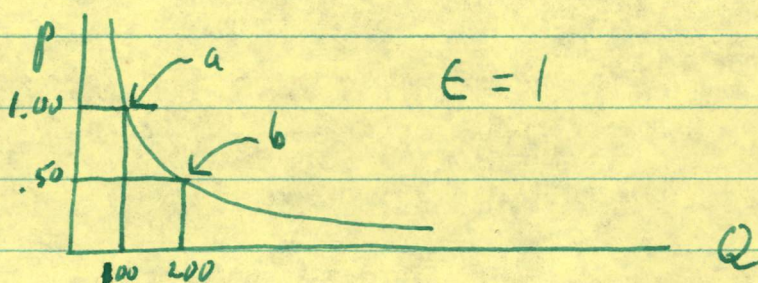


Sensitivity

"Arc-elasticity"

This illustrates why we need arc-elasticity when there are large changes in P and/or Q .

Consider a demand curve of "unit elasticity", $E = 1$. Every 1% drop in P causes a 1% rise of Q . Thus $P \cdot Q$ remains constant. The curve is a hyperbola:



Moving down the curve, a 1% ΔP gets smaller; a 1% ΔQ gets larger (because Q gets larger).

$$\text{at } a, P = \$1.00; Q = 100; PQ = \$100$$

$$\text{at } b, P = \$.50; Q = 200; PQ = \$100$$

If you move from a to b in one hop, $-\frac{\Delta P}{P} = \frac{.5}{1} = 50\%$.

$$\frac{\Delta Q}{Q} = \frac{100}{100} = 100\%. \quad \text{Simple } E = \frac{100\%}{50\%} = 2. \quad \text{The}$$

cumulative change in the bases distorts the value. To adjust, use arc- E .

$$\text{Arc-}E = \frac{\left(\frac{100}{150}\right)}{\left(\frac{50}{75}\right)} = 1$$

(over)

If you don't arc the ϵ you'll think you need $\epsilon = 2$ to accomplish a ΔQ which in fact only needs $\epsilon = 1$.

Or you'll think, after a sale with 50% off, you have $\epsilon = 2$, when in fact it is only $= 1$.

Thus you'll become biased against large price cuts, and in favor of small ones.

Qu. on 3d Quiz: Answer below

$$\text{Given: } -\frac{\Delta P}{P_1} = .3 \quad ; \quad \frac{\Delta Q}{Q_1} = .4$$

For ease, $P_1 = 1$, $Q_1 = 1$. Any other constant will cancel out anyway.

$$\text{Simple } \epsilon = \frac{.4}{.3} = 1.33, \text{ too high.}$$

$$\text{Arc } \epsilon = \frac{.4/1.2}{.3/.85} = 1.33 \times \frac{.85}{1.2} = .94$$

A quicker way to get arc- ϵ is with logs.
Try this:

$$\frac{\ln 200 - \ln 100}{\ln 1 - \ln \frac{1}{2}} = \frac{.6931}{.6931} = 1$$

$$\frac{\ln 140 - \ln 100}{\ln 100 - \ln 70} = \frac{.3365}{.3567} = .94$$

ϵ is the ratio of the changes in the natural logs, however large or small the change.
This also works with log base 10, but I recommend you stick with natural logs (LN), in general, because the change in the natural log is the fractional change in the number.

For some insight into why this works, try log base 2.

$$\frac{\frac{256 - 128}{128}}{\frac{8 - 4}{8}} = \frac{2 - 1}{1 - \frac{1}{2}} = 2$$

$$\frac{\log_2 256 - \log_2 128}{\log_2 8 - \log_2 4} = \frac{8 - 7}{3 - 2} = 1$$

Marvel + ponder! I leave it at that.

Summary Re log deriv. of IPF

$$1. \frac{d}{dx} \frac{u}{v} = \frac{u}{v} \left[\frac{u'}{u} - \frac{v'}{v} \right]$$

$$2. \frac{\frac{d}{dx} \frac{u}{v}}{\frac{u}{v}} = \frac{u'}{u} - \frac{v'}{v}$$

$$3. \frac{d}{di} e^{in} = n e^{in} \quad \text{"The } n^{\text{th}} \text{ rule of growth"}$$

It is $\frac{n}{100}$, rather than n , bcs. $di = \frac{1}{100}$

$$\frac{\frac{d}{di} e^{in}}{e^{in}} = n \quad \text{or} \quad \frac{d}{di} \ln e^{in} = \frac{d}{di} (in) = n$$

$$4. \frac{di}{di} = 1. \quad \text{When } di = .01, \quad di = .01 \times 1 = .01$$

$$5. \frac{d \ln i}{di} = \frac{1}{i} \quad \text{When } di = .01, \quad = \frac{1}{100 i}$$

$$5. \frac{\frac{d}{di} \frac{i}{1-e^{-in}}}{\frac{i}{1-e^{-in}}} = \frac{i'/i}{\frac{i}{1-e^{-in}}} - \frac{ne^{-in}}{1-e^{-in}} = i'/i - \frac{n}{e^{in}-1}$$

$$\text{When } di = .01, \quad = \frac{1}{100 i} - \frac{n}{100(e^{in}-1)}$$

6. IPF is much less sensitive to di than i is, and i than ~~the~~ cap. rate is.

Table

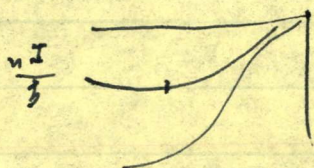
Fractional ~~to~~ change in IPF with change of i . $n = 25$

$$\text{Log Deriv. of } IPF = \frac{1}{100i} - \frac{n}{100(e^{10i} - 1)}$$

i	.04	.05	.06	.07	.08	.09	.10	.11	.12
$\frac{1}{100i}$	.250	.20	.167	.143	.125	.111	.10	.091	.083
$\frac{n}{100(e^{10i} - 1)}$	.146	.100	.072	.053	.039	.029	.022	.017	.013
Difference of above	.104	.100	.095	.090	.086	.082	.078	.074	.070
IPF	.064	.071	.078	.086	.094	.102	.110	.119	.127
$\frac{\Delta IPF}{IPF}$		.109	.099	.103	.103	.085	.078	.082	.067

$$\Delta\left(\frac{V}{g}\right) = \frac{g_0}{g_0} \left[\frac{1.03}{1.06} \right] \left[\frac{V_0}{g_0} \right] - \left[\frac{1.03}{1.06} \right] \left[\frac{V_0}{g_0} \right] = \frac{V_0}{g_0} \left[\frac{1.03}{1.06} - 1 \right]$$

$$\frac{g_0 (1.03)}{V_0 (1.06)} = \frac{g_0}{V_0} \times \frac{1.03}{1.06} = \frac{V_0}{g_0} (1 + 0.03 - 0.06)$$



$$\left[\frac{V}{n} - \frac{n}{V} \right] \frac{V}{n} =$$

$$\frac{\frac{V}{n}}{\frac{n}{V} - \frac{V}{n}} = \frac{\frac{V}{n}}{\frac{n}{V} - \frac{V}{n}} = \frac{V^2}{n^2 - nV}$$

$$\frac{V/n}{n/V} = \frac{V/n}{n/V} = \frac{V/n}{n/V} =$$

$$\left[\frac{V}{n} - \frac{n}{V} \right] \frac{V}{n} = \frac{V}{n} \frac{V}{n} - \frac{V}{n} = \frac{V^2}{n^2 - nV}$$

$$= \frac{d}{d} \left[\frac{1 - e^{-x}}{x} \right]$$

$$\frac{\frac{d}{di} \left(\frac{u}{v} \right)}{\frac{u}{v}} = \frac{u'}{u} - \frac{v'}{v}$$

$$\frac{u}{v} = \frac{i}{1 - e^{-i\eta}}$$

$$\frac{u'}{u} = \frac{i'/i} = \frac{i}{i}$$

$$\frac{v'}{v} = \frac{-ne^{-i\eta}}{1 - e^{-i\eta}} = \frac{n}{e^{i\eta} - 1}$$

$$\frac{u'}{u} - \frac{v'}{v} = \frac{i'}{i} - \frac{n}{e^{i\eta} - 1} = \frac{i'}{i} - \frac{25}{e^{25} - 1}$$

$$\# (i = .04)_v = \frac{.04}{.04} - \frac{25}{1.67} = 25 - 14.97 = 10.03$$

$$\frac{d e^{-i\eta}}{di} = e^{-i\eta} \frac{d(-i\eta)}{di} = e^{-i\eta} (-n) = -n e^{-i\eta}$$

$$1 - e^{-in}$$

$$\frac{de^{in}}{di} = n e^{in}$$

"n% rule of growth"
 $\ln e^{in} = in$
 $\log \text{ der.} = n$

	n	$i =$	.04	.05
e^{25i}	25		2.718	3.4903
Δe^{25i}				.772
$25 e^{25i}$				87.25

$$\frac{di}{di} = 1$$

But when di is $\frac{1}{100}$ of a unit, that means .01

$$\text{So } \frac{\frac{di}{di}}{i} = \frac{\cancel{.04}}{\cancel{.04} \cdot \frac{1}{100}} = 25$$

$$\ln e^{in} = in$$

$$\frac{d}{di} in = n$$

$$\Delta(in) = \Delta i \times n \quad \Delta i = .01, \text{ so } \Delta(in) = \frac{n}{100}$$

$$\text{Like wise, for } \frac{d}{di} (\ln i) = \frac{1}{i}$$

$$\Delta(\ln i) = \Delta i \cdot \frac{1}{i} = \frac{1}{100i} = .04$$

$$\text{Likewise, for } \frac{d}{di} (\ln(1 - e^{-in})) = \frac{n}{e^{in} - 1}$$

$$\Delta \ln(1 - e^{-in}) = \Delta i \frac{n}{e^{in} - 1} = \frac{n/100}{e^{in} - 1}$$

D. Sensitivity

Sensitivity analysis is the art of expressing how one variable responds to another. The calculus derivative is a first pass at sensitivity, but it is useful to divide the derivative by the function, getting the fractional rate of change. Multiplying that by 100 you get the percentage rate of change, a common and familiar concept.

The derivative divided by the function is also the derivative of the natural log of the function, called the "log derivative." You have the choice of taking the log first, and then the derivative, if that saves work.

A. The PIPO case

Take the basic compounding factor, with $P=1$ for simplicity:

$$F = In \quad (1, A)$$

$$LNF = nLNI = ni$$

$$dLNF/dn = i \quad (11)$$

For example, if n increases by one, $\Delta F/Fn = i$. Of course we already knew that, but it's reassuring when everything checks. In fact when anything doesn't check you have surely goofed.

But here is something we didn't know:

$$dLNF/di = n \quad (12)$$

Applying this is trickier, because i doesn't change by much, hardly ever by as much as one, which is 100%. It is more likely to change by .01, and although that doesn't look like much, empires have fallen from lesser causes -- consider the present scramble to refinance mortgages because rates have fallen a few points.

When i rises by .01, F rises by $n/100$, which is $n\%$. This is (almost) equally true if i rises from .01 to .02, or from .20 to .21. Thus if n is 25, every point rise in i makes F rise by about 25%.

This is called "The $n\%$ Rule of Thumb," and it brings out one of the vital facts of economics, that small differences of i result in large differences in future values, and the size of the

differences increases with the futurity, as measured by n . (It is a rule "of thumb" because it is less than 100% accurate, owing to the approximation that $LNI = i$; and it needs serious adjusting when n gets above 50 or so).

The $n\%$ Rule gets especially potent in discounting, when you divide by the compounding factor (or multiply by the discount factor, which amounts to the same thing). Higher interest rates depress present values in proportion to n , the futurity of the values being discounted. Let's show this directly. This time we let $F=1$.

$$P = I - n \quad (1, B)$$

$$LNP = -nLNI = -ni$$

$$dLNP/dn = -i \quad (13)$$

$$dLNP/di = -n \quad (14)$$

(14) says that a 1-point rise of i reduces P by $n\%$.

Another way to compare the growth of money at different rates, i and j , is to take their ratio. Let $J=1+j$, and $K=1+k$.

$$I_n/J_n = [I/J]^n = K^n \quad (15)$$

$$k = I/J - 1 = \text{-----} \quad (16)$$

(15) shows that the ratio of the sums growing at different rates is not constant, but keeps growing at the rate k , which (16) shows to be approximately the difference between i and j .

An excellent way to visualize this is to plot the growth curves on a semi-log scale, with n on the abscissa and F on the ordinate. The origin of the scale is where $n=0$, but $F=1$. (You get the same effect by plotting the logs of the growth curves on a common scale, but here we just treat the semi-log scale). On such a scale, equal vertical distances represent equal percentage increases (from the bottom of each vertical line), and every curve of constant percentage growth, such as (1,A), is a straight line.

The slope of the line is the percentage growth rate, so that doubling the growth rate doubles the slope. Growth curves with different rates, starting from the common origin, diverge like rays as n rises. The vertical distance between them increases

with $\underline{n}$. This shows that their ratios change with $\underline{n}$ -- not just their absolute differences. The highest curve becomes an ever higher multiple of the lower curves.

Discounting on the scale is the mirror image of compounding. Instead of growing the curves "decay" exponentially, forming straight lines sloping downwards to the right, starting from the origin.

These decay curves show the effect of interest rates on present values. Higher interest rates selectively downvalue future values in proportion to their futurity. The higher is $\underline{n}$, the more sensitive values are to $\underline{i}$.

B. The PVF case

Starting with (3,A), let $a=1$ for generality. We know (from the alternative derivation of (3,A) on p.4) that when $\underline{n}$ rises by one, P rises by $I-n$, which we shall call P' . Now we solve for the fractional increase, P'/P :

$$P'/P = \text{-----} = \text{-----} \quad (18)$$

(18) shows that P'/P is a sharply decreasing function of $\underline{n}$, approaching zero as an asymptote. This underscores the futility of building in durability beyond 50 years or so, especially at high interest rates.

C. The IPF case

For generality, set $P=1$. Take the log of both sides, and differentiate.

$$\text{LNa} = \text{LNi} - \text{LN}(1-I-n)$$

$$a'/a = d\text{LNa}/dn = \text{-----} \quad (18)$$

D. Elasticities

Elasticity is another useful measure of sensitivity. It is the ratio of the log derivatives of the dependent and independent variables. We already have the first, and the second is a piece of cake, it is simply the reciprocal of the independent variable, in this case $\underline{n}$. Dividing by $1/\underline{n}$ is the same as multiplying by $\underline{n}$, and there you have elasticity, generally described as the ratio of the percentage changes of two variables.

We can translate what we already have into elasticities immediately. Let E denote elasticity.

E of F wrt $n = n_i$

E of F wrt $i = i_n$ (same thing)

E of P wrt $n = -n_i$ (PIPO case P is referred to here) E of P wrt $i = -i_n$ "

Let PV denote the present value of a flow, as in (3,A), to distinguish it from P , used for the PIPO case.

E of PV wrt $n = \text{-----}$ (19)

E of IPF wrt $n = \text{-----}$ (20)

The ratio seen in (19) and (20) pops up often in finance. You can know how it behaves by realizing that it is the ratio of simple interest to compound interest. That means the top grows nearly as fast as the bottom for a few years, but then the bottom races far ahead and the ratio falls swiftly.

The trouble with these elasticities (and the log derivatives, too) is that they are true only at one point. If the changes are large, e.g. if you double n from 15 to 30, these point elasticities understate the response. They are accurate for small changes, however; and always useful to put you in the right ballpark.

Those are the basic tools. The applications and variations are endless. If the tools seem hard to learn, remember poor Mr. Carter. You can learn the cheap way from me, or the expensive way from your creditors. There are lots of folks out there eager to help you learn by taking your money.

May The Force be with you. Keep away from The Dark Side.