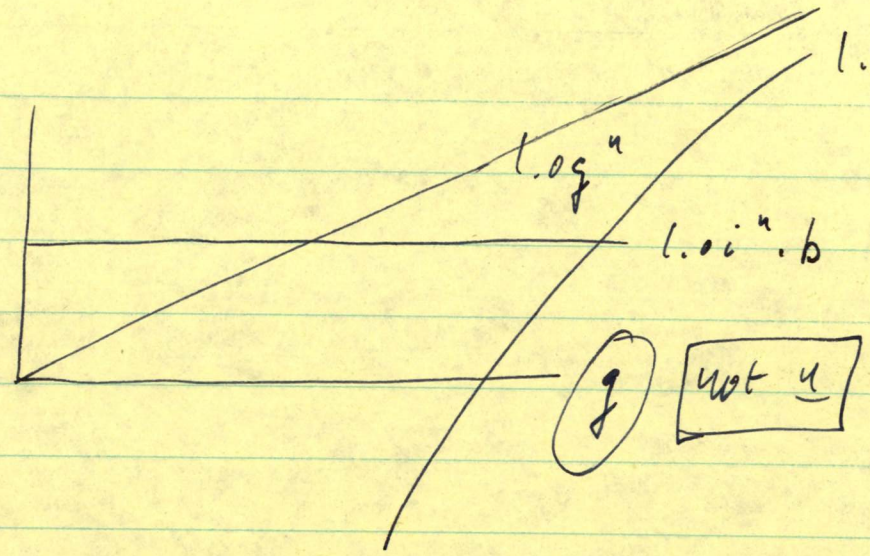
$r$ = eq. rate

$e$ = eq. stake as %.

$b$ = borrowed %

$$l.og^n - b\, l.oi^n = e\, l.or^n$$






Numerical e.g.,

$g = .072$

| $l.og^n$ | $b$ | $\frac{1}{4}\, l.0i^{10}$ | $\frac{3}{4}\, l.or^{10}$ |
|---|---|---|---|
| 2 | $\frac{1}{2}$ | $1\frac{1}{2}$ | |

So above confirms eqn.

$$l.og^n - b\, l.oi^n = e\, l.or^n$$

$$l.or^n = \frac{l.og^n - b\, l.oi^n}{e}$$

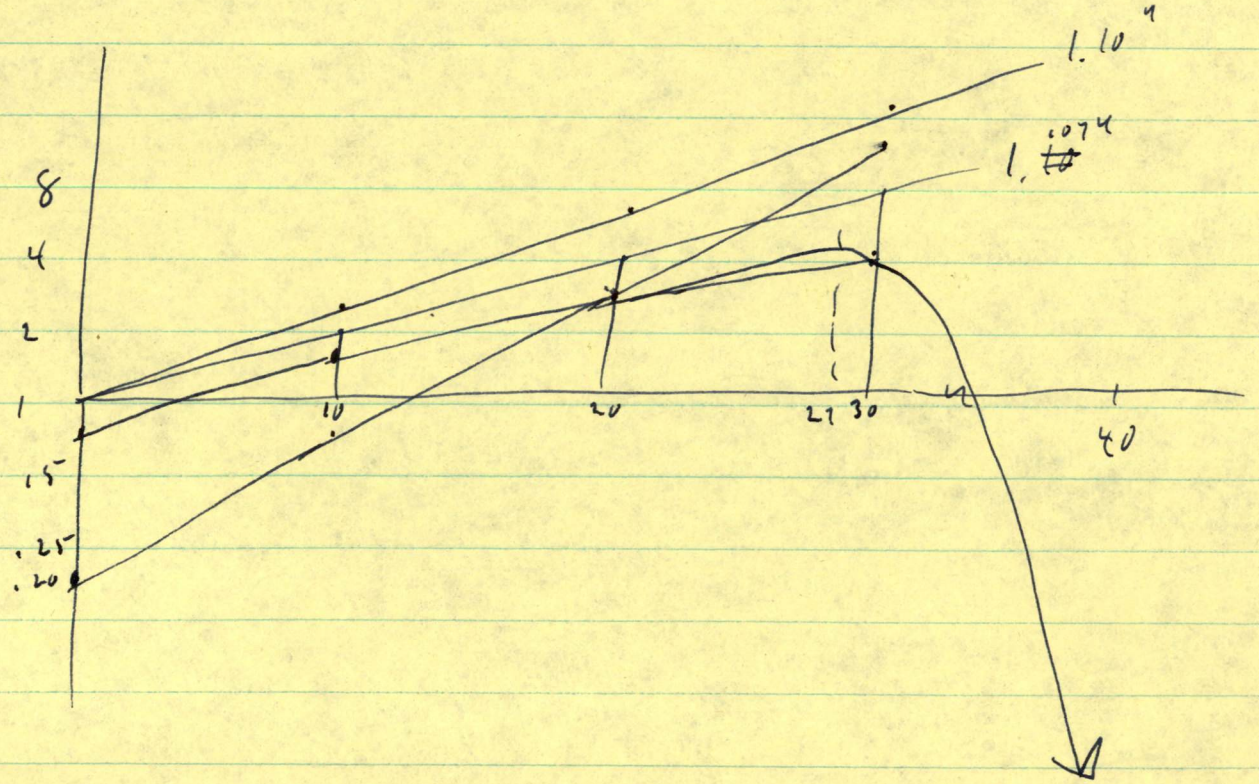
| $n$ | $l.0^n$ | $\frac{1}{4}\, l.20^n$ |
|---|---|---|
| 0 | 1 | $\frac{1}{4} = .25$ |
| 1 | 1.1 | .30 |
| 2 | 1.21 | .35 |

$$\frac{e\, l.or^n}{l.og^n} = 1 - b\,\frac{l.oi^n}{l.og^n}$$

$g = .10$ $\quad e = .80$

$i = .15$ $\quad b = .20$

| $n$ | $1.0g^n$ | $b\,1.0i^n$ | $e \cdot 1.0r^n$ | $1.0r^n$ | $r$ |
|---|---|---|---|---|---|
| (1) | (2) | (3) | (2) − (3) | (4)/e | (5) |
| | | | (4) | | |
| 1 | 1.1 | .230 | .870 | 1.088 | .088 |
| 2 | 1.21 | .265 | .946 | 1.182 | .087 |
| 3 | 1.331 | .304 | 1.027 | 1.284 | .087 |
| 5 | 1.611 | .402 | 1.208 | 1.510 | .086 |
| 10 | 2.594 | .809 | 1.785 | 2.231 | .084 |
| 20 | 6.727 | 3.273 | 3.454 | 4.318 | .076 |
| 27 | 13.1 | 8.1 | 4.4 | 5.5 | .065 |
| 30 | 17.5 | 13.2 | 4.2 | 5.2 | .051 |
| 40 | 45 | 53 | −8.3 | −10.4 | −.060 |



$1.0r^n$ appears to max + then plunge.

$1.0r^n$ max when $y' = 0$, $y = 1.0g^n - b\,1.0i^n$

$$\frac{dy}{dn} = g\,1.0g^n - b \cdot i \cdot 1.0i^n$$

$y' = 0$ when $g\,1.0g^n = bi\,1.0i^n$

$$\left(\frac{1.0i}{1.0g}\right)^n = \frac{g}{ib}$$

$$n(\ln 1.0i - \ln 1.0g) = \ln g - \ln i - \ln b$$

$$n = 27$$

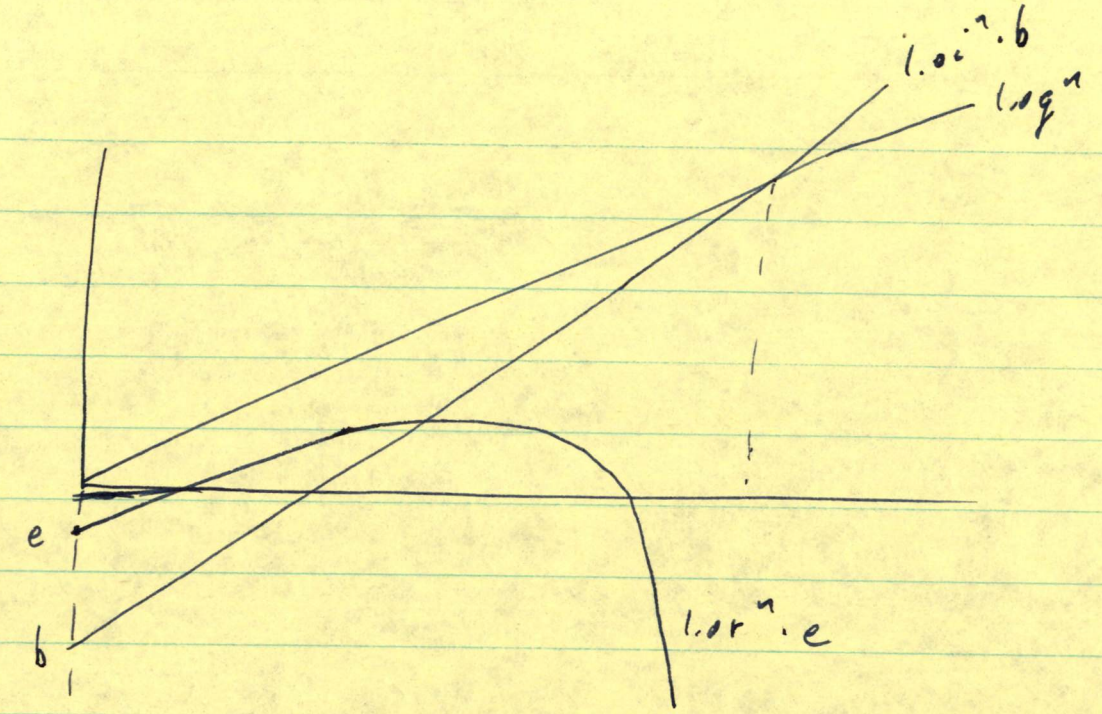
or more roughly.

$$b(1+i-g)^n = \frac{g}{i}$$

$$.2(1.05)^n = \frac{10}{11} = \frac{2}{3}$$

$$1.05^n = \frac{10}{3} =$$

$$n = \frac{\ln \frac{10}{3}}{\ln 1.05} = 21.5 \quad \text{too rough}$$




Do $r + i \stackrel{?}{=} g$

If $1.01^n + 1.01^n = 1.09^n$ No

$1.01^{50} + 1.09^{50} = 1.645 + 74.358 = 76.002$

$= 1.09047^{50}$

$\epsilon$ r/ PV wrt $\underline{n}$

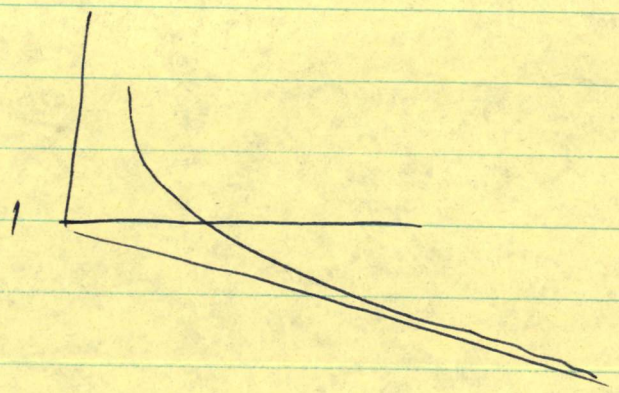
$$P = \frac{1 - 1.0r^{-n}}{r}$$

$$\frac{dP}{dr} = \frac{-1}{r}\left( - n\, 1.0r^{-n-1} \right)$$

$$\frac{P'}{P} = \frac{\frac{n}{r} 1.0r^{-n-1}}{\frac{1 - 1.0r^{-n}}{r}} = \frac{n\, 1.0r^{-1}}{1.0r^{n} - 1}$$

$$\frac{n'}{n} = \frac{1}{n}$$

$$G = \frac{1}{\left(1.0r^{n} - 1\right) 1.0r}$$



$$R = e^{pn}$$

$$\ln R = pn$$

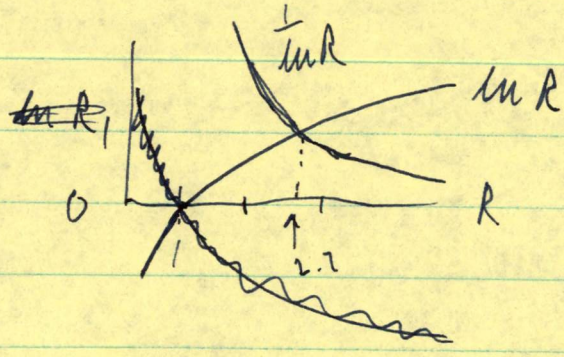
$$p = \frac{\ln R}{n}$$

$$\frac{dp}{dR} = \frac{1}{n}\,\frac{1}{R}$$

$$\frac{p'}{p} = \frac{\frac{1}{nR}}{\ln R/n} = \frac{1}{R \cdot \ln R}$$

$$\frac{R'}{R} = \frac{1}{R}$$

$$\therefore \text{ Elasticity of } p \text{ wrt } R = \frac{1}{R \cdot \ln R} \Big/ \frac{1}{R} = \frac{1}{\ln R}$$



$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{A'}{A} = \frac{2\pi r}{\pi r^2} = \frac{2}{r}$$

$$C = 2\pi r$$

$$C' = 2\pi$$

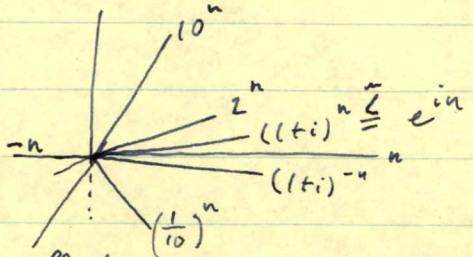
$$\frac{C'}{C} = \frac{1}{r}$$

$$\therefore \quad \epsilon = 2$$

(Needs another reorg.)

**Semi-log scale:** some useful ~~characteristics~~ properties.

1. The abscissa is an exponent. (which is also a log)
   Use when indep. vble. is an exponent, e.g.
   - growth rates over time.
   - compound interest
   - doubling, $2^n$
   - exponential decay, $y = (1+i)^{-n}$




2. Origin is $n=0$, but $y=1$. Ord. never $=0$.

3. Const. growth rate gives strt. line. (v.p.1B) (See p. 1A
   Use log base 2 to illustrate. But all bases fit same nos.
   Cf. const. g on common scale, rising curve; also show exp. decay, falling curve.

4. Any given rise on ord. has:
   - increasing absolute value as go up. Scale shows this
   - const. % value, (from its bottom)

   The rise ∴ shows the % growth of $y$, or $y'/y = g$. (over)

~~The slope of the curve is ∴ g.~~
~~$\Delta$ is rise/run. If $\Delta$run = 1, the rise is g; if run is n, rise is ng, and $\Delta = \frac{ng}{n} = g$.~~
~~One unit is 100%; i.e. one period's~~

5. Parallel lines have equal % growth.
   Upper curve is const. multiple of lower; lower is const. fraction of upper. (Useful for after-tax values).
   Cf. common scale, where only horiz. // have $= g$.

4 (cont.)

When $y = 1$, and $\Delta n = 1$, $\Delta y = g$, and $\lambda$ is $g$.

In general $\Delta y = g \cdot y$ (for $\Delta n = 1$)

Let $\lambda$ refer to physical slope, related to initial or common value of scale starting from one, and then $\lambda$ is always $g$.

For $\Delta n > 1$, $\Delta y = n\, g\, y$; but $\lambda$ remains constant

3, Appendix

A To prove $\log y = k \times \ln y$

$y = e^x$

$x = \ln y$

$y = e^{\ln y}$

$\log y = \ln y \log e = k \ln y \quad (k = \log e)$

$(\log e = .4343\ldots)$

NB. By extension, logs to any base are similarly related by a constant coefficient.

B. To prove $\log e \cdot \ln 10 = 1$.

$a = \log e \quad (= .4343\ldots)$

$b = \ln 10 \quad (= 2.3\ldots)$

$10^a = e$

$e^b = 10 \; ; \; e = 10^{1/b}$

$\therefore \; 10^a = 10^{1/b}$

$a = 1/b$

$\log e \cdot \ln 10 = 1 \qquad$ QED

C. Illustration

1. $1.072^{10} = 2^1$

$1.072^{20} = 2^2$

$1.072^{30} = 2^3$

etc.

2. $a^x = b^y$

$a^{2x} = b^{2y}$

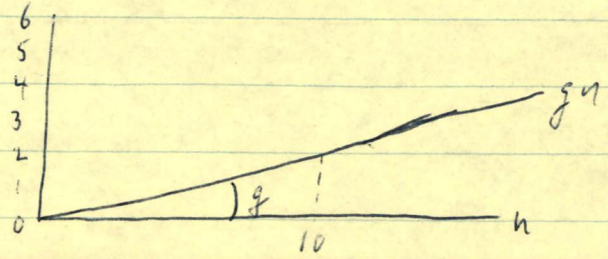
$a^{3x} = b^{3y}$

$a^{nx} = b^{ny}$

3. $y = e^{gn}$

$gn = \ln y$

$g = \dfrac{\ln y}{n}$



Re #3

set $Y = \log_{10} y$ ~~set~~ Call $\log_{10} A = B$
Call $\log_{10} b = M$

Suppose $y = Ab^x$ $b > 0$ $(x = \log_b \frac{y}{A})$
Then $\log_{10} y = \log_{10} A + x \log_{10} b$
Then $Y = Mx + B$
∴ if we plot $Y$ against $x$ we get a straight line!

Alternatively, suppose $y = Ax^n$ → for instance $y = P(1+i)^n$
Then $\log y = \log A + n \log x$ $Y$ as before
$Y = K + nX$ $X = \log x$
log-log paper ~~does~~ automatically, plots $Y$ against $X$

MacLaurin series

$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ ~~$x$ $-1 < x \leq 1$~~
$-1 < x \leq 1$

Works very badly if $|x|$ near 1.
Doesn't work at all if $|x| > 1$
Works beautifully if $|x|$ very, very small

suppose we increase $A$ by 2% to $(1.02)A$

$\log_e(1.02A) = \log_e A + \log_e 1.02$
$\log_e A + \log_e(1+.02) = \log_e A + .02 - \frac{(.02)^2}{2} + \frac{(.02)^3}{3}$

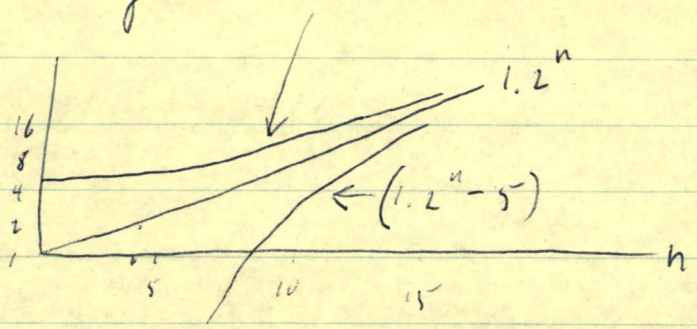
So $\log_e$ increases by approximately .02 in abs. value (but neg.)
with an error less than .0002

i.e. $\ln y$ is prop. to $n$.

6. Is same as plotting log on common scale
$y = a^n$; then $n = \log_a y = k \cdot \ln y$; so $\ln y = \frac{1}{k} \cdot n$, a strt. line.
Any given $\Delta \log$ is always same % $\Delta$ in $y$.
The % $\Delta$ is $g$, where $B$ = base, and $g = B - 1$
When plot $\ln$, $\Delta \ln = y'/y$ exactly. Worth proving & illustrating.

7. Any fixed amt is a smaller vert. distance as go up

E.g. $y = 5 + 1.2^n$.




? 8. ~~Logs being exponents, their values are roots of the fns.?~~
~~This seems to be wrong.~~

To be wrapped up.

$y = a^n$ $\qquad$ $y = e^{gn}$

$n = \log_a y$ $\qquad$ $gn = \ln y$

$y^{1/n} = a$

? But $y^{\frac{1}{n}} = n$

$y = 2^n$

$\log_2 y = n$

$y^{\frac{1}{n}} = 2$

8. When you plot $\ln y$, $\Delta \ln y = y'/y$ exactly, whereas other $\log_b = k \cdot y'/y$.

This is one reason for using + preferring base e.

The proof is a fundamental prop. of diff. calc., + could be omitted here. But seek for a simple heuristic showing. See 8A

Numerical table makes good drill or exercise for students. It also shows that relation is exact only for very small $\Delta$s.

| $y$ | $\ln y$ | $\Delta \ln y$ | ~~$\Delta y/y$~~ | $\Delta y^*/y$ |
|---|---|---|---|---|
| 0 | $-\infty$ | | | |
| 0.1 | -2.3026 | $\infty$ | | $\infty$ |
| 0.5 | -.6931472 | 1.6094379 | | .4 |
| 0.9 | -.1053605 | .5877867 | | .8 |
| 1.0 | 0 | .1053605 | | .1111 |
| 1.01 | .0099503 | .0099503 | | .01000 |
| 1.05 | .0487902 | .0388398 | | .0396040 |
| 1.10 | .0953102 | .0465200 | | .0476190 |
| 1.50 | .4054651 | .3101549 | | .3636... |
| 2.0 | .6931472 | .2876821 | | .3333... |
| $e=2.718$ | 1.0000 | .3068528 | ~~.35~~ | .35914... |
| 10.0 | 2.3025851 | 1.3025851 | | 2.6781944 |
| 100.0 | 4.6051702 | 2.3025851 | | 9.0000... |

$\Delta y^*/y$ is always larger than $\Delta \ln y$, bcs. the ln corresponds to $f = \infty$.

8A. To show that $\Delta \ln y = y'/y$

i) Continuous

If $y_n = e^{gn}$

$\Delta n = 1$

$y_{n+1} = e^{g(n+1)} = e^{gn+g}$ so $g = \Delta \ln y / \Delta n$

Similarly, $n = \Delta \ln y / \Delta g$. Note that $\Delta g$ is normally very small, e.g. .01, so this is probably much more accurate. E.g. with $n = 20$:

| $g_0$ | $\ln y = gn$ | $\Delta \ln y$ | $y = e^{gn}$ | $y'/y$ |
|---|---|---|---|---|
| .01 | .2 | .2 | 1.2214 | .2214 |
| .02 | .4 | .2 | 1.4918 | .2214 |
| .03 | .6 | .2 | 1.8221 | .2214 |
| .04 | .8 | .2 | 2.2255 | |
| .05 | 1.0 | .2 | 2.7182 | |
| .10 | 2.0 | | | |
| .15 | 3.0 | | | |
| .20 | 4.0 | | | |

Even here, the ~~rate~~ $\Delta \ln$ is 10% less than $\Delta y/y$. So, for discrete changes, $\Delta \ln$ only approximates $\Delta y/y$.

ii) Discrete $y_n = (1+g)^n$

$\Delta y = (1+g)^{n+1} - (1+g)^n = (1+g)^n (1+g-1) = g(1+g)^n$

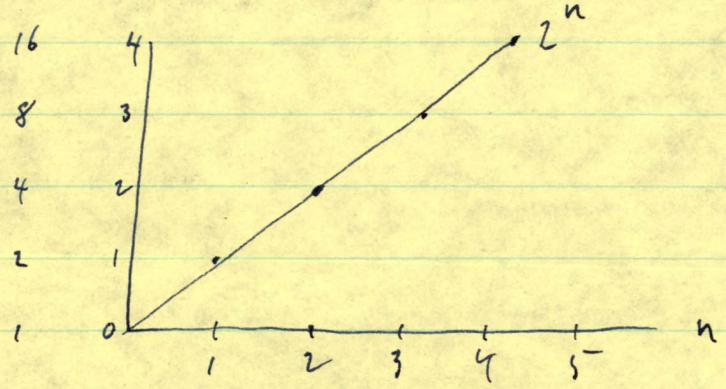
$\Delta y/y = g$ exactly & always.

| $n$ | $(1.05)^n$ | $\Delta y/y$ |
|---|---|---|
| 0 | 1 | |
| 1 | 1.05 | .05 |
| 2 | 1.1025 | .05 $(1+.05)^2 = 1 + .05 + .05 + .0025$ |

9. $\Delta$ of line is growth rate of $y$?
Cf. common scale, $\Delta$ is $\Delta y / \Delta n$.
Log scale, $\Delta$ is $\frac{\Delta y / y}{\Delta n}$

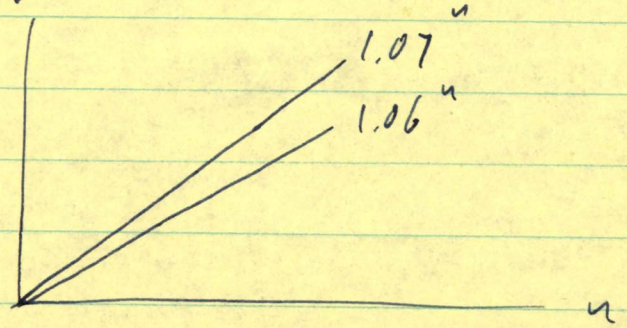
10. Rise, (or $\Delta$ when $\Delta n = 1$) is $(B-1)$, whereas $B$ = base.
Thus express base as $(1+g)$, and $g$ is growth rate, or $y'/y$

11. Superimpose a common scale on ord.




12. Slope of curve depends on base
Base = 1 means horiz.
Growth rate is $(B-1)$
Falling curve means $B < 1$ (or neg. exp.)

17. Rays diverge, shows $\Delta g$ % diff/s.




(Algebraically, $\left(\frac{1.07}{1.06}\right)^n \cong 1.01^n$)

$\ln(xy) \overset{?}{=} \ln x + \ln y$

$A^N = B$

$N \ln A = \ln B$

$N(1+n) \ln[A(1+a)] = \ln[B(1+b)]$

$(1+n) \dfrac{\ln[A(1+a)]}{\ln A} = \dfrac{\ln[B(1+b)]}{\ln B}$

$N(1+n)[\ln A + \ln(1+a)] = \ln B + \ln(1+b)$

$N[\ln A + \ln(1+a)] + Nn[\ ] - N \ln A = \ln B + \ln(1+b) - \ln B$

$N \ln(1+a) + Nn[\ln A + \ln(1+a)] = \ln(1+b)$

$N[\ln(1+a) + n \ln(1+a) + n \ln A] = \ln(1+b)$

More simplifying needed.

$(1+a)^{(1+n)} \overset{?}{=} 1+b$ No!

$10^2 = 100$

$10(1.05)^{2(1.08)} = 10.5^{2.16} = 160.61 = 100(1+.6061)$

$1.05^{1.08} = 1.054 \neq 1.6061$ No!

Ec. 162, Memo. Nov. 10, 1981, by M. Gaffney

Dedicated to Priscilla Douglas, who asked the right question.

This is for more serious students who want really and fully to understand replacement planning, and valuation of durable property, and the basic math. tools needed.

## 1. An oversimplified model of a typical asset.

We are considering buying or building a structure to generate cash flow on a site we own.

$a$ = annual cash flow
(Cash flow = gross receipts less running costs.)
$n$ = end of life
$C$ = Capital cost, time zero.
$G_1$ = PV of Generation #1
$S$ = Site Value in perpetuity

$$G_1 = -C + a\frac{1-(1+i)^{-n}}{i} \qquad (1)$$

$$S = G_1\left[\frac{1}{1-(1+i)^{-n}}\right]$$

$$= \frac{-C}{1-(1+i)^{-n}} + \frac{a}{i} \qquad (2)$$

The trouble with this oversimplified model of $S_1$ is the same as with $G_1$ alone, viz., it has no finite maximum w.r.t. $n$. It just says that if $a$ will run on indefinitely, then the less often we replace, the better.

To correct for this, we modify the model and let cash flow decline, or "decay", with $n$.

2. A simple but adequate model of a typical asset with decaying cash flow.

$d$ = decay factor, such that

$$\text{cash flow} = a\left[\frac{1}{1+d} + \frac{1}{(1+d)^2} + \dots + \frac{1}{(1+d)^n}\right]$$

$$G_1 = -C + a\,\frac{1-[(1+i)(1+d)]^{-n}}{d+i(1+d)} \qquad (3)$$

$$S = \frac{-C}{1-(1+i)^{-n}} + \frac{a}{d+i(1+d)} \times \frac{1-[(1+i)(1+d)]^{-n}}{1-(1+i)^{-n}} \qquad (4)$$

A skilled mathematician could inspect (4) carefully, and probably see $S$ rising to a max and then falling with $\underline{n}$. However, let us nail it down firmer.

3. Finding the max of a ratio.

It is intuitive, and you can demonstrate yourself, that a ratio rises in value when the numerator grows by a larger % than the denominator; and stops growing when the two % changes are the same.

$$\frac{u}{v} = \text{a ratio}$$

Ratio constant when:

$$\frac{du/dx}{u} = \frac{dv/dx}{v} \qquad (5)$$

Remember or look up the derivative of a ratio. Usually you find:

$$\frac{d\,\frac{u}{v}}{dx} = \frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2} \qquad (6)$$

But we may rewrite (6) so it immediately confirms (5).

$$\frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2} = \frac{u}{v}\left[\frac{\frac{du}{dx}}{u} - \frac{\frac{dv}{dx}}{v}\right] \qquad (7)$$

This may make you feel better. Our intuitive device is just simplified differential calculus.

## 4. Maximizing S

To maximize S, solve (5), using (4a)

$$S = \frac{-C + a\frac{1-[(1+i)(1+d)]^{-n}}{d+i(1+d)}}{1-(1+i)^{-n}} \qquad (4a)$$

$$\frac{du}{dn} = \frac{-a}{d+i(1+d)}\left[\text{[illegible]} - [(1+i)(1+d)]^{-n}[d+i(1+d)]\right]$$

$$= a[(1+i)(1+d)]^{-n} \qquad (8)$$

$$\frac{dv}{dn} = i(1+i)^{-n} \qquad (9)$$

$$\frac{du/dn}{u} = \frac{dv/dn}{v} \qquad \text{when}$$

$$\frac{a\left[(1+i)(1+d)\right]^{-u}}{-C + a\dfrac{1-\left[(1+i)(1+d)\right]^{-u}}{d+i(1+d)}} = \frac{i(1+i)^{-n}}{1-(1+i)^{-n}} = \frac{i}{(1+i)^{n}-1}$$

$$\frac{a\dfrac{(1+i)^{n}-1}{\left[(1+i)(1+d)\right]^{n}}}{a\dfrac{1-\left[(1+i)(1+d)\right]^{-n}}{d+i(1+d)} - C} = i \qquad (10)$$

The left side represents $\frac{du/dn}{u}$, and (once the denominator $> 0$) the derivative is positive so long as the left side is $> i$. It clearly is when $v = 0$, and for a reach of n-values greater than that, as it grows slowly. But then it clearly must peak and fall, because:

a) $v \to$ an asymptote or constant value

b) $u \to \frac{a}{(1+d)^{n}}$ and keeps getting smaller.

5. a simpler, marginal view of how to max w.r.t. $n$

The tangle of (10) can become suddenly clear and evident when we rearrange terms in such a way as to make it clear that the condition says that $G_1$ should be replaced in the year when the decaying cash flow ceases to earn interest on site value.

The denominator of (10) on the left side is $G_1$. Substituting,

$$a\frac{(1+i)^n-1}{(1+i)^n(1+d)^n} = G_1 \cdot i \qquad (11)$$

$$\frac{a}{(1+d)^n} = i\frac{G_1}{1-(1+i)^{-n}} = i \cdot S \qquad (12)$$

In (12), the condition for max $S$ is seen as a requirement that the current cash flow, the left side, just cover interest on the site. This makes intuitive sense in its own right.

So what most people do (if they think this

thought at all) is to figure they will demolish & replace capital when its cash flow no longer pays interest on the value of S. This is a useful rule of thumb to remember.

6. The ultimate need to max S w.r.t. n

The trouble is, logically it is circular, because S is unknown, it is what we are determining. Most people just take market value for S, and make out all right, but someone has to do the original thinking for others to follow. This person earns extra income by being first. She or he can spot bargains, avoid turkeys, etc.

This leads us back to (10). We need to solve it for n. The best way is to set up some standard tables, based on typical parameter values. Let:

$i = .05$ ‡

$C = 100$

$a = 20$

$d = .08$

Now we figure left side as function of n

‡ Using a low or real rate of interest has the same effect as letting a rise annually due to inflation. There are alternative adjustments for inflation.

| $n$ | $1.05^{-n}$ | $(1.05\times 1.08)^{-n}$ | $u$ | $v$ | $u/v$ |
|---|---|---|---|---|---|
| 22 | 2.925 | 15.90 | 2.42 | 39.87 | .0607 |
| 23 | 3.072 | 18.03 | 2.30 | 40.98 | .0561 |
| 24 | 3.225 | 20.45 | 2.18 | 41.96 | .0519 |
| 24.5 | 3.305 | 21.78 | 2.12 | 42.40 | .0499 |

So $S$ is max at $n = 24.5$

$$S = \frac{-100 + 20\,\dfrac{1-\frac{1}{21.78}}{.134}}{1-\frac{1}{3.305}} =$$

$$= \frac{-100 + 20\,\dfrac{.95}{.134}}{1-.303} = \frac{42.4}{.691} = 60.8$$

As a check, recall from #5, above, that at max $S$,

$$S \cdot i = \frac{a}{(1+d)^n} \qquad (12)$$

Here, $60.8 \times .05 = 3.040$

$20\,(1.08)^{-24.5} = 3.035$

That's pretty close, considering rounding and other approximations used here.

7. Sensitivity analysis: how is optimal replacement time ($n$) affected by changes in parameters ($a$, $C$, $i$, and $d$)?

I append some rudimentary sensitivities. I did this by programming (10) on a HP 38E, so I could proceed quickly by trial & error. With practise & experience & better algebra, no doubt we could find an easier way.

The outstanding finding is that $n$ is much more sensitive to $a/C$ than to $i$ or $d$.

In a few hours, one could work up a set of tables covering all the cases you would meet in a lifetime of making investment & replacement decisions.

| Case # | a/c | i | d | n |
|---|---|---|---|---|
| 1 | .20 | .05 | .08 | 24 + |
| 2 | .25 | .05 | .10 | 19 |
| 3a | .30 | .05 | .15 | 15 + |
| b | .30 | .04 | .15 | ~~15~~ - |
| c. | .30 | .03 | .15 | 14 |
| d | .30 | .02 | .15 | 13 + |
| e | .30 | .01 | .15 | ~~13~~ |
| 4a | .30 | .05 | .15 | ~~15.4~~ + |
| | | | .14 | 15.3 |
| | | | .13 | 15.2 |
| | | | .12 | 15.2 |
| | | | .11 | 15.3 |
| | | | .10 | 15.5 |
| | | | .09 | 15.8 |
| | | | .08 | 16.3 |
| | | | .07 | ~~16.8~~ |
| | | | .06 | 17.5 |

Sensitivity Analysis

Case

| | a/c | i | d | n |
|---|---|---|---|---|
| 5a | .30 | .05 | .15 | 15.4 |
| | .28 | | | 17 |
| | .26 | | | 19+ |
| | .24 | | | 22+ |
| | .22 | | | 29 |
| | .21 | | | 41 |
| | .20 | | | ∞ |

Conclusion: n is insensitive to i + d but highly sensitive to a/c.

(But when $\frac{a}{c} \to (i+d)$, n becomes sensitive to i + d)

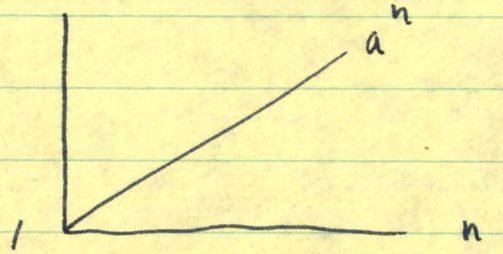
Semi-log scale, useful characteristics.

1. Same as plotting log on common scale
~~2, a. Const. rise has increasing value.~~ (distance up, not numerical value)
~~2, b. Δ ord.~~ is % growth, $y'/y = g$.
(So, ∴ is Δ log. Show by algebra + calculator)
(Use log base 2 as e.g.)
3. Strt. line has const. % growth, or $y'/y = g$
~~cf. common scale~~, exp. growth + decay.
4. Slope of line is growth rate
When $\Delta u = 1$, Δ ord. is $y'/y = g$

5. ∥ lines have = g
cf. common scale, where horiz. ∥ have = g.

6. Ord. never = 0
Lines do not begin from 0
For curves beginning from one, ord. shows % growth: (Ord. − 1) × 100 = % growth.
Let $V_0$ = value at $u = 0$, then
% growth = ~~(ord $V_0$)~~ $\frac{V_n - V_0}{V_0} \cdot 100 = (V_n/V_0 - 1) \cdot 100$

7. Useful when exponent is indep. vble. Gives st. line.
(This also tells us something about logs: any given Δ log is same % Δ in $y$.
8. Falling curve means base < 1




To prove: $\Delta \log y = y'/y$

$y = a^n$

$\log y = n \log a$

If $a$ is base of logs, then $\log y = n$

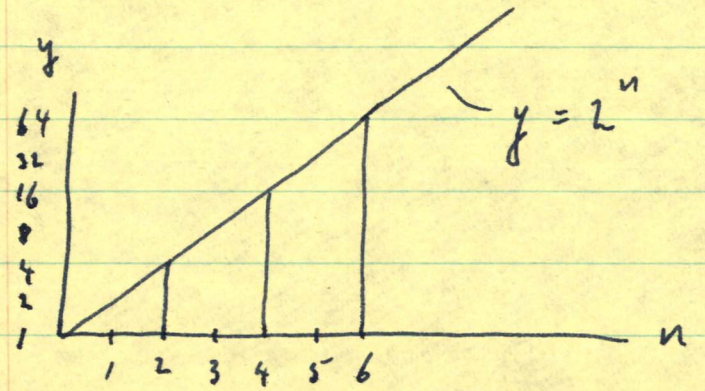
$$\frac{dy}{dn} = a^{n+1} - a^n = a^n(a-1)$$

Let $a = 1+g$

Then $a^n(a-1) = a^n \cdot g$

$$\frac{y'}{y} = \frac{a^n \cdot g}{a^n} = g$$

If $g < 0$, $y$ decays.




$y = 2^n$

$n = \log_2 y$

$$\frac{dy}{dn} = \log 2 \cdot 2^n = 2^n$$

$$\frac{y'}{y} = 1$$

---

$$y = (1+i)^n$$

$$\frac{dy}{dn} = (1+i)^n \ln(1+i) \cong i(1+i)^n$$

$$\frac{y'}{y} \cong i$$

---

To prove: $\ln(1+i) \cong i$

$$e^x = 1+i$$

$$x = \ln(1+i)$$

$$y = a^x$$

$$\Delta y = a^{x+1} - a^x = a^x(a-1)$$

Let $a = 1+g$

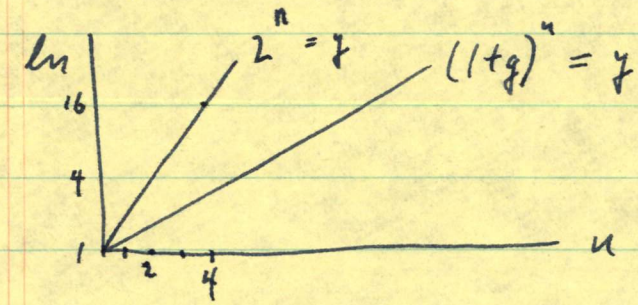
$$\Delta y = a^x \cdot g = (1+g)^x g$$

$$\frac{\Delta y}{y} = g \quad (= a-1)$$

But $g \doteq \ln(1+g)$

$$\ln y = x \ln a$$

$$\Delta \ln y = \ln a \, \Delta x$$




| n | $2^n$ | % $\Delta y$ | $1.02^n$ | % $\Delta y$ |
|---|---|---|---|---|
| 0 | 1 | | 1 | |
| 1 | 2 | 100 | 1.02 | .02 |
| 2 | 4 | 100 | 1.04 | .02 |
| 3 | 8 | 100 | 1.06 | |
| 4 | 16 | 100 | 1.08 | .02 |

$y = e^{gn}$ ; $\ln y = gn$

$\frac{dy}{dn} = e^{gn} \cdot g$

$\frac{dy}{dg} = e^{gn} \cdot n$

| $g$ | $e^g$ | $\frac{dy}{dg}$ | $y'/y$ | $e^{50g}$ | $\frac{dy}{g}$ | $g'/y$ |
|---|---|---|---|---|---|---|
| 0 | 1 | | | 1 | | |
| .01 | 1.01 | .01 | 100% | 1.50 | .50 | .50 |
| .02 | 1.02 | .01 | 100% | 2. | .505 | .50 |
| .03 | 1.03 | .01 | 100% | | | .50 |

Interpn. of $y'/y = 4 = 50$

$y' = .50$

$y = 1$

$y'/y = 50\%$, regardless of what $g$ is.

How does $n = 50$ become $50\%$?

Qu: What is slope or rise of semi-log scale?

1. Discrete growth: $y = (1+g)^n$ or $a^n$

$$\Delta y = a^{n+1} - a^n = a^n(a-1) = a^n \cdot g = y \cdot g$$

$$\frac{\Delta y}{y} = g$$

Since $\Delta y$ is always a const. % $= g$ of its bottom $= y$, $\therefore$ $\Delta y$ is always same rise on log scale.

Variations {
2. Cont. growth, $y = e^{gn}$
3. Graphing the log on common scale.

Qu. Is the slope "actually" $g$? In what sense?

The ~~slope is~~ rise is $\Delta y = g \cdot y$

When $y = 1$, $\Delta y = g$

In general, $\Delta y = g \cdot y$ (for $\Delta n = 1$)

~~But if $y_0 = 1$ then $\frac{\Delta y}{\Delta n} = g$ no matter how big $\Delta y$ + $\Delta n$ are~~

# 1. Reciprocal of log

$y = a^n$ $\qquad n = \log_a y$

$a = y^{\frac{1}{n}}$ $\qquad \frac{1}{n} = \log_y a$

E.g. $\log_{10} e = .4343\ldots$

$\ln_e 10 = 2.3\ldots = 1/.4343\ldots$

# 2. Log scale is $\propto n$, a direct proof.

A. Discrete form

$y = (1+g)^n$

$n = \log_{1+g} y = \log_{1+g} e \ln y = k \ln y$

$1/k = \log_e (1+g) = \ln(1+g) \doteq g$. $\quad \therefore n \doteq \ln y / g$

B. Continuous form

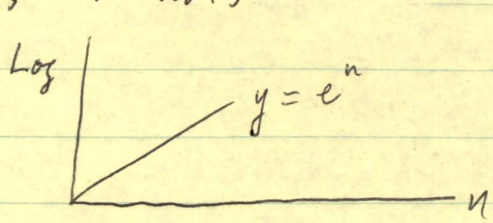
$y = e^{gn} = (e^g)^n$

$n = \log_{e^g} y = \log_{e^g} e \ln y = \ln y / \ln e^g = \ln y / g$

$ng = \ln y$

# 3. Logs + roots

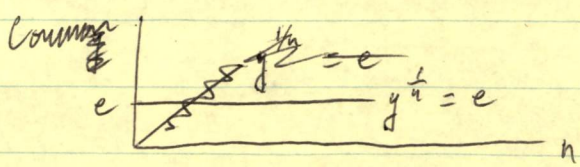
If no use?




$y = e^n$

$n = \ln y$

$y^{\frac{1}{n}} = e$ (or $e^g$ or $(1+g)$)



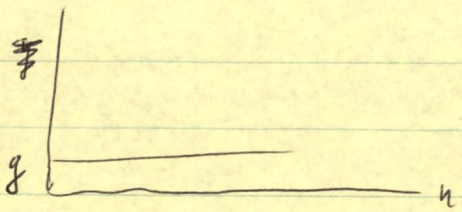

Thus we get strt. line by plotting nth root of y. Yes, but the line is long.

3, Logs & roots, cont.

Now make g the slope

A. Discrete form $y = (1+g)^n$

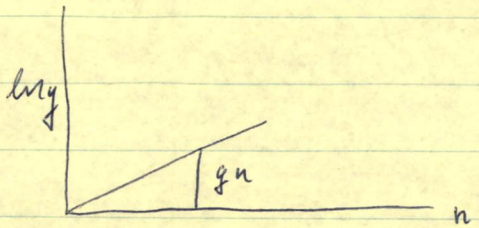
$y^{\frac{1}{n}} - 1 = g$




Seems to lead nowhere.

$y = (1+g)^n$

$y = e^{gn}$




$gn = \ln y$

slope is $gn/n = g$

n is indep. vble., g is const., hence g & n are not parallel, on graph. $y'$ is $dy/dn$, a particl

$\frac{d(\ln y)}{dn} = \frac{y'}{y}$ is the rule. This seems to say that

the $\Delta \ln = \%\ \Delta y$. True, for very small $\Delta y$

| y | ln y | Δ ln y | log y | Δ log y |
|---|---|---|---|---|
| 0 | $-\infty$ | | | |
| 1 | 0 | ∞ | 0 | |
| 1.05 | .0487902 | .0487902 | .0211893 | |
| 1.10 | .0953102 | .0465200 | | |
| 1.20 | | | | |
| 1.50 | | | | |
| 1.75 | | | | |
| 2.00 | | | | |

$x^{-n} = \left(\frac{1}{x}\right)^n$

$10^{-n} = \left(\frac{1}{10}\right)^n$

$y$

0

1

$1.01$

$1.01^2$

$1.01^3$

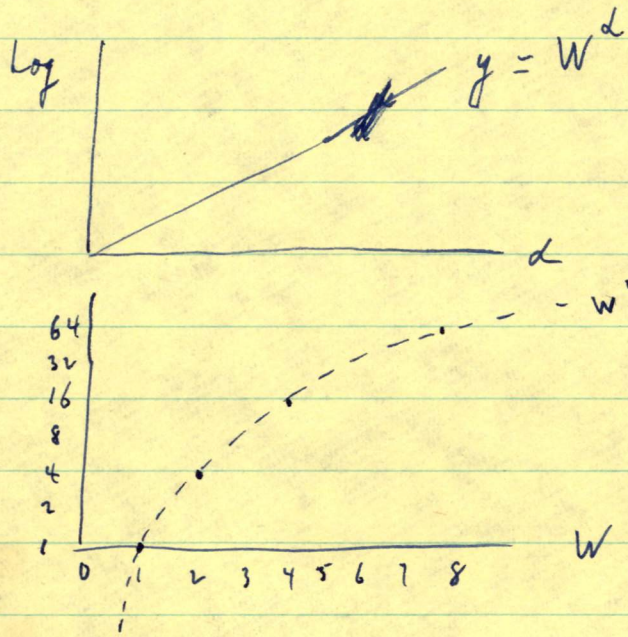
If $y = 1.01^n$

$\ln y = n \cdot \ln(1.01) \cong n \frac{1}{100} = .01 \cdot n = g \cdot n$

$\Delta \ln y = .01 \cdot \Delta n = g \cdot \Delta n$

$$\frac{\Delta \ln y}{\Delta n} = g$$

## Log scales

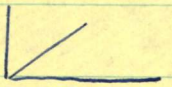



$y = w^2$

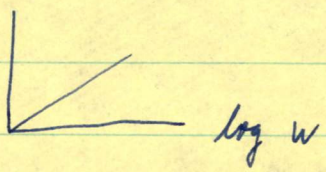
$\ln y = 2(\ln w)$ But $\ln w$ rises much slower than $w$ itself, ∴ this curve bends to right.

| w | $y = w^2$ | $\ln y$ |
|---|---|---|
| 0 | 0 | |
| 1 | 1 | 0 |
| 2 | 4 | |
| 3 | 9 | |
| 4 | 16 | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | 64 | |

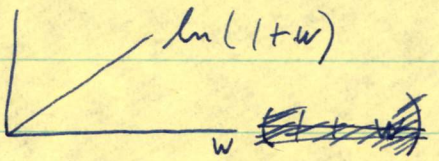
To get a strt. line, 3 ways

1. 

Growth curves

2. 

Elasticity

3. 

$(w < .15)$

?

$(1+w)^{\alpha} \cong e^{\alpha w}$

$y = 1+w \cong e^{w}$

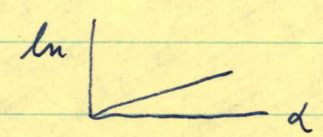
$\ln y \cong w$

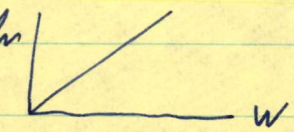
$\ln(1+w) \cong w$

3, A

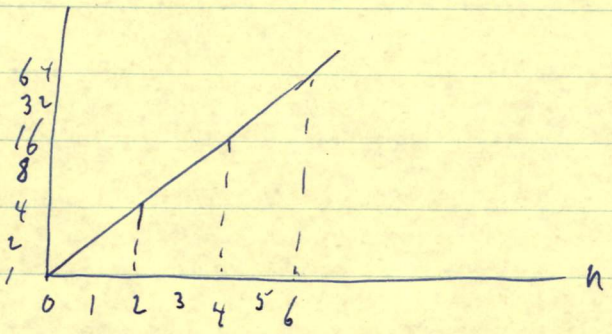
$y = e^{\alpha w}$

$\ln y = \alpha w$



$$y = 2^n$$




The physical slope, $\Delta_1$, is const.
The value slope, $\Delta y / \Delta n$, keeps rising.

$$y = (1+g)^n$$

$g = y^{\frac{1}{n}} - 1$ is const.

$y'/y$ is const. Call this the "% slope". It also $= \Delta \ln y$.
Note that $n$ is $\log_2 y$, and $\therefore\ n = k \ln y$

$$y' = 2^n \ln 2$$

$$\frac{y'}{y} = \ln 2 = k \log_2 2 = k \quad \text{a const.}$$

$$n = \log_2 y = k \ln y$$

$$\ln y = n \cdot \left(\frac{1}{k}\right) = n \cdot K, \text{ (another const.)}$$

Thus shows that $\ln y = c \cdot \frac{y'}{y}$

(notes)

To prove how logs to different bases are related.

Let $y = e^x$

$y = k a^x$

$e^x = k a^x$

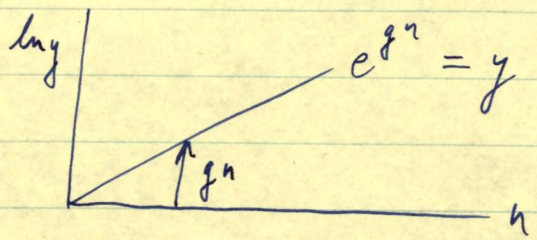
$x = \ln k + x \ln a$

?

---

$y = e^{gn}$

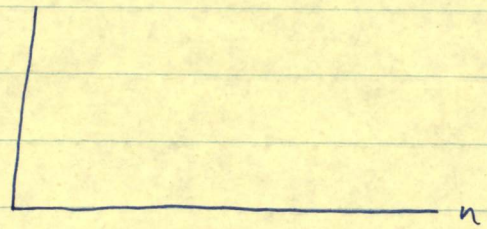
$gn = \ln y$




? ∴ $\lambda$ of curve is always $= g$

---

$y = 2^n$




For every $\Delta n = 1$, $\Delta y = 2y$, so $\Delta y / y = 2$

$$y = (1+g)^n$$

$$\Delta y = (1+g)^{n+1} - (1+g)^n = (1+g)^n(1+g-1) = g(1+g)^n = g \cdot y$$

That shows that $\frac{\Delta y}{\Delta n} = g \cdot y$

Every time $\Delta n = 1$, $\Delta y / y$ is a const. %.

But:

a. On a log scale, every given rise is a const. %

b. On a log, " " " " " "

E.g., if $n$ is the $\log_2$ of $y$, the %, is low

if $n$ is " " $(1+g)$ " " " $g$

If the base is $e$, then $\log_e$ is a const. $\times \log_{(1+g)}$.

| | $f$ | $f'$ | $f'/f$ | $I$ | $I'$ | $I'/I$ | $\frac{I'/I}{f'/f}$ |
|---|---|---|---|---|---|---|---|
| $b$ | $b^2$ | $\Delta(b^2)$ | | $b^3$ | $\Delta(b^3)$ | | |
| | | | | ~~(A6)~~ K | | | |
| 30 | 900 | – | 27.0 → | ~~[illegible]~~ 24.8 | | | |
| 31 | 961 | 61 | .063 | ~~[illegible]~~ 32.8 | 2.8 | .104 | 1.5 |
| 32 | 1024 | 63 | .066 | ~~[illegible]~~ 35.9 | 3.0 | .101 | ( |
| 33 | 1089 | 65 | .063 | ~~[illegible]~~ 39.3 | 3.1 | .095 | ( |
| 34 | 1156 | 67 | .062 | ~~[illegible]~~ 42.9 | 3.4 | .095 | ( |
| 35 | 1225 | 69 | .060 | | 3.6 | .092 | ( |

| $x$ | $(y)$ $x^2$ | $\Delta y$ | $2x \cdot \Delta x$ | $\Delta y / y$ | $\frac{2}{x} \cdot \Delta x$ | $\ln y$ | $\Delta \ln y$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | — | | | $\infty$ | | |
| .1 | .01 | .01 | .02 | $\infty$ | 2.000 | -4.6052 | $\infty$ |
| .2 | .04 | .03 | .04 | 3.0 | 1.000 | -3.2189 | 1.3863 |
| .3 | .09 | .05 | .06 | 1.25 | .6667 | -2.4079 | .8110 |
| .4 | .16 | .07 | .08 | .7778 | .5000 | -1.8326 | .5754 |
| .5 | .25 | .09 | .10 | .5625 | .4000 | -1.3863 | .4463 |
| — | | | | | | | |
| .9 | .81 | — | .18 | | .2222 | -.2107 | — |
| 1.0 | 1.0 | .19 | .20 | .2346 | .2000 | 0 | .2107 |
| 1.1 | 1.21 | .21 | .22 | .2100 | .1818 | .1906 | .1906 |
| 1.2 | 1.44 | .23 | .24 | .1901 | .1667 | .3646 | .1740 |
| 1.3 | 1.69 | .25 | .26 | .1736 | .1538 | .5247 | .1601 |
| — | | | | | | | |
| 2 | 4 | 3 | 4 | 3.0 | 1.0 | 1.3863 | 1.3863 |
| 3 | 9 | 5 | 6 | 1.2500 | .6667 | 2.1972 | .8109 |
| 4 | 16 | 7 | 8 | .7778 | .5000 | 2.7726 | .5754 |
| 5 | 25 | 9 | 10 | .5625 | .4000 | 3.2189 | .4463 |
| — | | | | | | | |
| 50 | 2500 | — | 100 | — | .0400 | 7.8240 | — |
| 51 | 2601 | 101 | 102 | .0404 | .0392 | 7.8637 | .0396 |
| 52 | 2704 | 103 | 104 | .0396 | .0385 | 7.9025 | .0388 |
| 53 | 2809 | 105 | 106 | .0388 | .0377 | 7.9406 | .0381 |
| 54 | 2916 | 107 | 108 | .0381 | .0370 | 7.9780 | .0374 |
| 55 | 3025 | 109 | 110 | .0374 | .0364 | 8.0147 | .0367 |

# Sensitivity

In discounting we divide, so we are interested in % change: $\frac{d \ln[(1+i)^n]}{dn + di}$ or $\frac{y'}{y}$.

$$\ln(1+i)^n = n \ln(1+i) = np$$

Use calcs, $\ln(1+i) \cong i$

A. $\frac{d(np)}{dp} = n$

| n | Δp | Δp · n |
|---|---|---|
| 1 | .01 | .01 |
| 10 | | .10 |
| 20 | | .20 |
| 50 | | .50 |

E.g. ~~$\frac{1.01^{50}}{1.08^{50}} = \frac{29.457}{46.902} = .628$~~

Rules are approximate bcs. you are impatient w. arithmetic needed to make them more exact.

$\frac{1.08^{50}}{1.07^{50}} = \frac{46.902}{29.457} = 1.59$

$1.01^{50} = 1.64$

The rule understates the sensitivity

B. $\frac{d(up)}{dn} = p$

$$\frac{P_i}{a} = 1 - (1+i)^{-n}$$

$$1 - \frac{P_i}{a} = (1+i)^{-n}$$

$$(1+i)^{n} = \frac{1}{1 - \frac{P_i}{a}}$$

$$n \ln(1+i) = -\ln\left(1 - \frac{P_i}{a}\right)$$

or $$\frac{\frac{1}{a - P_i}}{a} = \frac{a}{a - P_i}$$

$$n \ln(1+i) = \ln a - \ln(a - P_i)$$

$$n = \frac{-\ln\left(1 - \frac{P_i}{a}\right)}{\ln(1+i)}$$

If $a = P_i$, $n = \frac{-\ln 0}{\ln(1+i)} = \infty$

If $i = 0$, $n = \frac{-\ln 1}{\ln 1}$ ($\lim$ as $i \to 0$)

Review. Ec. 162A, 10/21/82
15 min. before midterm.

1. $y = \frac{u}{v}$

$y_u = y\left(1 + \frac{y'}{y}\right) \doteq y(1 + g - d)$

E.g. $y = 1$

$g = .20$

$d = .13$

$y_u \doteq 1 + .20 - .13 = 1.07$

Precisely,

$y_u = \frac{1.20}{1.13} = 1.062$

2. %'s are a way of standardizing info.
We are interested in % changes a lot.

$\frac{y'}{y}$ is % change of a fn.

$\frac{d \ln y}{dx} = \frac{y'}{y}$

A practical application, compound interest

$y = e^{ix}$

$ix = \ln y$

$\Delta(ix) =$ % change in $y$

a. If $x$ is the variable

$$\frac{d(ix)}{dx} = i$$

Big deal! We already knew that.

b. If $i$ is the variable.

$$\frac{d(ix)}{di} = x$$

If $x = 20$, and $\Delta i = 1\%$

$$d(ix) = di \cdot x = .01 \cdot 20 = .20 = 20\%$$

Now this is a useful rule!
Change i.r. by $1\%$, reduce $PV_x$ by $x\%$.

cf. $\frac{1-d}{1+i}$ vs. $\frac{1}{1+i+d}$ vs. $\frac{1}{(1+d)(1+i)}$

1. $\frac{1-d}{1+i} \stackrel{?}{=} \frac{1}{1+i+d}$

$$1+i+d-d-di-d^2 \stackrel{?}{=} 1+i$$

$$-i-d \stackrel{?}{=} 0$$

$$-d \stackrel{?}{=} i$$

$$-d < i$$

$$\therefore \frac{1-d}{1+i} < \frac{1}{1+i+d}$$

At $i = .05$,

| ~~d~~ | $\frac{1-d}{1.05}$ | $\frac{1}{1.05+d}$ | $\frac{1}{1.05(1+d)}$ |
|---|---|---|---|
| .05 | .9048 | .9091 | .9070 |
| .25 | .7143 | .7692 | .7619 |
| .50 | .3226 | .6452 | .6349 |
| | | .4878 | .4762 |
| 1.00 | 0 | ~~.49~~ | ~~.48~~ |

Remarkably close

2. $\frac{1}{1+i+d} \stackrel{?}{=} \frac{1}{(1+d)(1+i)}$

$1+i+d+di \stackrel{?}{=} 1+i+d$

$di > 0$

$\therefore$

$\frac{1}{1+i+d} > \frac{1}{(1+d)(1+i)}$

However, this difference is very small.

3. Conclusion. To adjust for decay, ~~[illegible]~~

- ~~Rarely~~ ~~[illegible]~~ subtract $d$ from num.
- ~~Usually~~ add $d$ to denom.
- ~~For generality~~ use $\frac{1}{(1+i)(1+d)}$

- If $d$ is under 10%, and $n$ is small, all 3 forms are nearly the same
- For larger values of $d+n$, $(1-d)$ decays much faster than the other 2
- ~~[illegible]~~ The other 2 are remarkably close.

$i = .05$

| $d$ | $\frac{1-d}{1.05}$ | $\frac{1-.05-d}{}$ | $\frac{1}{1.05+d}$ | $\frac{1}{1.05(1+d)}$ |
|---|---|---|---|---|
| .01 | .9429 | .9400 | .9434 | .9430 |
| .02 | .9333 | .9300 | .9346 | .9337 |
| .03 | .9238 | .9200 | .9259 | .9246 |
| .04 | | | | |
| .05 | .9048 | .9000 | .9091 | .9070 |
| .06 | | | | |
| .07 | .8857 | .8800 | .8929 | .8901 |
| .10 | .8571 | .8500 | .8696 | .8658 |
| .15 | .8095 | .8000 | .8333 | .8282 |
| .20 | .7619 | .7500 | .8000 | .7937 |
| .30 | .6667 | .6500 | .7407 | .7326 |
| .50 | .4762 | .4500 | .6452 | .6349 |

---

| | | | | |
|---|---|---|---|---|
| −.01 | .9619 | .9600 | .9615 | .9620 |
| −.05 | 1.000 | 1.000 | 1.000 | 1.0025 |
| −.10 | 1.0476 | 1.0500 | 1.0526 | 1.0582 |
| −.30 | 1.2381 | 1.2500 | 1.333 | 1.3605 |
| −.50 | 1.4286 | 1.4500 | 1.8182 | 1.9048 |

$$y = (1+i)^n$$

$$\frac{dy}{dn} = n(1+i)^{n-1}$$

$$\frac{dy/dn}{y} = \frac{n}{1+i} \doteq n$$

$$\frac{dy}{di} = n(1+i)^{n-1}\,(1+i)^n \ln(1+i)\,\frac{dn}{di}$$

$$\frac{dy}{dn} = (1+i)^n \ln(1+i)\,\frac{dy}{dn}$$

$$\frac{dy}{di} = n(1+i)^{n-1} \cdot \frac{d(1+i)}{di} \quad ; \quad dy = di\left(n(1+i)^{n-1}\right)$$

$$\frac{\frac{dy}{di}}{y} = \frac{n}{1+i}$$

$$y = e^{in}$$

$$\frac{dy}{di} = e^{in} \cdot \frac{d(in)}{di} = n e^{in} \quad ; \quad dy = di\left(n e^{in}\right)$$

$$\frac{\frac{di}{di}}{i} = \frac{1}{i}$$

$$\frac{\frac{dy/di}{y}}{\frac{di/di}{i}} = \frac{n}{\frac{1}{i}} = ni$$

As a, if $i = .05$

$\Delta i = .01$

$\Delta i / i = .20$

$n = 10$

$\epsilon = n i = .5$

a 20% $\Delta i$ gives a 10% $\Delta y$

$1.05^{10} = 1.63$

$1.06^{10} = 1.79$

$\frac{1.79}{1.63} = 1.10$ check

$\left(\frac{1.05}{1.05}\right)^{10} \doteq 1.01^{10} \doteq 1.10$

5. Logs  Glossed over as misc., p. 45

Really are more important.

a. $y = \ln x$

$y' = \frac{1}{x}$

b. ~~$y = \ln u \quad (u = f(x))$~~

$y = f(x)$

$\frac{d \ln y}{dx} = \frac{y'}{y}$

~~$\frac{dy}{dx} = \frac{d(\ln u)}{du} \cdot \frac{du}{dx}$~~

c. $\ln(1+i) \doteq i$

3, p. 45

~~$y = (1+i)^n$~~

~~$\frac{dy}{dt} = (1+i)^n \ln(1+i) \doteq i(1+i)^n$~~

d. $y = a^x$

$y' = a^x \ln a$

If $a = 1+i$

Then $y' = (1+i)^x \ln(1+i) = (1+i)^x \cdot i$

e. From (d)

$$y = e^x$$
$$y' = e^x \ln(e) = e^x$$

$$y = e^{ax}$$

$$y' = a e^{ax}$$

Last is handy in sensitivity analysis,

re effect of $i$ + $n$ on future values, or discount factors.

$$F = (1+i)^n \cong e^{in}$$

$$dF/dn = i e^{in}$$

$$dF/di = n e^{in}$$

$$(iz)^{n+1} - (iz)^n = (iz)^n(iz-1) = i(iz)^n$$

$$(iz)^{-(n+1)} - (iz)^{-n} = \frac{(iz)^{-1} - 1}{(iz)^n} = \frac{1 - iz}{iz}(iz)^{-n}$$

$$= \frac{-i}{iz}(iz)^{-n}$$

$$= -i(iz)^{-(n+1)}$$

✓ $$(iz)^{-n} - (iz)^{-(n-1)} = \frac{1 - (iz)^1}{(iz)^n} = -i(iz)^{-n}$$

4. Sensitivity of F to n.

Comp. int. growth factor, $G_n \equiv (1+i)^n$

$\Delta G/\Delta n = i(1+i)^n$

$\frac{\Delta G/\Delta n}{G_n} = i$

| n | $G_n$ | Current yield $\frac{\Delta G/\Delta n}{G_n}$ | $\frac{\Delta G/\Delta n}{G_n}$ | $\frac{G_n}{G_{n-1}}$ |
|---|---|---|---|---|
| 0 | 1 | $i$ | $i$ | |
| 1 | $1+i$ | $i(1+i)$ | ~~$i(1+i)$~~ | |
| 2 | $(1+i)^2$ | $i(1+i)^2$ | $i$ | |

5. Sensitivity of PV discounts to n

$\Delta(G^{-1})/\Delta n = G^{-n} - G^{-n} = \frac{1-G^1}{G^n} = \frac{-i}{(1+i)^n} = -i\,G^{-n}$

$\frac{\Delta(G^{-1})/\Delta n}{G_{n-1}} =$

| n | $(1+i)^{-n}$ | $\Delta(1+i)^{-n}/\Delta n$ | $\frac{\Delta(1+i)^{-n}/\Delta n}{(1+i)^{-n}}$ |
|---|---|---|---|
| 0 | 1 | ~~0~~ | |
| 1 | .9524 | -.0476 | -.05 |
| 2 | .9070 | -.0454 | -.05 |
| 3 | .8638 | -.0432 | -.05 |
| 4 | .8227 | -.0411 | -.05 |
| 5 | .7835 | -.0392 | -.05 |

4. $G_n \equiv (1+i)^n$

$$\frac{\Delta G}{\Delta n} = i(1+i)^n$$

$$\frac{\Delta G/\Delta n}{G_n} = i$$

| $n$ | $G_n$ | $\Delta G/\Delta n$ | $\frac{\Delta G/\Delta n}{G_{n-1}}$ |
|---|---|---|---|
| 0 | 1 | | |
| 1 | $1+i$ | $i$ | $i$ |
| 2 | $(1+i)^2$ | $i(1+i)$ | $i$ |

5.

| $n$ | $1/G_n$ | $\Delta(\frac{1}{G})/\Delta n$ | $\frac{\Delta(G^{-1})/\Delta n}{G_{n-1}^{-1}}$ |
|---|---|---|---|
| 0 | 1 | | |
| 1 | $(1+i)^{-1}$ | $-\frac{i}{1+i}$ | $-\frac{i}{1+i}$ |
| 2 | $(1+i)^{-2}$ | $-\frac{i}{(1+i)^2}$ | |

$$\Delta(G^{-1})/\Delta n = (1+i)^{-n} - (1+i)^{-n} = \frac{1-(-i)}{(1+i)^n} = \frac{-i}{(1+i)^n}$$

4. $\dfrac{d\,(1+i)^n}{dn} = \ln(1+i)(1+i)^n \cong i(1+i)^n$

$\dfrac{d\,(1+i)^{-n}}{dn} \cong -i(1+i)^{-n}$

$\ln (1+i)^n \equiv G$

$\dfrac{\frac{dG}{dn}}{G} = i$

$\dfrac{d\frac{1}{G}}{\frac{1}{G}} = -i$

So, what happens to PV when $\Delta n = 1$? It drops by $i$%. To confirm, complete ffg. table, using calc.

| $n$ | $F$ | $(1+.05)^{-n}$ | $PV_n$ | $PV_n - PV_{n-1} = \Delta PV$ | $\frac{\Delta PV}{PV_{n-1}}$ |
|---|---|---|---|---|---|
| 1 | 100 | .9524 | 95.24 | -4.76 | ~~.05~~ ~~.0476~~ |
| 2 | | | 90.70 | -4.54 | .01 ~~.0477~~ |
| 3 | | | 86.38 | -4.32 | .01 ~~.0476~~ |
| 4 | | | | | |
| . | | | | | |
| . | | | | | |
| . | | | | | |

1. Derive e by //. steps.

a, $1.00001^{100,000} = 2.71828\,82\,37$

b, $.99999^{-100,000} = 2.71828\,29\,54\,20$

c, add a + b, ÷ 2 = 2.71828 18 29

d. Q. e.

$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$

You have just found $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$. In a few moments you have duplicated work that took John Napier many years.

2. $\lim_{f \to \infty} \left(1 + \frac{i}{f}\right)^{fi} = e^{i}$ etc.

3. $\therefore (1+i)^{n} \doteq e^{ni}$

$\therefore$ PV sensitivity to n & i are //

$$P_n = (1+i)^{-n}$$

$$P_{n-1} = (1+i)^{1-n}$$

$$\frac{P_n}{P_{n-1}} = \frac{(1+i)^{-n}}{(1+i)^{1-n}} = (1+i)^{-1}$$

$$\frac{P_{n-1}}{P_n} = 1+i$$

$$\frac{P_0}{P_1} = \frac{1}{(1+i)^{-1}} = 1+i$$

$$\frac{P_1}{P_2} = \frac{(1+i)^{-1}}{(1+i)^{-2}} = 1+i \quad \text{etc.}$$

$$\Delta P = (1+i)^{-n} - (1+i)^{1-n} = \frac{1}{(1+i)^n} - \frac{1+i}{(1+i)^n} = \frac{-i}{(1+i)^n}$$

$$\frac{\Delta P}{P_{n-1}} = \frac{\frac{-i}{(1+i)^n}}{(1+i)^{1-n}} = \frac{-i}{1+i}$$

---

$$\Delta \frac{1}{G} / \Delta n = \frac{1}{1+i} - \frac{1+i}{1+i} = \frac{-i}{1+i}$$

$$\frac{\Delta (1+i)^{-n}}{\Delta n} =$$

$$(1+i)^{-n} - (1+i)^{1-n} = \frac{1}{(1+i)^n} - \frac{1+i}{(1+i)^n} = \frac{-i}{(1+i)^n}$$

or

$$= \frac{(1+i)^{-1}}{(1+i)^{n-1}} - \frac{1}{(1+i)^{n-1}} = \frac{\frac{1}{1+i} - \frac{1+i}{1+i}}{(1+i)^{n-1}} = \frac{-i/(1+i)}{(1+i)^{n-1}}$$

Logs — Things to org. & straighten out for presentation.

$A(1+r)$

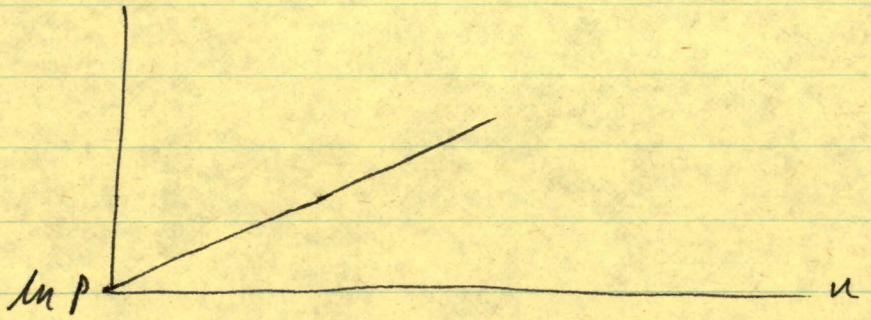
$$\ln[A(1+r)] = \ln A + \ln(1+r)$$

$$\ln\left[(1+r)^{n-1}(1+r)\right] = (n-1)\ln(1+r) + \ln(1+r)$$

---

Let $F = P(1+r)^n$

$$\ln F = \ln P + n\ln(1+r) \cong \ln P + nr$$



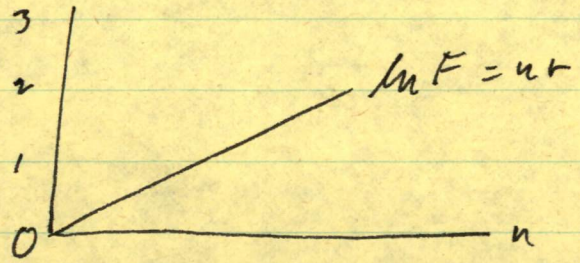
$F = Pe^{nr}$

$\ln F = \ln P + nr$

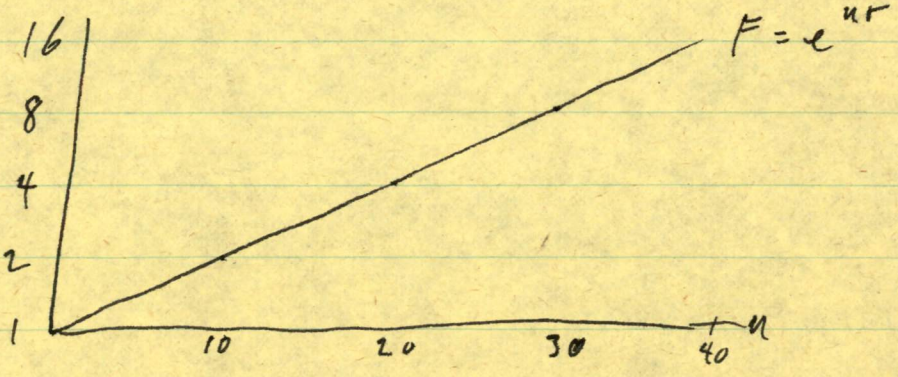
$\ln F - \ln P = nr$

If $P = 1$, then

$\ln F = nr$




Semilog paper is more useful, bec. it immediately translates $\ln F$ into $F$



1. The absolute $\Delta$ of the log is the % $\Delta$ of the fn.

Thus, $F = e^{in}$

$\ln F = in$

If $\Delta n = 1$ then $\Delta(in) = i$

and $\Delta F = Fi$

[By calc.,

$$\frac{dF}{dn} = ie^{in} = iF \quad ]$$

If $\Delta i = .01$, then $\Delta(in) = .01n$

and $\Delta F = F(.01n)$

2. If $P = AK^{\alpha}B^{\beta}$

Then $\alpha$ & $\beta$ are elasts. of prodn. $\equiv$ the input-elast. of output  Show this.

3. If $C = k P^{\frac{1}{2}}$

Then $\frac{MC}{aC} = \frac{1}{2}$

$$\frac{\Delta C/\Delta P}{\frac{C}{P}} = \frac{\frac{\Delta C}{C}}{\frac{\Delta P}{P}} = \text{the output-elast. of input}$$

Study qn. 10 is exasperating.

$\text{Log}_2 y$ increases from 10.0 to 10.1

$\frac{10.1}{10.0} = 1.01$, a 1% change, but a change of .1

$\Delta \ln y * \cdot \log_2 e = \Delta \log_2 y$

$\Delta \ln y = \frac{\Delta \log_2 y}{\log_2 e} = \frac{.1}{1.4427} = .0693$

Check

$y_1 = 2^{10} = 1024$

$y_2 = 2^{10.1} = 1091.5$

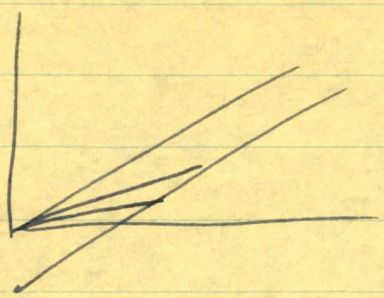
$1091.5 / 1024 = 1.0718..$

(Error due to finite size of change, ~~[illegible]~~ and or rounding)

$\text{Log}_2 1024 = 10$

$\text{Log}_e 1024 = \frac{10}{1.4427} = 6.931$ This checks

Study Qu. 12



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Points from reading, Baumol, Chs. 1-4

1. p. 20, $\Sigma$. Think of tabj paper, reticulated. and of areas under curves. E.g. a normal forest. " " integral calc.

2. p. 23, = mgnl princ, Think of best mgnl. alt. as MC

3 p. 39. Says mgnl. anal. cannot cope w. corner maxima.

I doubt this. Problem is the assm. of fixed capac. max. which creates corner.

4. Ch. 4. Among diff. rules, leaves out $\frac{d \ln a}{da} = \frac{1}{a}$

5. NB p. 60, General diff. This takes us from partial to gnl. or system anal.

GNP and., if time.