

Different effects of Δi and Δt on bldg. (I)

A difference between prop. tax & i.r. is that you can stop the tax by destroying the property. This could rarely happen, e.g. with attractive plantings.

A good rule is not to assess & tax the public face of priv. property. Thus push it into sewing neighbors.

Used

Sheaf LP - 1

~~Good for Appendix~~

1

I deem to use in Lyon paper

1. You can use lots of capt. on land, & be labor-int. if capt. is recovered quickly. This is best of all, in fact.

a tomato grower, or berry grower

a bakery, or food mill,

a factory

Capt. is inventories, turning fast, complements labor a great deal & hardly subs. for it at all.

2. $I_{int.} = 1 - \frac{R \cdot C}{R} \frac{m}{e^{m-1}}$ may overstate intensity badly, bcs. it values a internally. i.e. it uses an income approach to land valuation, rather than an o.c. or bldg. &c. - resid. or land-residual concept, method.

Better is to ~~define land~~ begin w. land-intensity,
 $LF = \frac{\sum_1^m a}{FVR_m}$ where a is o.c. ~~This is same as~~

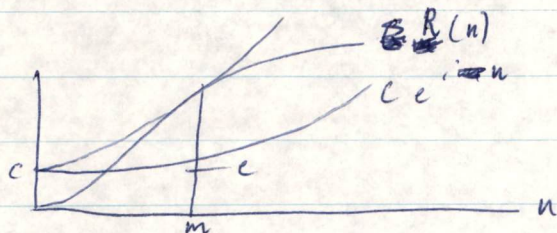
~~other def~~

Then $I = 1 - \frac{\sum_1^m a}{FVR}$. This is same as other def. when internal a = o.c. ~~But~~ But when o.c. > ~~int~~ internal a, then this def gives lower I.

~~By the way~~ When I show how I derive I, it is evident why these are identical. $I = \frac{e^{im} + 5(e^{im-1}) - ma}{f_m}$

See #3.

3. Deriving $I = 1 - \frac{R-C}{R} \frac{mi}{e^{im}-1}$



$$I = \frac{ce^{im}}{R(n)} = \frac{ce^{im}}{ce^{im} + S(e^{im}-1)} = \frac{c}{c + S(1-e^{-im})} = \frac{c_0}{PV_0 R(n)} = \frac{c}{R}$$

Not quite. Above understates I by treating int. on deferred rent as a land input, when it is a capital input. (alho it is a cap. input uniquely derived from land ownership, giving it some special distributive position.)

To add to numerator the interest earned by deferred rent, a. That is the sum of ydy rent, a, plus interest:

Less the sum of rent: ma

Now,

$$I = \frac{ce^{im} + a \frac{e^{im}-1}{i} - ma}{R(n)}$$

But $R(n) = ce^{im} + a \frac{e^{im}-1}{i}$

So $I = 1 - \frac{ma}{R(n)}$, i.e. the ~~complement~~ complement of land share

We can formalize this by using $a = (R(n) - ce^{im}) \frac{i}{e^{im}-1}$

$$I = 1 - \frac{m}{R(m)} (R_0^m - C e^{im}) \frac{i}{e^{im} - 1} = 1 - \frac{R_0 - C}{R_0} \frac{e^{im} - 1}{e^{im} - 1}$$

This formula not limited to PIPD. R_0 means PVR, all revs. at all times.

$$\sum [R_n e^{(m-n)i}]$$

(See i & t file for more detail)

4. $Q > \frac{C}{R}$, and the higher m , the greater the excess.

So $\frac{C}{R}$ as a measure of intensity understates intensity of long cycles relative to short. Same time, it overstates labor's share in the labor-capl. mix.

5. A low labor share when cycles are short is pure land effect. As rich farm land, where landlord gets half of yty. cycle.

But that is less common way of reducing labor's share. More common is the long cycle, like timber, or mining, or p.w. Here rent plus rent-derived capital add most of the value.

✓ "Rent-derived capl." is capl., but not a product of labor. So it is distinguishable from labor-derived capl. It does not ~~create~~ ^{create} capl.

- ✓✓✓ 6. Rent-derived capt. is always implicitly expensed.
But labor input into durable capt. is usually capt., for inc. tax.
 Owner's labor input is also implicitly expensed.

7. Sometimes there is no unambiguous value for m , as when bldg. is remodeled or restored often.

Then, select most reasonable value for m , and evaluate I for that m .

C will include remodeling costs, disc.

R " " resulting higher revs, plus final salvage value at m

8. Labor-derived capt. is created by actively employing men, and deliberately choosing to invest rather than consume.

Land-derived capt. is created passively, by abstaining from putting land to more active use.

✓ Taxing land-derived capt. $\therefore$ has some of the qualities of taxing land itself.

Labor-derived capt. is the lion's share in early yrs. of a cycle. Land-derived capt. is very small. It piles up in later yrs.

✓ Ratio of labor-derived to land-derived capt.
$$\frac{ce^{im}}{s(e^{im}-1)}$$

$$= \frac{e}{s(1-e^{-im})}$$
, a decreasing fn. of m .

9. $\psi = \frac{i_m}{e^{i_m} - 1}$ is the % of the "surplus" that is rent.

10. When, ^{in defining intensity} we avoid land-residual method, then what is residual?

It has to be management, the responsible agency.

Or, internal rate of return on owned assets - a good compromise, bes. you can let S be hard, as well as C , and S is known in advance, while i is disclosed by ex post.

$$\text{Now, let } I = 1 - \frac{m \cdot S \cdot i}{\sum [R_n e^{i(m-n)}]} \quad (10)$$

Here, if m drags on too long, i falls. Denom. is more sensitive to i , so it falls more than num. (% wise), and I falls. Indicating that long m makes mix more land-rich.

But why shldn't capt. be awarded its o.c., too?
For one thing, this wd. let all users have $I=1$ simply by arbitrarily evaluating I over $m=\infty$

More thought needed - or fix this one.

Proble, fallacy is to assume sum of land-int. plus non-land int. = 1. When revs. full share of covering inputs, then the sum > 1 . This follows when nothing is valued internally; all o.c.s. are hard. To the enterp. responsible, something is always soft. It is the

outside observer who makes this judgment.

Intensity shd. be neutral as to gm. of what inputs are soft & hard. So take them all as hard, and presume an output yielding o.c. returns to all. Now, Int. is a ratio of inputs

- ✓ 11. c_n shd. be added to C_0 , rather than deducted from R_n , for full picture of intensity.
Re trees, esp c_m , harvest labor, is very large.

✓ 12. Labor-share = $\frac{c}{FVR} = \frac{c}{ce^{im} + S(e^{im} - 1)}$
C

a. If $S=0$, labor-share (λ) unaffected by c

b. λ rises with:

- c/s
- fall of m
- fall of i
-

c. λ is more sensitive to m when S/c is large, and when m is small.

i.e., if opn. is land-heavy, then added time cuts labor share way down, e.g. going from one year to two.

Say, going from double cropping to single. Or from annual crop to biennial. (even tho capt. is tied up for one yr. land tied up for 2.)

Whenever cycles are short, say under 20 yrs., labor/land falls badly when m rises. Examples:

- mining (?)
- vines, alfalfa, perennials, etc. Cattle, sheep.
- Any farm capt. that gets away from short cycles.
- inventories, incl. & comm.

As they turn faster they Δ labor/land, sharply.
(Why don't they Δ labor/capt. just as much?)

~~Return to~~ ~~to~~

They do, but the ratios are different. That is, when m is low, capt. input has to be low relative to C ; but land input can be high rel. to C , e.g. $a = C$.

The capt. input is $C(e^{im} - 1)$. Rewrite ~~it~~ so:

$$\text{Labor share} = \frac{C}{\underbrace{C}_{\text{Labor}} + \underbrace{C(e^{im} - 1)}_{\text{Cap.}} + \underbrace{S(e^{im} - 1)}_{\text{Land}}}$$

So the real point here is that when ~~the~~ cycles are short, capt. input is small, so even by cutting it in half there is little gain in labor's share, ~~or~~ as a % But land's share may now be very large, and ~~there~~ there is now scope for further large Δ in labor's share by shortening cycles to very small periods, by cutting land's share in half.

E.g.

<u>Period, yrs.</u>	<u>Labor₀</u>	<u>Rent (a)</u>	<u>FVP @ 5%</u>
.18 4	100	100	$122 + \frac{100}{.05} \times .22 = 122 + 440 = 562$
.32 2	100	100	$110 + \frac{100}{.05} \times .10 = 110 + 200 = 310$
.49 1	100	100	$105 + \frac{100}{.05} \times .05 = 105 + 100 = 205$



Labor Share

(By shorter cycles, labor gains not bcs land gets less, but there is more gross yield.)

Putting pressure on capl. can only help labor up to a point. When cycles get short, it calls for pressure on land to keep on helping labor.

✓ Pressure on land also helps when cycles are long, by helping appreciate how much capl. is tied up. The inputs in a tree are not just capl. but rent, and the rent is compounded.

13. When $I = 1 - \frac{\sum a}{FVR}$, then in every yr., near end of life,

when $R_n < a$, it reduces I .

Excessive life lowers I if a is o.c. & i is residual
But " " raises I if i is o.c. & a is " "

14. A bldg. of long life may complement labor greatly, despite its own slowness, by abetting labor inside it.
Factories; offices.

So may vines & orchards.

But trees & cattle hardly do.

15. C_0 is capt. "investment", but not capt. input, or capt. cost.
An invt. is not a cost.

✓ So stop using C_0 . Let it be W , wages

16. Hi i makes people cut trees sooner, but not demolish bldgs. sooner. Bcs. hi i lowers land rent. Shorter ec. life, longer carcass life.

Hi i means bldgs. are built with lower C_0 ... & less often.
And w. less floor space for labor to work in.
So for high t on bldgs.

i to help labor, make it costlier to hold land, rather than capt.

17. Bldgs., like trees, (may) store up early rents for later realization.
 Carry reduced rev. in early yrs. to get more later, invest not just stored labor, but land.

E.g. - loss of ~~in~~ cash flow or service during constrn.
 - thick walls, to last, reduce floor space
 - loss of versatility, choice

Thus in my formulae re bldgs, land input is treated same as for trees.

✓ 18. "Full demand" of a site w. a high dam, or tall bldg., may get into increments of capt. that are submarginal now, in static sense, & are recognized as such, but are in antic. of future Δ demand. (Like 4065 Lockhaven)

In early years, receipts fail to cover int. & rent. These are then stored up in bldg - the deficits

This is one aspect of longevity. — ~~& also of~~ This kind of intensity is not good for labor at all.

19. Higher perceived h.c. for land, faster recycling of capt., several e.g.s.
 a. Mines. Higher ore value, higher urge to recover surplus, as well as capt.

b. Trees. Higher perceived a means more capt. in trees. Hence higher urge to cut — altho this will mainly affect ex ante choices, add to perceived cost of long cycles.

In s.r., tho, higher a adds to cost at mgn.

c. Bldgs. Earlier dems. And probly early recovery of capt., i.e. cash flow $> i + a$. Probly that is needed to realize a.

20. Slow turnover of inventories & warehouse needs.
Slow return of capt. in bldgs & road needs -- & utility needs.

(Low load factor also & warehouse needs)

Re PV of property taxes on bldgs. Cfs. SF. vs SL deprecn. methods.

Very Good, Save. All the way through

$$B_n = \text{bldg value yr } n = \text{tax base}$$

Assume const. cf, hence SF deprecn. (Temporarily only. This is an easy way to forget the facts.)

$$B_n = cf \cdot \frac{1 - e^{-i(m-n)}}{i} = cf \frac{1 - e^{-in} \cdot e^{-im}}{i}$$

$$cf = B_0 \frac{i}{1 - e^{-im}}$$

$$B_n = B_0 \frac{1 - e^{-i(m-n)}}{1 - e^{-im}} = B_0 \frac{1 - e^{-im} e^{-ni}}{1 - e^{-im}}$$

$$T_n = t B_n$$

$$PVT_{\#} = \sum (t B_n e^{-in}) = t B_0 \sum_1^m \frac{1 - e^{-i(m-n)}}{1 - e^{-im}} e^{-in} = t B_0 \sum_1^m \frac{e^{-in} - e^{-im}}{1 - e^{-im}}$$

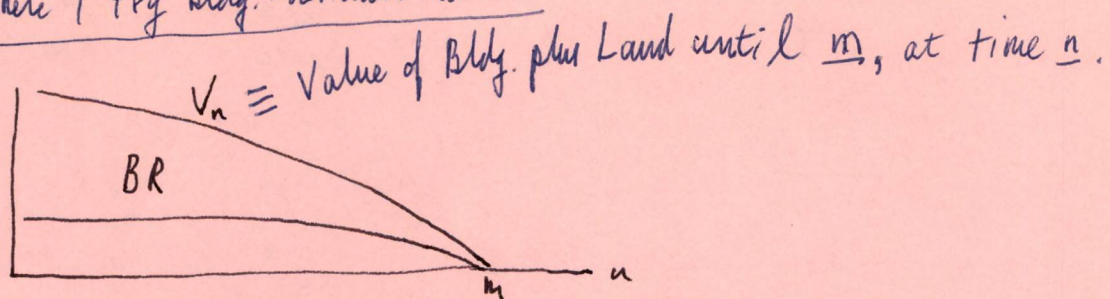
$$= \frac{t B_0}{1 - e^{-im}} \left[\sum_{n=1}^m e^{-in} - \sum_{n=1}^m e^{-im} \right] = \frac{t B}{1 - e^{-im}} \left[\frac{1 - e^{-im}}{i} - m e^{-im} \right]$$

$$= \frac{t B_0}{i} \left[\frac{1}{e^{im} - 1} - m \right]$$

Site value $\equiv \hat{S} = PVR - C_0 - t C \left[\frac{1}{i} - \frac{m}{e^{im} - 1} \right] + S_m e^{-im}$

Caution: B_n is defined above using land-residual method, implicitly, or no-land method. This could be misleading. It, along w. SF. method, understates deprecn.

So here I try bldg. - residual method.



BR = Bldg. Residual

$$BR_n = V_n - a \frac{1 - e^{-i(m-n)}}{i} = V_n - S + S e^{-i(m-n)}$$

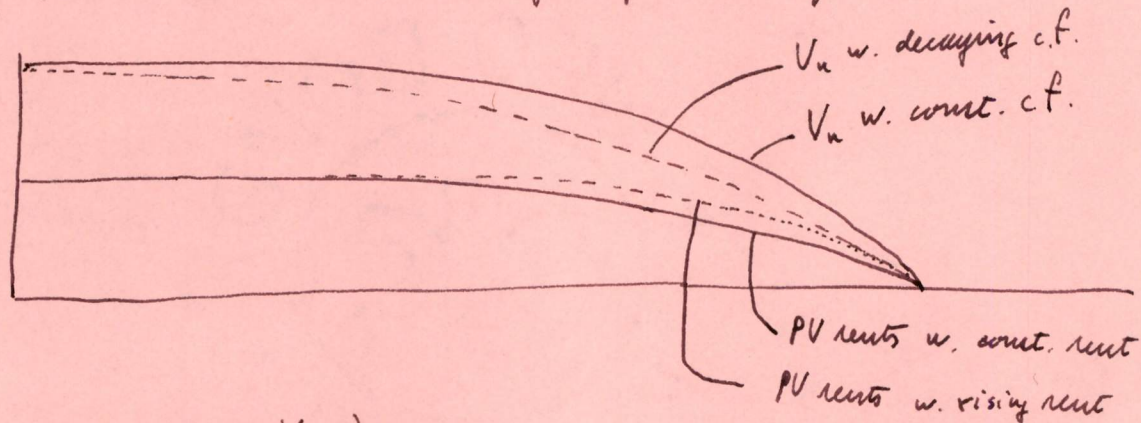
Let's see how bldg. resid. method affects deprecn. when cf is constant.

Note: V_n is value derived from exist'g bldg. Mkt. value of land & bldg. = $V_n + S_m e^{-i(m-n)}$

$$V_n = cf \frac{1 - e^{-i(m-n)}}{i} - a \frac{1 - e^{-i(m-n)}}{i}$$

$$= (cf - a) \frac{1 - e^{-i(m-n)}}{i}$$

So st pattern of deprecn. remains unchanged, merely using smaller annuity. That wd. be unusual. Normally, cash flow decays and land rent rises, so bldg. depreciates faster:



$$V_n = cf \frac{1 - e^{-(i+d)(m-n)}}{i+d} - a \frac{1 - e^{-(i-g)(m-n)}}{i-g}$$

or

$$V_n = cf(1-d)^n \frac{1 - e^{-i(m-n)}}{i} - a(1+g)^n \frac{1 - e^{-i(m-n)}}{i}$$

m	$\frac{e^{im}}{e^{im}-1}$	$\frac{e^{im}-1}{i}$	$\frac{m}{e^{im}-1}$	$\frac{1}{i} - \frac{m}{e^{im}-1}$
1	$1+i$	i	$\frac{1}{i}$	0

2	$1+2i+i^2$	$2i+i^2$	$\frac{1}{i+i^2/2} < \frac{1}{i}$	> 0
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10

20

∞

0

$\frac{1}{i}$

a simple expression for ~~making bldg. deprec. factor~~ ~~taking these factors into acct.~~ is
 stat. time deprecn. (Altho it shd rarely be used, since it is so seductively easy a
 way to slip into land-resid. method.)

$$B_n = B_0 - \frac{B_0}{m} \cdot n = B_0 \left(1 - \frac{n}{m}\right) = B_0 \frac{m-n}{m} = \frac{B_0}{m} (m-n)$$

$$t_n B_n = t B_0 \left(1 - \frac{n}{m}\right)$$

$$PV \Sigma T_n = t B_0 \sum_{n=1}^m \left(1 - \frac{n}{m}\right) e^{-in} = t B_0 \left[\frac{1 - e^{-im}}{i} - \frac{1}{m} \sum_{n=1}^m \frac{n}{e^{in}} \right] \quad (1)$$

~~This assumes a given value for m.~~

~~$$\sum_{n=1}^m \frac{n}{e^{in}} = \frac{1}{e^i} + \frac{2}{e^{2i}} + \frac{3}{e^{3i}} + \dots + \frac{m}{e^{mi}}$$~~

~~| n | $\frac{n+1}{n}$ | $= n+1$ |
|-----|-----------------|---------|
| 1 | $\frac{2}{1}$ | $= 2$ |
| 2 | $\frac{3}{2}$ | $= 3$ |
| 3 | $\frac{4}{3}$ | $= 4$ |
| 4 | $\frac{5}{4}$ | $= 5$ |~~

~~$$r = \frac{\frac{n+1}{n}}{e^i}$$~~

~~$$\Sigma = \frac{1}{e^i} \frac{1 - \left(\frac{\frac{n+1}{n}}{e^i}\right)^m}{1 - \frac{\frac{n+1}{n}}{e^i}}$$~~

But n is not constant,
 so Σ can't be valued.

$$\sum_{i=1}^m \frac{n}{e^{in}} = \frac{1}{e^i} + \frac{1+1}{e^{2i}} + \frac{1+1+1}{e^{3i}} + \dots + \frac{m}{e^{mi}}$$

$$= \left(\frac{1}{e^i} + \frac{1}{e^{2i}} + \dots + \frac{1}{e^{mi}} \right) + \left(\frac{1}{e^{2i}} + \frac{1}{e^{3i}} + \dots + \frac{1}{e^{mi}} \right) + \dots + \left(\frac{1}{e^{mi}} \right)$$

$$= \sum_{i=1}^m \frac{1}{e^{ni}} + \sum_{i=2}^m \frac{1}{e^{ni}} + \sum_{i=3}^m \frac{1}{e^{ni}} + \dots + \sum_{i=m-1}^m \frac{1}{e^{ni}}$$

$$= \frac{1}{e^i} \frac{1 - e^{-i(m-1)}}{i} + \frac{1}{e^{2i}} \frac{1 - e^{-i(m-2)}}{i} + \dots + \frac{1}{e^{mi}} \frac{1 - e^{-i(m-m)}}{i}$$

$$= \frac{1}{i} \left[\sum_{i=1}^m 1 - \sum_{i=1}^m e^{-i(m-n)} \right]$$

$$= \frac{1}{i} \left[m - e^{-im} \sum_{i=1}^m e^{in} \right]$$

$$= \frac{1}{i} \left[m - e^{-im} \frac{e^{im} - 1}{i} \right] = \frac{1}{i} \left[m - \frac{1 - e^{-im}}{i} \right] \quad (2)$$

Subbing (2) in (1)

$$PV \sum_{i=1}^m \frac{1}{e^{in}} = +B_0 \left[\frac{1 - e^{-im}}{i} - \frac{1}{i} \left(1 - \frac{1 - e^{-im}}{mi} \right) \right] \quad (3)$$

~~$e^i - e^{-im}$~~

$$= \frac{e^{-i} - e^{-im}}{i} + \frac{e^{-2i} - e^{-im}}{i} + \dots + \frac{e^{-mi} - e^{-im}}{i}$$

$$= \frac{1}{i} \left[\sum_{i=1}^m e^{-ni} - \sum_{i=1}^m e^{-im} \right]$$

$$= \frac{1}{i} \left[\frac{1 - e^{-im}}{i} - m e^{-im} \right]$$

(2)
(See attached letter from Mutt for exceptions.)

Subbing (2) in (1)

$$PV \sum_{i=1}^m T_n = t B_0 \left[\frac{1 - e^{-im}}{i} - \frac{1}{i} \left(\frac{1 - e^{-im}}{mi} - e^{-im} \right) \right] \quad (3)$$

$$= t B_0 \left[1 - \frac{1 - e^{-im}}{mi} \right]$$

(When $m=0$, ~~a nonsense case~~ series begins with $n=1$)

$$PV \sum_{i=1}^m T_n = t B_0 \left[\frac{1}{i} - \frac{1}{i} \left(\frac{0}{0} - 1 \right) \right] = t B \frac{1}{i} \quad ?$$

When $m = \infty$

$$PV \sum_{i=1}^m T_n = t B_0 \left[\frac{1}{i} - \frac{1}{i} \left(\frac{1}{\infty} - 0 \right) \right] = t B_0 \frac{1}{i} \quad \text{As it should.}$$

When $m = 1$

$$PV \sum_{i=1}^m T_n = t B_0 \left[\frac{1}{e^i} - \frac{1}{i} \left(\frac{1}{e^i} - \frac{1}{e^i} \right) \right] = t B_0 \frac{1}{e^i} \quad \text{as it should}$$

Incidental insight.

$$\sum_{n=1}^m \frac{n}{e^{in}} = \frac{1}{i} \left[\frac{1 - e^{-im}}{i} - \frac{m}{e^{im}} \right]$$

When $m = \infty$,

$$\sum_{n=1}^{\infty} \frac{n}{e^{in}} = \frac{1}{i} \left(\frac{1}{i} - 0 \right) = \frac{1}{i^2}$$

This interesting summation is also regroupable as:

$$\sum_{n=1}^m \frac{n}{e^{in}} = \frac{e^{im} - (1 + mi)}{e^{im} \cdot i^2}$$

Top is excess of compound over simple interest.

$$\int u dv = uv - \int v du$$

$$u = n$$

$$du = dn$$

$$dv = \frac{n}{e^{in}}$$

$$v = \frac{e^{-in}}{-i}$$

$$\int = \frac{n}{e^{in}} - \int n$$

You can also reach this by integrating, using rule above plus:

$$\int n e^{in} dn = \frac{e^{in}}{i^2} (in - 1)$$

But it takes some doing.

With this faster deprec. (SL), S is less affected by t

T is less; and $\frac{dT}{dt}$ is less, ($T \equiv \sum PVT_n$)

PVT with SL is how much less? Take ratio

$$\frac{SL}{\text{Sly. Fund}} = \frac{\frac{e^{im} - 1 - mi}{e^{im} \cdot i^2}}{\frac{e^{im} - 1 - mi}{i(e^{im} - 1)}} = \frac{e^{im} - 1}{e^{im} \cdot i} = \frac{1 - e^{-im}}{i}$$

$$\frac{SL}{SF} = \frac{1 - e^{-im}}{i} - \frac{1}{i}$$

$$\frac{e^{im} - 1}{e^{im} \cdot mi} - \frac{mi}{e^{im} \cdot mi}$$

$$\frac{PV\&T, SL}{PV\&T, SF} = \frac{\frac{1}{i} \left[1 - \frac{1 - e^{-im}}{mi} \right]}{\frac{1}{i} \left[1 - \frac{mi}{e^{im} - 1} \right]}$$

$$= \frac{\frac{mi - 1 + e^{-im}}{mi}}{\frac{e^{im} - 1 - mi}{e^{im} - 1}} = \frac{e^{im} \cdot mi - e^{im} + 1 - mi + 1 - e^{-im}}{e^{im} \cdot mi - mi - m^2 i^2}$$

$$= \frac{mi(e^{im} - 1) - (e^{im} - 1) - (e^{-im} - 1)}{mi(e^{im} - 1) - m^2 i^2}$$

Try subtracting instead, get the difference

$$\begin{aligned} SF - SL &= \frac{1}{i} \left[1 - \frac{mi}{e^{im} - 1} - 1 + \frac{1 - e^{-im}}{mi} \right] = \frac{1}{i} \left[\frac{1 - e^{-im}}{mi} - \frac{mi}{e^{im} - 1} \right] \\ &= \frac{1}{i} \left[\frac{e^{im} - 1}{e^{im} \cdot mi} - \frac{mi}{e^{im} - 1} \right] = \frac{1}{i} \left[\frac{(e^{im} - 1)^2 - e^{im} m^2 i^2}{e^{im} \cdot mi (e^{im} - 1)} \right] \end{aligned}$$

$$= \frac{1}{i} \frac{e^{im} - 1 - 1 + e^{-im} - m^2 i^2}{(e^{im} - 1)^2}$$

SF > SL if

$$e^{im} + e^{-im} - 2 > m^2 i^2$$

$$m=1, \quad 1+i + \frac{1}{1+i} - 2 > \text{fig } i^2$$

To p. 8, bottom

$$S_L = \frac{1 - e^{-im}}{i} - \frac{1}{mi} \left[\frac{1 - e^{-im}}{i} - me^{-im} \right]$$

$$SF \quad \frac{1}{i} = \frac{m}{e^{im} - 1}$$

$$= \frac{1 - e^{-im}}{i} - \frac{1}{mi} \left[\frac{e^{im} - 1 - mi}{i e^{im}} \right]$$

$$\frac{e^{im} - 1 - mi}{i(e^{im} - 1)}$$

$$= \frac{\frac{e^{im} - 1}{e^{im}}}{\frac{e^{im} - 1 - mi}{e^{im} - 1}} - \frac{1}{mi} \frac{\cancel{e^{im}} - 1}{\cancel{e^{im}}}$$

$$= \frac{(e^{im} - 1)^2}{e^{im} - e^{im} - mi e^{im}} - \frac{1}{m} \frac{1 - e^{-im}}{i}$$

$$= \frac{(e^{im} - 1)^2}{e^{im}(e^{im} - 1 - mi)} - \frac{(e^{im} - 1)}{e^{im}(mi)}$$

$$= \frac{\cancel{e^{im}}(mi)(e^{im} - 1)^2 - \cancel{e^{im}}(e^{im} - 1 - mi)(e^{im} - 1)}{e^{im} mi (e^{im} - 1 - mi)}$$

$$= \frac{(e^{im} - 1)[mi e^{im} - \cancel{e^{im}} - e^{im} + 1 + mi]}{(e^{im} - 1 - mi) mi e^{im}} = \frac{(e^{im} - 1)(mi - 1 + e^{-im})}{(e^{im} - 1 - mi) mi}$$

$$= \frac{e^{im} mi - e^{im} + 1 - mi + 1 - e^{-im}}{(e^{im} - 1 - mi) mi}$$

From p. 7b

$$\frac{1 + 2i + i^2 + 1}{1+i} = \frac{2(1+i) + i^2}{1+i} = 2 + \frac{i^2}{1+i}$$

$$1+i + \frac{1}{1+i} - 2 = \frac{(i-1)(i+1) + 1}{1+i} = \frac{i^2}{1+i} < i^2$$

So $S_F < S_L$ when $m=1$. Odd

PVΣT

1. Using SF deprecn

$$PV\Sigma T = \frac{tB_0}{i} \left[\frac{1}{i} - \frac{m}{e^{im} - 1} \right] \quad (\text{from p.1})$$

when $m=1$, $PV\Sigma T = 0$

$$m=2, PV\Sigma T = \frac{tB_0}{i} \left[\frac{1}{i} - \frac{2}{2i + i^2} \right] \quad \text{but} = \frac{tB_0}{i} \left[\frac{1}{i} - \frac{1}{i + i^2/2} \right]$$

$$= \frac{tB_0}{i} \left[\frac{i + i^2/2 - 1}{i(i + i^2/2)} \right] = \frac{tB_0}{i} \frac{i/2}{i + i^2/2} = \frac{tB_0}{i} \frac{1/2}{1 + i/2} = \frac{tB_0}{2 + i}$$

Hardly seems possible. Double check p.1, SF formula.

No, it's ok. When $m=2$, you tax it once, when $B_n \approx \frac{1}{2} B_0$.

2. Using SL deprecn.

$$PV\Sigma T = \frac{tB_0}{i} \left[1 - \frac{1 - e^{-im}}{mi} \right] \quad (\text{from p.5})$$

when $m=1$,

$$PV\Sigma T = \frac{tB_0}{i} \left(1 - \frac{1 - e^{-i}}{i} \right) = \frac{tB_0}{i} \frac{e^{-i}}{1 + i}$$

My 2 fns seem to begin with incompatible first terms.

When $m = 20$, $i = .01$,

$$SF_i[\] = \frac{1}{\cancel{1.01}} - \frac{20 \times .01}{3} = 1 - \frac{1.4}{3} = 1 - .46\bar{3} = .53\bar{3}$$

$$SL_i[\] = 1 - \frac{1 - \frac{1}{4}}{1.4} = 1 - \frac{3}{5.6} = 1 - .54 = .46$$

So PVET is lower under SL method, but not by much.
But the leverage effect on S is much greater, if $\frac{c}{E} \rightarrow 1$.

$$\frac{\frac{e^{im} - 1 - i + e^{-im} - m^2 i^2}{mi(e^{im} - 1)}}{\frac{e^{im} - 1 - mi}{\frac{e^{im}}{e^{im} - 1}}} =$$

$$\begin{array}{r} .53\bar{6} \\ 56 \overline{) 30.0} \\ \underline{280} \\ 200 \\ \underline{168} \\ 320 \end{array}$$

Present Value of Property Taxes on Building.

A. Sinking Fund Depreciation assumed

$$B_n = B_0 \frac{1 - e^{-i(m-n)}}{1 - e^{-im}}$$

$$PVT_A = t B_0 \left[\frac{1}{i} - \frac{m}{e^{im} - 1} \right] = B_0 \frac{t}{i} \left[1 - \frac{mi}{e^{im} - 1} \right]$$

B. Str. Line Deprec. Assumed

$$B_n = B_0 \left(1 - \frac{n}{m} \right)$$

$$PVT_B = t B_0 \left[\frac{1}{i} - \frac{1 - e^{-im}}{mi} \right] = B_0 \frac{t}{i} \left[1 - \frac{1 - e^{-im}}{mi} \right]$$

[] $\equiv$ discount coefficient from depreciation, "DCD"

DCDs

$\frac{m}{\downarrow}$	$i \rightarrow .02$				$.04$				$.08$			
	e^{im}	e^{-im}	SF	SL	e^{im}	e^{-im}	SF	SL	e^{im}	e^{-im}	SF	SL
20	1.49	.673	.184	.187	2.21	.453	.34	.32	4.66	.215	.563	.57
40	2.21	.453	.34	.32	4.88	.205	.588	.50	21.72	.046	.85	.70
60	3.28	.305	.472	.42	10.56	.095	.749	.62	102.	.01	.95	.79
80	4.88	.205	.588	.50	23.04	.043	.855	.70	471.	.002	.986	.85

The S.F. deprec. is probably misleading here, because a higher i makes it change, when only greater discount of future is intended.

✓ SL is indep. of i . so cd. SF be, by using arbitrary i , different from discount i .

$$\therefore 1 - \frac{.785}{1.6} = 1 - .49 = .51$$

$$1 - \frac{.954}{3.2} = 1 - .30 = .70$$

$$1 - \frac{.99}{4.8} = 1 - .21 = .79$$

$$1 - \frac{.99}{6.4} = 1 - .15 = .85$$

$$\therefore 1 - \frac{.547}{.8} =$$

$$1 - \frac{.795}{1.6}$$

$$1 - \frac{.905}{2.4} = 1 - .38 = .62$$

$$1 - \frac{.951}{3.2} = 1 - .30 = .70$$

$$\therefore 1 - \frac{.327}{.4} = 1 - .813 = .187$$

$$1 - \frac{.547}{.8} = 1 - .68 = .32$$

$$1 - \frac{\cancel{.58} \cdot .695}{1.2} = 1 - .58 = .42$$

$$1 - \frac{.795}{1.6} = 1 - .50 = .50$$

$$\therefore 1 - \frac{1.6}{3.66} = 1 - .437 = .563$$

$$1 - \frac{3.2}{20.7} = 1 -$$

$$1 - \frac{4.8}{101} = 1 - .048 = \overset{.952}{\cancel{.995}}$$

$$1 - \frac{6.4}{470} = 1 - .014 = .986$$

$$\therefore 1 - \frac{.8}{1.21}$$

~~$$1 - \frac{1.2}{1.21}$$~~

$$1 - \frac{2.4}{9.56} = 1 - .251 = .749$$

$$1 - \frac{3.2}{22} = 1 - .145 = .855$$

$$1 - \frac{.4}{.49} = 1 - .816 = .184$$

$$1 - \frac{.8}{1.21} = 1 - \frac{.66}{.34} = .34$$

$$1 - \frac{1.2}{2.28} = 1 - .528 = .472$$

$$1 - \frac{1.6}{3.88} = 1 - .412 = .588$$

2

Lessons from table of DCDs of SF and SL deprecn.

1. Faster deprecn. (SL) reduces PV ΣT from the no-deprecn. level of $B_0 \frac{t}{i}$ by materially more than slower deprecn. (SF)

So the assessor can materially lower impact of bldg. tax by an announced policy of deprecg. bldgs. fast. And of raising land assets. during mid life of bldgs., which has same effect —
a big pt. to get across.

2. My simple example where $T = B_0 \frac{t}{i}$ overstates impact of tax.

The overstatement, for rough purposes, is only serious for low i and short lives.

3. At higher i , SF deprecn. gets slower.

Thus, as you increase the discount of future taxes, you also increase the base. This reduces sensitivity of PV ΣT to higher i , and understates reduction of PVT with Δi .

Slower deprecn. means ΣB_n is more in future, hence is more sensitive to Δi than SL. But shift of SF schedule w. Δi offsets and conceals this.

4. Taxes have less PV at higher i .

But not as much less as you wd. think from $\frac{t}{i} B_0$.

5. The advantage of faster deprec. is greater at lower i .

Both burdens are higher at lower i , but slower deprec. weights the burden to the future, makes PVT more sensitive to Δi

But this is concealed by #3. At lower i , SF deprec. gets faster

6. The advantage of faster deprec. is greater at longer lives.

This is only weakly shown in table, but warrants further study.



It looks plausible from graphs.



at low i and m , SF deprec. is nearly same as SL

$$B_n = B_0 \frac{1 - e^{-i(m-n)}}{1 - e^{-im}} = B_0 \frac{1 - e^{-im} \cdot e^{in}}{1 - e^{-im}} = \frac{\frac{e^{i(m-n)} - 1}{e^{i(m-n)}}}{\frac{e^{im} - 1}{e^{im}}}$$

$$\text{if } B_n = \frac{e^{i(m-n)} - 1}{e^{i(m-n)} - e^{-in}} = \frac{e^{i(m-n)} - 1}{e^{im} - 1} e^{in}$$

if $m=2$
 $n=1$

$$B_n = \frac{i}{1+i - \frac{1}{1+i}} = \frac{i}{i + \frac{i}{1+i}} = \frac{1}{1 + \frac{1}{1+i}} = \frac{1+i}{2+i} \approx \frac{1}{2}$$

This probably applies to inc. tax deprecn., too.

P.S. Your summary of my remarks re annuity is correct. # Dec 3, 1972

Dear Mary,

In your Nov. 19th letter you wrote

$$\sum_1^m n e^{-in} = \frac{1}{i} \left[\frac{1 - e^{-im}}{i} - m e^{-im} \right]$$

This is not ^{exactly} correct, as you can see by taking $m = 1$. You would then have

$$e^{-i} = \frac{1}{i} \left[\frac{1 - e^{-i}}{i} - e^{-i} \right]$$

If one ~~sets $e^{-i} = 1 + i + i^2$~~ solves this for e^{-i} one gets $(1 + i + i^2)^{-1}$
or $e^i = 1 + i + i^2$

which, of course, is not true, albeit approximately so for small i .
 $1 + i + \frac{i^2}{2}$ might be even closer.

(exactly)
the _{correct} formula is

$$\sum_{n=1}^m n e^{-in} = \frac{e^{-i}}{(1 - e^{-i})^2} \left[1 - (m+1) e^{-mi} + m e^{-(m+1)i} \right]$$

Setting $e^{-i} = r$, (and using k instead of n)
this reads

$$\sum_{k=1}^m k r^k = \frac{r}{(1-r)^2} \left[1 - (m+1) r^m + m r^{m+1} \right]$$

A proof could begin

$$\sum_{k=1}^m k r^k = \sum_{k=1}^m \sum_{j=1}^k r^k = \sum_{j=1}^m \sum_{k=j}^m r^k = \text{etc.}$$

A different proof would begin (this time using x for $-i$ & waiting to the end to set $x = -i$)
(over)

$$\sum_{k=0}^m k e^{xk} = \frac{d}{dx} \sum_{k=0}^m e^{xk} = \frac{d}{dx} \left[\frac{1 - e^{x(m+1)}}{1 - e^x} \right] = e^x$$

doesn't change sum
to add $0 \cdot e^{x \cdot 0}$

more of a trick. If
The 2nd proof is ~~trickier~~, but gives the
same answer.

Any more news of when you'll be where?

Matt

P.S. The exact formula reduces to yours
by two approximations, as follows:

$$\frac{e^{-i}}{(1 - e^{-i})^2} = \frac{e^{-i}}{1 - 2e^{-i} + e^{-2i}} = \frac{1}{e^i - 2 + e^{-i}}$$

setting $e^i \approx 1 + i + \frac{i^2}{2}$ & $e^{-i} \approx 1 - i + \frac{i^2}{2}$ this becomes $\approx \frac{1}{i^2}$

~~Continuing in this vein, and at one point se~~

why
your formula then becomes $\approx \frac{1}{i^2} \left[1 - (m+1)e^{-mi} + me^{-(m+1)i} \right]$

$$= \frac{1}{i^2} \left\{ \cancel{1 - e^{-mi}} + me^{-mi} [e^{-i} - 1] \right\}$$

↑ Approximating $e^{-i} \approx 1$
by i we
get your formula.

Some sinking fund worth

a simple assum. is tax is based on C for m yrs.

Then $tC \frac{e^{im}-1}{i}$ is maturity value of taxes, and

$$S_0 = \frac{MVR - Ce^{im} - tC \frac{e^{im}-1}{i}}{e^{im}} + S_m e^{-im} \quad (3)$$

a depreciating C wd. be:

By Tax base, $H_n = C \left[\frac{1 - e^{-i(m-n)}}{1 - e^{-im}} \right] = \frac{C [1 - e^{i(n-m)}]}{1 - e^{-im}}$

Under sinking fund deprecn.

$$a \frac{e^{im}-1}{i} = C$$

$$\text{so } a = C \frac{i}{e^{im}-1}$$

$$H_n = V_n = C - a \frac{e^{in}-1}{i} = C - C \frac{e^{in}-1}{e^{im}-1} = C \left[1 - \frac{e^{in}-1}{e^{im}-1} \right]$$

$$= C \frac{e^{im}-1 - e^{in} + 1}{e^{im}-1} = C \frac{e^{im}-e^{in}}{e^{im}-1} = C \frac{1 - e^{i(n-m)}}{1 - e^{-im}}$$

✓ Check RIMS — and TISTOC. How come?

never seen this before?

~~$$\text{OK } \frac{e^{in}}{e} \cdot e^{-im} = e^{in-im} = e^{i(n-m)}$$~~

Note 1 can derive SF deprecn. another way now.

Convert C to a level annuity returning C in m yrs.

$$L = C \frac{i}{1 - e^{-im}}$$

Take PV of this annuity in yr. n

$$PV = C \frac{i}{1 - e^{-im}} \cdot \frac{1 - e^{-i(m-n)}}{i} = C \frac{1 - e^{-i(m-n)}}{1 - e^{-im}}$$

$$\text{Tax base, } H_n = C \frac{1 - e^{-i(m-n)}}{1 - e^{-im}}$$

$$\text{Value of taxes at } m \text{ is } \sum_{n=0}^m [t H_n e^{-i(m-n)}]$$

$$= t \sum_{n=0}^m \frac{e^{-i(m-n)} - 1}{1 - e^{-im}}$$

a difficult summation?

(9a)

9a

$$V_0 = a \frac{1 - e^{-mi}}{i}$$

$$V_n = a \frac{1 - e^{-i(m-n)}}{i} = a \frac{1 - e^{ni} \cdot e^{-mi}}{i}$$

$$\text{Deprec.} = -\frac{dV}{du} = \frac{a}{i} \left(-e^{-mi} i e^{ni} \right) = a e^{i(n-m)} = b e^{in}$$

$$b = V \frac{i}{e^{im} - 1} = \text{s.f. input}$$

$$a = V \frac{i}{1 - e^{-im}} = \text{crf input} = b e^{im}$$

$$V_i + b = V_i + V \frac{i}{e^{im} - 1} = V_i \left[\frac{e^{im} - 1 + 1}{e^{im} - 1} \right] = V \frac{i}{1 - e^{-im}} = a$$

$$\text{So } b = a - V_i$$

and

$$b = a e^{-im}$$

$$\therefore a e^{-im} = a - V_i$$

$$a = \frac{-V_i}{e^{-im} - 1} = V \frac{i}{1 - e^{-im}}$$

Proposition. The decline in value caused by an increase of i can be estimated tolerably accurately by capitalizing an ~~stream~~ ^{1% of V ,} ~~of future~~ annuity of ~~the~~ ^{1%} and subtracting it. That is, the higher interest is treated as a higher additional cost, rather than a higher cap. rate.

Original value, $V = \frac{a}{i}$

→ New value, $\bar{V} = \frac{a}{i + \Delta i}$ So simple!
 $\frac{\bar{V}}{V} = \frac{i}{i + \Delta i}$

Alternate estimate, $\bar{V} = V - \frac{\Delta i V}{i} = V \left(1 - \frac{\Delta i}{i}\right) = \frac{a}{i} \left(1 - \frac{\Delta i}{i}\right)$

$\frac{a}{i + \Delta i} \stackrel{?}{=} \frac{a}{i} \left(1 - \frac{\Delta i}{i}\right) = \cancel{\frac{a}{i} \frac{i - \Delta i}{i}}$

~~$i^2 = i^2 - \Delta i^2$~~

$i^2 \stackrel{?}{=} i^2 - (\Delta i)^2$

$i^2 > i^2 - (\Delta i)^2$ but not by much, unless Δi is large

relative to i .

My method of estimate gives a value which is too low.

The reason of course is that my method bases the interest cost on too high a base, $V = \frac{a}{i}$

I would get the right value by adjusting, and subtracting the capitalized value of Δi % of $\bar{V}$.

$$\bar{V} = V - \frac{\Delta i \bar{V}}{i} = \frac{Vi - \Delta i \bar{V}}{i}$$

$$\bar{V}(i + \Delta i) = Vi$$

$$\bar{V} = \frac{Vi}{i + \Delta i} = \frac{a}{i + \Delta i}$$

→ To work method, $\therefore$, $\bar{V} = V - \frac{i}{i + \Delta i} V = V \left[1 - \frac{i}{i + \Delta i} \right] = V \frac{\Delta i}{i + \Delta i}$

$X \equiv$ error of estimate of ΔR

$$R = \frac{1}{i}$$

$$-\Delta R = \frac{1}{i} - \frac{1}{i + \Delta i} = \frac{i + \Delta i - i}{i(i + \Delta i)} = \frac{\Delta i}{i(i + \Delta i)} = R \frac{\Delta i}{i + \Delta i}$$

$$\text{est. of } -\Delta R = \frac{+\Delta i}{i^2} = R \frac{\Delta i}{i}$$

$$X = \cancel{\frac{\Delta i}{i^2}} - \frac{1}{i} + \frac{1}{i + \Delta i}$$

$$= \frac{\Delta i - i}{i^2} + \frac{1}{i + \Delta i} = \frac{(\Delta i)^2 - i^2 + i^2}{i^2(i + \Delta i)} = \frac{(\Delta i)^2}{i^2(i + \Delta i)}$$

$$\frac{X}{R} = \frac{X}{\frac{1}{i}} = \frac{(\Delta i)^2}{i(i + \Delta i)} \approx \frac{.0001}{.0030} \quad \text{So ~~what~~ error as a$$

fraction of R is small.

$$\frac{X}{-\Delta R} = \frac{\frac{(\Delta i)^2}{i^2(i + \Delta i)}}{\frac{\Delta i}{i(i + \Delta i)}} = \frac{\Delta i}{i} \approx \frac{.01}{.05} = 20\%, \text{ so}$$

error as a fraction of ΔR is large.

To correct estimate, however, is duct soup.

Change est. from $\frac{R \frac{\Delta i}{i}}{\frac{\Delta i}{i^2}}$ to $\frac{R \frac{\Delta i}{i + \Delta i}}{\frac{\Delta i}{i(i + \Delta i)}}$, a slightly lower

figure, which is absolutely correct.

And if the larger figure is wanted, i.e. R rather than ΔR , the estimate is tolerable to 3% or so.