



Introduction to Finance for Managerial Economics

By Dr. Mason Gaffney

Article citation:

Gaffney, Mason. "Introduction to Finance for Managerial Economics." Econ 162A, Fall 1984. Study guide.

Fair Use Disclaimer

The Robert Schalkenbach Foundation, a non profit private foundation, is publishing this article, at no cost, pursuant to the Fair Use provisions of Section 107 of the Copyright Act which states that under certain circumstances, the fair use of a copyrighted material for purposes of criticism, comment, teaching, scholarship and research is not an infringement of copyright. If you have any concerns about the publication of this article, please contact us at info@schalkenbach.org.

The Robert Schalkenbach Foundation's Digital Collections and Archives is an extension of the Robert Schalkenbach Foundation's Engel Georgist Library. It collects, organizes, preserves, and provides free online access to researchers and the general public of Georgist literature including books, newspapers, pamphlets, scholarly articles and archival materials.

RSF is particularly proud to offer free online access to the works of the late Georgist scholar Mason Gaffney, made possible through the generosity of the Danica Foundation and with the consent of the Mason and Letitia Gaffney Trust.

Robert Schalkenbach Foundation: <https://schalkenbach.org/>

Engel Georgist Library: <https://www.librarycat.org/lib/EngelGeorgistLibrary>

RSF Digital Collections and Archives: <https://schalkenbach.starter1ua.preservica.com/>

Installation I, pp. 1-10
II, pp. 11-18

page 1 of 18

INTRODUCTION TO FINANCE FOR MANAGERIAL ECONOMICS

Notes by Mason Gaffney for use in Econ. 162A, Fall, 1984

1. Compounding.

The compound interest expression is $F = P(1+i)^n$, where P is present value, F is future value, i is the interest rate per period, and n is the number of periods. We have in class so far used the year as the period, but months are also commonly used.

To simplify writing, we are letting $I = 1+i$. Then our basic expression becomes simply:

$$F = PI^n \quad (1)$$

(1) is the basic building block of which all the mathematics of finance is constructed.

In words, (1) says that "Money makes money, and the money money makes makes more money." Ben Franklin said that, and you can't get much more concise with words. (1) is a lot easier to work with. Be thankful therefore for mathematics, and not afraid, for it is there to help you. Math is often abused by pseudo-economists and other charlatans who use it to create artificial problems. Our objective is to use it to solve real problems, to make life easier rather than harder.

Figure 1, "Growth of Money at Compound Interest", ^(not reproduced here) shows what the function looks like. The curve there is called a curve of exponential growth. It grows at a constant percentage each year. It is said to have a constant growth "rate". "Rate" means ratio, and implies a numerator and a denominator. In this case the numerator is absolute growth, and the denominator is the growth achieved at the end of the previous period. Thus the constant growth rate means a rising absolute growth.

2. Commensurability

The interest rate, through Equation (1), is the rate of exchange between values at different times. It makes them commensurable. "Measurement is Science", according to some. Money makes apples and oranges commensurable, as you all know. But we have to supplement the dollar measure with interest rates, in order to make future apples commensurable with present apples. We will see that the present apple is worth a lot more.

3. Discounting

Discounting is compounding backwards. You use it when you know the future value, F , and want to find the present value. The most common basing point for making values commensurable is, as you might expect, the present. Solving (1) for P we have:

$$P = FI^{-n} \quad (2)$$

Figure 2, "Present Value of Given Future Values", shows the shape of (2). The curve sloping downwards, and then leveling off, is I^{-n} . It is called a curve of exponential decay, and is the reciprocal of exponential growth.

Each point on I^{-n} is found by dividing one (the level curve on top) by I^n . You may visualize the process by following the dash lines. Dividing by I^n is compounding backwards, along a compound growth curve, to the y-intercept. That value of y is then projected to the right, out to n , to form a point on the discount curve.

4. Doubling time, the rule of 72, and the great power of accumulation

Money at compound interest doubles in a number of years equal to 72 divided by the percentage rate of interest. Thus, at 10% money doubles every 7.2 years; at 7.2%, every ten years, and so on. Today with a little effort you can secure 14.4%, at which money doubles every 5 years (except for the tax bite, which we'll discuss separately, later.)

The rule is approximate, but accurate enough for most purposes.

Test this by raising 1.072 to the 60th power, and see how close you get to 64.

In 60 years at 7.2% money will double 6 times: one \$ grows into \$64. Of course you have to resist the temptation to spend the interest, and that is not as easy as it sounds. But given persistent will power, and the absence of calamities and blunders and confiscations, it happens.

The corollary is that a value of \$64 expected 60 years in the future is only worth \$1 today. If that future value is something you think highly of, like basic food for the poor, or natural beauty, and the present value is thought of in terms of something trivial, flighty, vulgar or superfluous, that trade-off may offend you. It has many conservationists and humanitarians. But it is simply a logical extension of consumer sovereignty in the time dimension. Projecting consumer sovereignty in time does not create consumer follies, it simply dramatizes them. In a market system we accept the folly of some to allow freedom for all, and because we know that one man's meat is another man's poison. When we don't like the results of consumer sovereignty we modify it through regulation or discriminatory taxation, which is appropriate and workable. It is not a good idea to get this mixed up with compounding and discounting, however. In Managerial Economics we see this as part of the system in which we are managing resources through time for the greatest economic return. It means we sacrifice very large future values in order to gain much smaller present values. The idea is that by reinvesting the proceeds of the present saving along the fastest growth path we will end up with more in the future than if we had not made the sacrifice.

5. The use of semi-log scales, introduction.

Semi-log scales are very handy to use in learning to understand and work with compound interest and discount. Here are some of their advantages and characteristics.

a. They keep the small and the large numbers on one sheet of paper, without losing accuracy for the small numbers.

b. They help us analyze sensitivity, our first and present

purpose, in percentage terms. "Sensitivity" means how much a function changes in response to changes in its independent variables.

c. The curve of exponential growth becomes a straight line on the semi-log scale. The percentage growth rate is i . The slope of the straight line is proportional to i , so that doubling i also doubles the slope of the line. This is obviously true for $n=1$, and since the lines are straight it is true throughout. Each time n increases by one, y increases by iy^n (you can also call that $100y\%$ of y^n , which says it with some excess motion, but yet may be clearer by virtue of long usage.)

The slope of the curve is always iy^n . It is straight because iy^n/y^n remains constant throughout the curve. That is the slope divided by the ordinate, which is the percentage growth rate. It forms a straight line because on the semi-log scale a given vertical distance always represents a given percentage growth from its base.

d. Figure 3, "Growth of Money on a Semi-log Scale", shows two growth paths, at 7.2% and at 3.6%. Half the rate means exactly half the geometric slope, throughout. Half the geometric slope would mean half the ordinate on a common scale, but on the log scale it means a decreasing fraction of the ordinate as n grows. Thus at $n = 20$ the lower curve is $2/4$ or $1/2$ the upper curve. But at $n = 40$ it is $4/16$ or $1/4$ of the upper curve. At $n = 80$ it is $16/256$ or $1/16$ of the upper curve.

If you want to see a curve equal to half the upper curve at all points, it will be parallel to the upper curve, and its y -intercept is at $1/2$. It intersects the 1.036^n curve where $n = 20$. See (6,a,ii) below for more on this.

e. (d) means that cutting i in half doubles the doubling time from 10 to 20 years, as shown on Figure 3. For any future value, F , $ni = k$, a constant. (For those who care, the constant is $\ln F +$ a very small fudge factor which we needn't get into here.) We have been working with $F = 2$. $\ln 2 = .69$, to which we add a small fudge factor of .03 to get our rule of 72.

f. 1.072^n is the locus of all values whose present value (PV)

is one when discounted at 7.2%. That just says that discounting is backwards compounding. But this is an approach to sensitivity analysis. Discount a number of points on the upper curve back to the present at a lower discount rate, e.g. 3.6%. This means drawing a set of lines parallel to the lower curve, from points on the upper curve, back to the ordinate. They all have different y-intercepts now, which are their present values. The further to the right they are, the higher are the y-intercepts, showing they have been increased by higher percentages. Thus, for $n = 20$, PV rises from 1 to 2, or by 100%. For $n = 40$, PV rises from 1 to 4, or by 300%.

So if you expect interest rates to fall in the future, invest today in long maturities and you will really cash in -- if you are right. If you expect rates to rise, stick with the fast payout, or with cash. If you are really cocksure, and a gambler, don't settle for just not buying long maturities, but sell them short. This last move calls for deep conviction, and/or a mood of reckless abandon. We do not speak here of the psychology of it, but only the mathematical framework.

g. Using the semi-log scale gives the same curve as graphing the log of a function on a common scale. This tells you a lot about logs and how they behave, and what they mean.

A rule of algebra is that $\ln I^n = n \ln I$. Thus for an exponential function the graph of the log is a straight line, since $\ln I$ is a constant, and n is the abscissa. The straight line starts from the origin, just as on the semi-log scale.

A very useful and close approximation is that $\ln I \approx i$. The note explains why, but you may take it on faith if you prefer.* Thus the curve of the log of I^n is simply ni . Every time n increases by one, the curve increases by i . So i is not only proportional to the slope, i is the slope. This makes it clear why the slope increases directly with i .

h. It follows from (g) that $d \ln y / dx = dy / dx / y$. Which is to say that the absolute increase in the log is the fractional increase in the function. That's the way the semi-log scale works, we have seen; and now we see that the log on the common scale works just

the same.

That is not just the way the graph works; that is the way that logs themselves work.

Thus, using a semi-log scale you quickly get a secure base for working with logs themselves. Understanding logs in some fundamental sense will come more slowly, as you work with them. But you can work with them right away, that is the important thing. When in doubt, check what you are doing with the semi-log graph. It will not mislead you.

6. Sensitivity analysis of compounding and discounting

a. Using semi-log scale

i) Comparing 1.072^n and 1.036^n

The vertical difference of 1.072^n and 1.036^n waxes wider as n increases. On the semi-log scale, this means that the percentage difference rises.

If the ordinate on Figure 3 were a common scale the lower line would be always just half the upper one in numerical value, because geometrically it is always just half the height. But on the semi-log scale, where the ordinate scale is logarithmic, the lower line changes from being much more than half when n is small to much less than half when n is large. It is exactly half when n is about 20, and the ratio is $4/2$. Table 1, "Value of 1.036^n as a fraction of 1.072^n ", gives the numbers.

Table 1: Value of 1.036^n as a fraction of 1.072^n

n	1.072^n	1.036^n	$1.036^n/1.072^n$
10	2.	1.42	.71
20	4.	2.00	.50
30	8.	2.89	.36
40	16.	4.00	.25
50	32.	5.86	.18
60	64.	8.00	.12

70 128. 11.90 .09

ii) Line of constant fractional value

A line whose value is any constant fraction of the top line is parallel to it. A line plotting $1/2$ of the top line in Figure 3 begins from $1/2$ when $n=1$, and intersects the 1.036^n line at $n=20$. Draw this on Figure 3 yourself. Don't fear to "cross the equator" and go below the abscissa -- it's only a number.

(Optional. Draw a set of rays from the origin to this 50% line, at various values of n . The slopes of these rays are the rates of return that would be earned on an investment whose payoff was always $1/2$ of 1.072^n .)

iii) Discounting and present value

On the semi-log scale, the reciprocal of a function is the mirror image of the function below the abscissa. Thus discount curves are the mirror image of the corresponding compounding curves. And all the rules above, from (5,a) forwards, apply to discount curves as well, mutatis mutandis. ("Mutatis mutandis" is a handy Latin phrase which means "when you have made the appropriate and obvious adaptations, so don't come at me with any dumb quibbles." Those old Romans sure packed a lot into a few words.)

As to sensitivity we are now ready for a big payoff. It follows that the PV of any FV is the more sensitive to i the farther in the future it is. More tersely, the sensitivity of P to i increases with n .

Draw the mirror images of the curves on Figure 3, and work out a table corresponding to Table 1. If your credit is shaky and you have to pay a premium above the prime to borrow, what does your table tell you about what kind of assets you cannot afford to buy?

Look out for one of the "mutatis" which you have to make when you go below the abscissa and look at discounting. A given vertical distance is always the same percentage change we have said, but that means percentage of the value at the bottom of the vertical reach, not the top. Going from 1 down to $1/2$ is 100% of

1/2, but only 50% of one. There cannot be a 100% or more drop from any value on the log scale, because it never gets down to zero.

Don't despair, there are two solutions. One, whenever possible express any % change as an increase from the lower value. Two, if you have to go from the top value downwards, use the rule that when a denominator rises by $\frac{x}{1+x}$, the quotient falls by $x/(1+x)$.

b. Sensitivity, using calculus

We have seen above (5,h) that a change in $\ln y$ equals the fractional change in y itself. In calculus that is called the "log derivative".

$$\frac{d \ln y}{dx} = \frac{dy/dx}{y} \quad (3)$$

It is reassuring that the calculus rule comports with what we infer from the semi-log scale.

Our "y" is I^n . The shortest way to the log derivative is first to find the log.

$$\ln I^n = n \ln I \quad (4)$$

$$\text{But } \ln I \approx i, \text{ so } \ln I^n \approx ni \quad (5)$$

(Optional note. For those who care, and remember how the Napierian constant e is derived, here is why $\ln I \approx i$.

$$(1+i)^n \approx e^{in}$$

$$1+i \approx e^i$$

$$i \approx \ln(1+i) = \ln I \quad)$$

$$\frac{d \ln I^n}{di} \approx \frac{d(ni)}{di} = n \quad (6)$$

(6) says that when i rises by a percentage point (e.g. from .07 to .08) then I^n rises by about $n\%$. Table 2, "Sensitivity of I^n to i ", shows this for $n = 20$.

Table 2: Sensitivity of I^n to i , when $n = 20$

i	$(1+i)^{20}$	% rise
.00	1.000	----
.01	1.220	22.0
.02	1.486	21.8
.03	1.806	21.5
.04	2.191	21.3
.05	2.653	21.1
.06	3.207	20.9
.07	3.870	20.7
.08	4.661	20.4
.09	5.604	20.2
.10	6.727	20.0
.11	8.062	19.9
.12	9.646	19.7
.13	11.523	19.5

Column 2 looks like a chaotic jumble of numbers, but Column 3 finds order in chaos. The rise is always about 20%. (You may ignore the minor drift from 22% down to 19.5% -- it has a technical cause which need not concern us here.) It gives an accurate, memorable rule of thumb you can always retain and use. The rule is in the paragraph above the table. To repeat, a rise of one point in i gives a rise of $n\%$ in I^n . Memorize that. (Only remember it is approximate, not exact.)

We need to label that rule so we can retrieve it from our memory banks at will. Let's christen it "The $n\%$ rule of growth".

From the viewpoint of the aggressive and/or acquisitive investor, let's apply that rule to compounding. "The $n\%$ rule of growth" says that you can increase your future wealth by $n\%$ for every rise of one point in your rate of return on investment. Raising the rate from 20% to 21% is just as effective as raising it from 0% to 1%: there are no diminishing returns to this effect. And it is so simple to remember. Our travail has led to a simple, useful end.

Applied to discounting and valuation, The $n\%$ rule of growth says that every drop in the discount rate of one % point raises the PV of 5-year futures by 5%; of 30-year futures by 30%; of 100-year futures by 100%; and so on. The drift away from exactitude which we observe in Table 2 means that for abnormally high values of n and i there is less precision than Einstein would want, but still enough for most business purposes.

Do be cautious however not to apply the rule to drops in PV caused by rises in i . Thus when $n=100$ a rise of i by one point does not cause a drop in PV of 100%. Rather it doubles the compounding factor which is the divisor, and doubling a divisor cuts a quotient only by 50%. Watch that. Sorry that life never goes on being as simple as the simplest rules we can devise, but that's the way it is and you had best learn to handle it that way.

c. Sensitivity to i

Equation (6) is symmetrical between n and i . (6) says the sensitivity of I^n to i is n . The symmetry of it tells us therefore that the sensitivity of I^n to n is i . Actually we already knew that, that is what compound interest is all about. It just says that when n grows by one (i.e. another year passes) our money grows by $100i\%$.

(To be continued. Don't miss next session's thrilling installment.)

INTRODUCTION TO FINANCE FOR MANAGERIAL ECONOMICS

Notes by Mason Gaffney for use in Econ. 162A, Fall, 1984

Installment #2

7. Capitalization in perpetuity

Another basic building block of finance is the valuation of a perpetual income stream, or "perpetuity". This is simply the inverse of calculating the annual income, a , from a known principal, P . Everyone knows that $Pi=a$. Therefore, if a is known, and P is not:

$$P = a/i \quad (7)$$

(7) is called "capitalization" because it converts a flow (a) into the equivalent fund of "capital" at time zero.

(Note -- slightly digressive, but important. The analogy to capital is misleading, but we are stuck with the misnomer. The value of land titles may serve the individual like real capital, it is true, because A can liquidate land by selling to B. But society as a whole cannot liquidate land in the sense of speeding up the flow of services and drawing down the "capital" value for early consumption, like a stock of canned goods. Neither can society ever recover the principal from cash flow of land: because the cash flow just covers interest on the principal, not any recovery of principal, ever. We will see that the cash flow of real capital includes recovery of principal. Thus all capital is ultimately liquidated by sale to consumers, whereas the "capitalized" value of land titles is only liquidated by sale to another landholder. To think that land values are social or group capital is a "fallacy of composition" that comes to light whenever large numbers of people try to draw on this "capital" at the same time. The history of banking panics, or collective failures, is one of the inability of many sellers to dump land at the same time. Notice any bank failures lately? Be warned!)

In (7), P is quite sensitive to i . The most orderly way of expressing this particular sensitivity is in terms of "elasticity", which is the % change in P caused by a % change in i . (You may wonder why I choose elasticity here when I did not do so before, in Table 2 and Eqn. (6). The reason is that we found an orderly rule there without going to that extra trouble. You may convert (6) to an elasticity rule if you like, by dividing it by $1/i$).

(7) is so simple it is intuitively obvious that an $x\%$ increase of i causes a corresponding $x\%$ decrease in P , so long as x is small. This is just the same relationship that you should remember from price theory when you studied elasticity of demand and learned about elasticities equal to one, when price times quantity was constant. In the present subject case it is Pi which is a constant, being always equal to a .

I suggest you confirm this by calculus. Take the log derivative of P w.r.t. (with respect to) i . You should get $-1/i$. Take next the log derivative of i w.r.t. i : you should get $1/i$. To get elasticity divide, and get -1 .

If you have ever found elasticities hard to wrap your mind around, note here that elasticity is a much more advanced notion than that of the mere derivative in calculus. Elasticity is the ratio of two log derivatives. Sounds pretty tough, and yet you mastered it in price theory -- you just didn't know what a complexity you were mastering.

What if the $x\%$ increase in i is not small? What if x is 100%, for instance? Remember in price theory how they distinguished between what they called "arc elasticity" and point elasticity? It is the same here, but there is a simple algebraic expression to give it exactitude. The percentage response of P to i is not exactly $-x$ when x is large. It is $-x/(1+x)$.

To prove that, let x be the fractional increase of i , and y be the corresponding fractional change in P . Then:

$$P(1+y) = a/i(1+x) \quad (8)$$

Divide (8) by (7)

$$1+y = 1/(1+x) \quad (9)$$

$$y = 1/(1+x) - (1+x)/(1+x) = -x/(1+x) \quad (10)$$

I suggest you remember this rule in the simple form, that elasticity here equals -1. As for Eqn.(10), that is a general rule applicable to all cases of large changes in divisors, and you should remember that as a general algebraic rule, and not just in connection with this case.

The sensitivity of a/i to i is comparable to the sensitivity shown in Table 2 when raising i from .05 to .06, a 20% increase, raises the discount factor by about 20% also. Thus at a 5% rate of interest the sensitivity of a perpetuity is the same as that of a single or point payment due in 20 years. Going to Eqn. (6) you can infer that it is also the same as a 50% increase when $n = 50$; a 10% increase when $n = 10$; and so on.

8. Present value of level annuity for finite years of life

Land lasts forever, and many common shares are valued as if they will. But depreciable capital yields net cash flows for only finite lives; so do many financial instruments. Thus there is a need to value cash flows (and service flows, too, like the service of owner-occupied housing) over finite lives. The general formula for this follows, and is derived by taking a perpetuity, as in (7), and subtracting from it another perpetuity which begins n years future and is therefore discounted back over n years.

$$P = a/i - (a/i)I^{-n} = a(1-I^{-n})/i \quad (11)$$

(11) is the "Present Value Formula". It converts a flow of payments, a , into the equivalent fund, lump sum or principal at time zero, the present. It specifies the number of years, n , over which the flow continues. It assumes that the flow is level or even. ((11) must and can be modified when it is not.)

((11) is extremely general and useful, and we will conjure with it for several pages.

Another way to derive (11) is as the sum of the present values of all the individual payments, each one being discounted individually to the present. This method is worth showing, for the added insight it gives.

To simplify this we let $a = 1$. That means we can multiply our final result by a and thus put a back in.

$$P = I^{-1} + I^{-2} + \dots + I^{-n} \quad (12)$$

(12) is a geometric progression, whose first term is I^{-1} and whose rate of progression is also I^{-1} . The algebraic formula for the sum of a geometric progression (which you had in high school) is:

$$S = b(1-r^n)/(1-r) \quad (13)$$

where S is the sum, b is the first term, r is the rate of progression, and n is the number of terms. Substituting we have:

$$S = I^{-1}(1-I^{-n})/(1-I^{-1}) = (1-I^{-n})/(I-1) = (1-I^{-n})/i \quad (14)$$

The last equation is the same as (11), as originally derived in the first way, once we recognize that S is P ; and multiply by a .

Figure 4, "Present Value as the Area under a Discount Curve", illustrates what the algebra has done. It discounts each future payment according to its years of futurity; and then cumulates the discounted values to a sum, which is Present Value of the annuity. In the class demonstration with the programmed calculator, we added the columns under the curve one at a time. Then we compared the result with Eqn. (11) and the results were the same, to 8 places or so. Thus the PV formula is accurate, and saves a lot of work. Figure 5, "Cumulative Value of P with increase of n ", takes the area under the curve of Fig. 4 and makes it a curve, labelled $(1-I^{-n})/i$. This curve gives the cumulation of all the increments to PV from discounting each succeeding annual payment. Above it there is a straight line, n , which shows the cumulative sum of all the payments, without discounting them.

9. Some practical inferences from Sect.8 and Eqn. 13, the PV

formula

a. Most of the PV of the annuity comes from the early years. The PV curve is concave downwards to the right, and nearly flattens out after a while. It is approaching an asymptotic limit, which is a/i , as n gets very large.

b. It follows from a) that doubling the life of capital from a small base, say from one year to two, nearly doubles the PV. Doubling life from 20 years to 40 does not nearly double the PV. And doubling from 50 years to 100 adds very little to PV.

Putting that more generally, the elasticity of PV w.r.t. n is a decreasing function of n . In Table 3, "Elasticity (ep) of P w.r.t. n ", n increases by doubling, i.e. in steps of 100%. dP/P is thus the ep, since $dn/n = 1$.

Table 3: Arc Elasticity (ep) of P w.r.t. n

<u>n</u>	<u>i = .05</u>		<u>i = .10</u>	
	<u>P</u>	<u>dP/P</u>	<u>P</u>	<u>dP/P</u>
1	.95	---	.91	---
2	1.86	.96	1.74	.91
4	3.55	.91	3.17	.82
8	6.146	.82	5.33	.68
16	10.84	.68	7.82	.47
32	15.80	.46	9.53	.22
64	19.12	.21	9.98	.05

You might think about Table 3 when you're deciding how much it is worth spending today to make a product last for the second 50 years. And it is not just simple durability, it is also capacity. Many buildings and engineering works are built with excess capacity, in expectation of higher future demand. You have to balance the extra cost of that extra capacity against the fact that its extra value doesn't come into play until after several years.

Even more generally we can find an algebraic expression for elasticity (ep) at a point. It is clear from (12) that the increment to PV is I^{-n} whenever n increases by one. The fractional increase in PV is therefore:

$$dP/P = I^{-n}/(1-I^{-n})/i = i/(I^n-1) \quad (15)$$

The fractional increase in n w.r.t. n is always just $1/n$. To get ep , divide by that, and we have:

$$ep = ni/(I^n-1) \quad (16)$$

You can also struggle with calculus and get (16) as the ratio of the log derivatives of PV and n , and it would be a revelatory experience which I commend to you. However the answer would be no different.

Now (16) probably looks to you like mush when you first eyeball it, but there are nice easy ways to grasp it. First, I trust you have gathered that " ep " is for epsilon, the conventional notation for elasticity, since I have no epsilon symbol on this machine which I am stroking at 1 A.M. at home. On the right side, ni is simple interest; and I^n-1 is compound interest. Thus ep is just the ratio of simple to compound interest, which we may call the "simp/comp ratio". (If you are into baiting reds you may call it the reciprocal of the com/simp ratio.) This ratio pops up often in finance, so much so that I have it all programmed on my calculator and can give you a table of numbers in Table 4, "Point Elasticity of PV w.r.t. n ". Absent the table you can easily remember that which is not otherwise obvious, that the ep is always less than one, because simp is always less than comp.

Table 4: Point Elasticity of PV w.r.t. n

n	$i=$	<u>.02</u>	<u>.05</u>	<u>.10</u>
2		.99	.98	.95
4		.97	.93	.86
8		.93	.84	.70
16		.86	.68	.45
32		.72	.42	.16
64		.50	.15	.01

c. PV is also sensitive to i , of course. Here, however, the sensitivity is negative, that is higher i means lower PV. If you want the practise of taking the log derivative of PV w.r.t. i , I recommend it, you will gain a lot of insight. Remember that $(1+i)^{-n}$ is also expressed as e^{-in} , which tells you that the effects of i and n on the discount factor are parallel. However the effects on PV are not parallel because i also appears alone in the basement. Taking the log derivative this causes the subtraction of $1/i$ from an expression otherwise parallel to the one in (15); dividing by $1/i$ makes that become one. The upshot is that ep of PV w.r.t. i is $\text{simp/comp} - 1$:

$$\text{ep} = ni/(I^n - 1) - 1 \quad (17)$$

And the upshot of that is that you can construct your own table simply by subtracting 1 from the corresponding figures in Table 4. That operation of course makes the figures negative, as they should be, indicating that PV is a decreasing function of i . Perform this exercise, and note how ep changes with both n and i . Now you are really into sensitivity analysis.

d. Share of interest in cash flow.

How much of cash flow (c.f.) is income to the investor? How much is recovery of P ? (P stands not just for Present Value but also for Principal, or Purchase Price -- handy that they all start with P).

See Fig. 5, "Cumulative Value of PV with increase of n ". n is the cumulative cash flow. P is the lower curve, concave down to the right. It tells you what P is normally (in equilibrium) required to buy an income stream whose total value is n .

The curve gives a conspectus. It is most useful when used in double harness with the curves' algebraic definitions, which in turn generate real numbers (the numbers are no more real than the algebra, but may seem that way because you are more used to them). When you have graphs, equations and numerical tables reinforcing each other you really get a handle on things.

All the cash flow above P has to be interest. It is $n-P$. The share of interest in the total cash flow is therefore:

$$\text{Share of interest} = (n-P)/n = 1-P/n = 1 - (1-I^{-n})/ni \quad (18)$$

Table 5, "Share of Interest in Cash Flow", gives the numbers from (18).

Table 5: Share of interest in cash flow

$i=$	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>
$\frac{n}{5}$				
5	.13	.24	.33	.40
10	.23	.39	.50	.58
20	.38	.57	.69	.76
40	.57	.76	.83	.88
80	.76	.88	.92	.94

Table 5 says that if you borrow at 10%, amortized over 40 years, then 76% of all the payments you make go to pay interest. The other 24% pays off the loan.

That's the bad news. The good news is that if you buy a house for \$100,000 and it yields you a service flow worth \$10,000/year, then in 40 years you will have received \$400,000 from it, for your outlay of 1/4 that much. (The even better news is that the service flow is not taxable, while the interest on your loan is deductible. But wait, there is more bad news: that is why houses cost so much to buy.)

An operation is "capital-intensive" when a high share of cash flow goes to feed the banker or investor. That occurs basically when payments are long deferred, strung out in the far future. Normally that occurs when capital is durable, heavy and expensive, like a Rolls-Royce (life, 45 years; $P = \$95k + \text{tax}$; is it worth it?)

e. Depreciation. (To be continued in the next thrilling installment).

INTRODUCTION TO FINANCE FOR MANAGERIAL ECONOMICS

Notes by Mason Gaffney for use in Econ. 162, Fall, 1984

Installment #3, pp. 1-8

#4, pp. 9-10

9. Practical Inferences (cont.)

e. Depreciation and Amortization, Level Annuity

How much of an annual level payment received from an asset of finite flows is depreciation? The same formula shows amortization of the principal of a debt, using level payments for debt service (the normal way, aka installment plan payment).

Here are three approaches to answering the question, all giving the same answer.

One) Depreciation (D) is the loss in the PV of an asset. Current annual Depreciation is the loss of PV over one year.

To keep the years straight, use this notation:

D = Depreciation

n = any current year

w = whole life

b = balance of life = w-n

Current annual Depreciation is the excess of PV when b years of life remain over PV when (b-1) years of life remain. Here is the algebra:

$$\frac{D}{a} = \frac{1 - I^{-b}}{i} - \frac{1 - I^{-(b-1)}}{i} = \frac{I^{-(b-1)} - I^{-b}}{i} = \frac{I^{-b}(I - 1)}{i} = I^{-b}$$

$$D = a I^{-b} \quad (19)$$

The result, (19), is much simpler than the process of reaching it. Current Depreciation is simply the basic single payment or PIPO Discount Factor, (our elemental building block), multiplied times the level payment a. That means the share of a that is Depreciation is simply the Discount Factor itself. (The

Discount Factor (DF) is always less than one.)

Two) In this, the second approach we first pay interest on the unrecovered balance, then impute the residual share of a to Capital Recovery (KR). Unrecovered Balance, aka outstanding debt, is the original principal of a loan less the sum of paybacks previously made.

The math is the same when we apply it to real capital (K) stored up in plant and equipment. The unrecovered balance is the original capital input, K_0 , less the sum of Capital Consumption Allowances (KCAs) previously made.

The unrecovered balance, P_n , is the PV of a b -year annuity. " b " stands for the balance of life, i.e. the years of life remaining. That is neither the original whole-life, w , nor the number of years that have passed, n -- don't get mixed up about that. Remember it this way: $w-n=b$.

In any current year, n :

$$\frac{\text{Interest}}{a} = \frac{P_n i}{a} = \frac{1 - I^{-b}}{i} \times i = 1 - I^{-b}$$

$$\frac{D}{a} = 1 - \frac{P_n i}{a} = 1 - [1 - I^{-b}] = I^{-b} \quad (20)$$

Again, the share of Depreciation is the DF. The shares of the DF and of Interest sum up to one. They are "complements." "Complement" is a good word to know: if you already know it, accept my compliments, and spell each word right.

Three) Graphically, it looks like this:

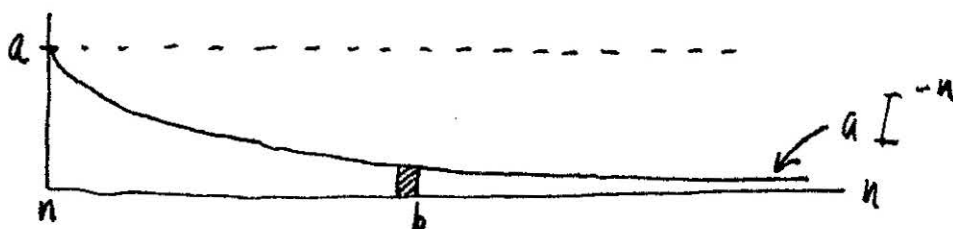


Figure 6: Value of Depreciation over One Year

The shaded column is Depreciation when b years of life remain. The origin of the graph is not the original time zero, but the current year, n .

However you figure it, the share of the level payment that is a payback of Principal is the DF. In numbers, that can be very small. For example, on a 40-year mortgage, at 11% interest, the share of the first payment that is payback is only 1.54%. That is not 1.54% of the debt, but of the first payment on the debt. As a share of the debt it is much smaller: 1.54% of 11.17%, or .17%. Over 99% of the first payments are pure interest.

Many debtors are appalled when they first learn that. However, looking at it from the bankers' viewpoint, they have problems too: they aren't recovering their money any faster than the debtors are reducing their debts. Bankers can get in deep trouble tying up their funds in long-term loans at fixed rates of interest. All it takes is one of three things: a rise of current interest rates; inflation; or a drop in collateral values. All three have happened within recent memory; many lending institutions, notably S&Ls, have failed.

10. Appreciation: the PV of a rising cash flow or service flow.

Here we deal with a perpetual asset, like land or common stock. We also assume tenure institutions that grant land to private owners in "fee simple," meaning the private tenure is to last as long as the land. (Why the legal detail? Many interests in land are terminable; fee simple is not universal or inevitable or a "natural order" of things. It is a human institution of fairly recent origin and an uncertain future.)

Let cash flow begin with \$1, and rise yearly by a fraction, g .

$$CF = 1 + (1+g) + (1+g)^2 + (1+g)^3 + \dots + (1+g)^\infty$$

$$\text{Then } PV = I^{-1} + I^{-1}(1+g) + I^{-2}(1+g)^2 + \dots + [(1+g)/I]^\infty$$

Summing the series:

$$PV = I^{-1} \frac{1}{1 - I^{-1}(1+g)} = \frac{1}{1+i-1-g} = \frac{1}{i-g} \quad (21)$$

Hereafter use P alone for PV .

Solving (21) for P_i :

$$P_i = 1 + P_g \quad (22)$$

P_g is ΔP by obvious inference from the equations above (21) which define cash flow year by year. Work this out as an exercise.

Thus, P_i equals the sum of CF plus current appreciation. Note the symmetry and consistency of this with (20), where P_i equals CF less Depreciation.

(21) and (22) are "factors" like the PVF previously studied. They are true for each \$1 of CF. For other values of CF multiply the initial CF by the relevant factor.

11. Periodic Payments and Repeating Cycles

Go back to Equation (11); solve it for a/i :

$$a/i = \frac{P}{1 - \tau^{-n}} \quad (23)$$

" a " is the n -year-long annuity corresponding to P , i.e. having a PV of P . Dividing a by i "capitalizes" it, i.e. gives the PV of a perpetual annuity of a . (Cf. (7), supra.)

Thus the right side of (23) is the present value of a perpetual succession of payments of value P , received every n years. Note how this jibes with our earlier look at the Land Value Multiplier, which we previously derived in two different ways. This is restated in §12, following.

12. Positive NPV and land values.

Net Present Value, NPV, means the present discounted value of future cash flows less the capital costs (K_0) at time zero that make these cash flows happen. What happens if investments yield positive NPVs regularly and routinely? That sounds too good to be true. If it were true, anyone could build new capital and sell it immediately for more than cost, quickly growing infinitely rich.

What prevents it is there is an invisible input that makes these favorable results possible. That input is land. Land is not really invisible, but often unobserved. ("My dear Watson, you see, but you do not observe.")

What happens is, the possibility of getting positive NPVs creates demand for the land on which such favorable results are possible. The selling price of land rises to soak up the surplus.

Land lasts forever, so its value soaks up the surplus from all future cycles, not just the first. The NPV of the first "Generation" or cycle of investment and recovery is dated at time zero, the present. The NPV of the second cycle is dated at the end of the first one; of the third cycle at the end of the second, and so on. From these dates each should be discounted back to time zero and added to the NPV of the first cycle.

Performing this summation we get Land Value (B). "B" is from the German Bodenwerte, or site value. Why German? There are four reasons. One, we also use the German K for Kapital. Two, Bodenwerte is already standard usage in forestry, where the math was originally worked out by Martin Faustmann in 1849. Three, I want to reserve G (as in Ground) to mean Generation. Four, I want to reserve T (as in Terra) to mean Taxes.

$$B = NPV [1 + I^{-n} + I^{-2n} + \dots + I^{-\infty}] \quad (23A)$$

$$= NPV \frac{1}{1 - I^{-n}}$$

(23A) consists of NPV of Generation #1 times the rest of the right side. The latter is the reciprocal of the denominator. Call this reciprocal the "Land Value Multiplier" (LVM). To find land value, multiply NPV of Generation #1 by LVM. LVM is a coefficient greater than one which shows how much greater land value (B) is than the NPV of Generation #1.

$$LVM = \frac{1}{1 - I^{-n}} \quad (24)$$

For values of LVM, see Table 6.

Table 6: Land Value Multipliers

i=	.02	.04	.08	.12
n				
1	51.00	26.00	13.50	9.33
5	10.61	5.62	3.13	2.31
10	5.57	3.08	1.86	1.47
20	3.06	1.84	1.27	1.12
40	1.83	1.26	1.05	1.01

To find Land Value, multiply NPV of Generation #1 by the appropriate figure.

To adjust for anticipated inflation, the easy way is to use a real interest rate. That means subtract the inflation rate from the nominal interest rate. Thus, .04 is more realistic than .12.

In the absence of general inflation, there is often a real inflation of land rents, even in constant dollars. To adjust for that, subtract the real inflation from the .04. Thus, .02 might be most appropriate. Note how this raises the LVM.

13. Maturity and Replacement

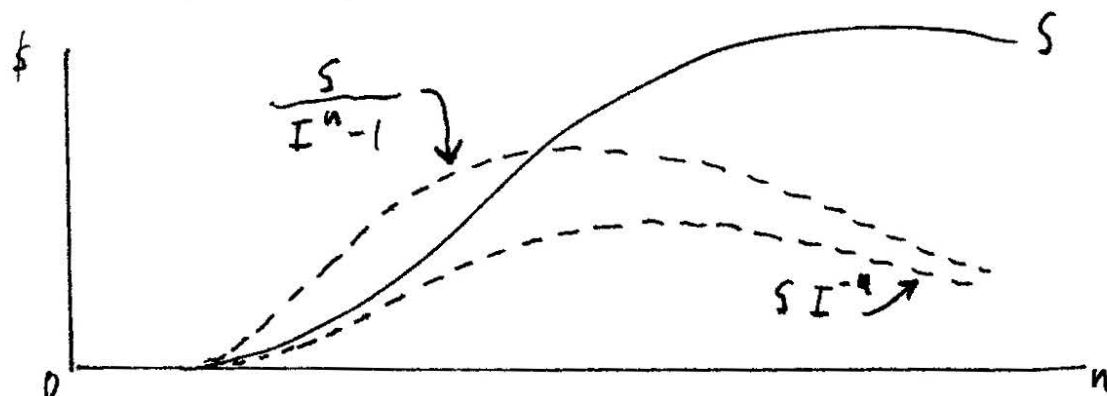


Figure 7: Growth of Stumpage, Discounted Stumpage, and Land Value

S = Stumpage, meaning the market value of a tree "on the stump" before cutting.

Different people have lined up behind at least four alternative maximizing solutions to the question, "When to cut the tree?" This is more than a theory question: the parties also actually manage timber in the four different ways, and it makes years of difference.

To keep this simple assume $K_0 = 0$.

Here are the four alternatives:

A. Max S . You have to be a pretty deep ecologist to go for this. It ignores the economic costs of time, both interest and ground-rent.

B. Max S/n . S/n is aka Mean Annual Increment (MAI).

C. Max $S I^{-n}$, that is the PV of Generation #1 (G_1).

$$D. \text{ Max } B = \frac{S I^{-n}}{1 - I^{-n}} \quad (25)$$

(25) is the sum of the NPVs of all future generations, aka Land Value, which we are calling Bodenwerte (B).

(25) in the form above shows its relationship to the formula of (C). By clearing the fraction, however, we write it more simply, in (25A) below:

$$B = \frac{S}{I^n - 1} \quad (25A)$$

(25A) is a curve shown in Fig. 7.

The economic solution is (D), same as max

(25A), aka Faustmann's Solution. This solution advances the date of maturity, compared to the other three maximizing solutions. (Solutions to what? Remember the problem: "When to cut a tree?"

14. The marginal meaning of maximizing solutions.

Every maximizing "solution" or performance standard is also a marginal criterion. Let us see what marginal conditions correspond with each of the four maximizing solutions.

A) $S'=0$ (S' is shorthand for dS/dn , or annual growth)

B) $S'/S=1/n$

C) $S'/S=i$

D) $S'/S = \log$ derivative of the denominator of (25A). That denominator is $(I^n - 1)$.

Its log derivative is its derivative divided by itself, and that is approximately:

$$\frac{i I^n}{I^n - 1} = \frac{i}{1 - I^{-n}} > i \quad (26)$$

(26) is approximate because we simplify it by using the useful approximation that $\ln I = i$.

(26) is the same as the Installment Plan Factor (IPF). Figure 8 shows its relationship to n and i .

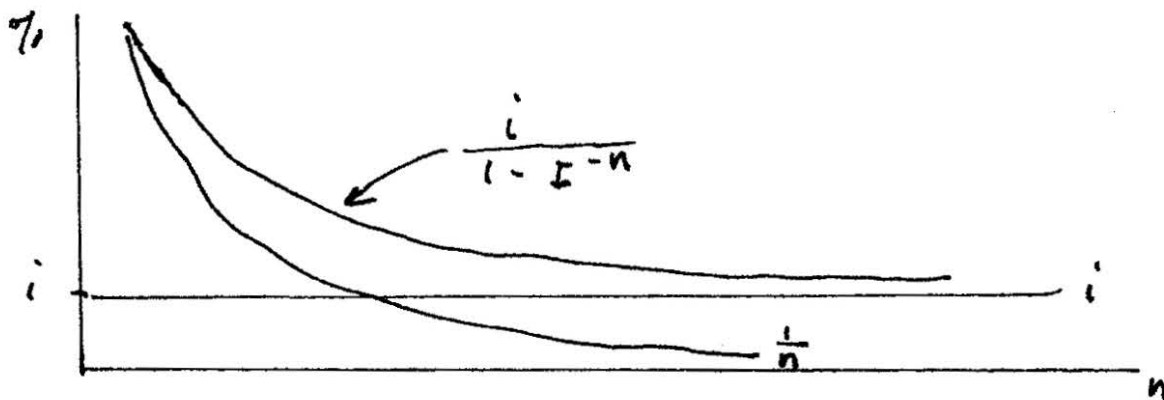


Figure 8: Marginal Costs of Time, Expressed as Percentages

Figure 8 shows how the true marginal cost of time (TMCT) relates to i and to $1/n$.

-- TMCT is always higher than either i or $1/n$.

-- For low n , TMCT $\rightarrow 1/n$

-- For high n , TMCT $\rightarrow i$

To use Figure 8 to find cutting time, overlay one more curve, the curve of S'/S . This curve is negatively sloped, i.e. it falls as n rises. The optimal time to cut a tree is when the S'/S curve crosses the TMCT curve.

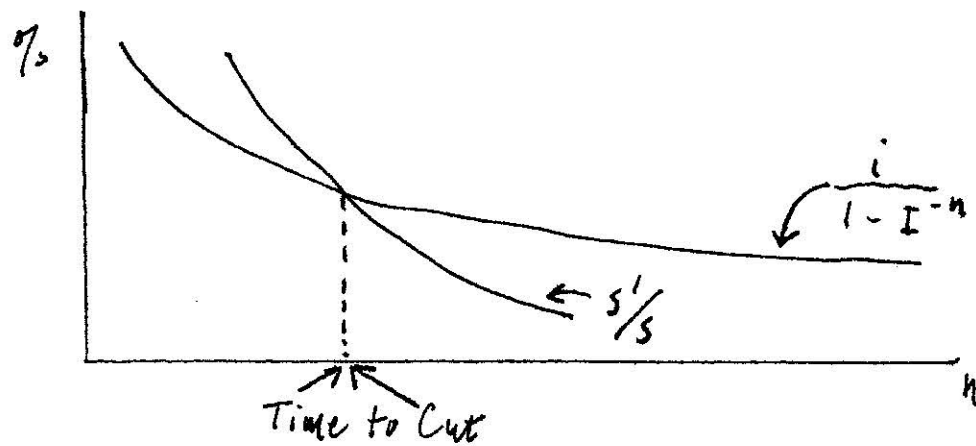


Fig. 9: Optimal Harvest Time

INTRODUCTION TO FINANCE FOR MANAGERIAL ECONOMICS

INSTALLMENT #4

Ec. 162A, F84, M.G.

Marginal solution D to the tree harvesting problem has an intuitive interpretation that makes it easier to remember. We will show that it says simply that the marginal cost of time is $i(S+B)$. That is, tree growth must be fast enough to pay interest on both Stumpage and Land Value.

Let us begin by looking at simple-minded Marginal Solution C. C is popular because its intuitive interpretation is so simple, so easy for economists to grasp. It says the marginal cost of time is just iS , interest on Stumpage alone. The trouble with that is, it assumes the land is free, and treats it that way in the last few years (as well as in the early years, when there should be an economic force prodding owners to reforest cleared land quickly -- but that is another story).

Rewriting the log derivative equation of D, above, the marginal condition is:

$$S' = Si \frac{1}{1 - I^{-n}} > Si \quad (26)$$

The purpose of (26) is to factor out Si , in order to identify the other element on which solution D requires the tree to earn interest. We know there is another element, because S' in (26) is greater than Si . But we cannot immediately see what that element is, because the information comes to us in the denominator of a ratio, rather than as something we add.

Method: add and subtract I^{-n} to and from the numerator, letting us then factor out 1.

$$S' = Si \frac{1 - I^{-n} + I^{-n}}{1 - I^{-n}} = Si \left[1 + \frac{I^{-n}}{1 - I^{-n}} \right] = i \left[S + \frac{S}{I^n - 1} \right] \quad (27)$$

But $\frac{S}{I^n - 1}$, the last term, is Land Value, B.

So Solution D boils down to this:

$$S' = i(S+B) \quad (28)$$

The left side of (28), S' , is the marginal product of time - the yearly growth of S .

The right side of (28), $i(S+L)$, is the marginal cost of time -- the foregone gain of tying up the Stumpage and the Land for another year. Each of these has an opportunity cost. The Stumpage could be liquidated for cash and reinvested. The Land could be sold to another investor and the cash reinvested. From a social viewpoint the land could not be liquidated, but it still has an opportunity cost: it could be cleared and put to use growing the next crop. Its annual rent would be a , and $a=Si$.