

*Economic Policy, Tax Policy
Invest Turnover and Capital Turnover*

THE INFLUENCE OF FEDERAL TAX POLICIES ON REAL ESTATE DEVELOPMENT
INVESTMENT TURNOVER VS. CAPITAL TURNOVER

BY

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Economic Policy, Tax Policy, Investment
Turnover and Capital Turnover

I Introduction

This paper is the first working report of a study* of the hierarchy of relationships between national economic goals, Treasury goals, Treasury tools (tax laws, codes and enforcement) in accomplishing these goals. The paper concentrates on buildings and land, though we feel the concepts are generalizable to other types of assets.

We hope not to induce too much dispute if we indicate that national economic goals include 1) a high rate of growth in real gross national product with a minimum rate of price increase - these growth rates to be stable from period to period, and 2) full employment. Looking at the growth in GNP more carefully we conclude that what is specifically desired is growth of national product net of depreciation; and that in the capital investment component, net new investment is preferred over replacement. The two goals of capital employment labor employment argues that resource management should have the goal of equimarginal productivity of labor and capital. Our politico-economic consensus in this country is that it is appropriate to minimize the necessary corrective governmental activity to accomplish national goals. This presumes it is better to accomplish these goals by the private sector if possible.

* The study is itself a part of a longer term. The Real Estate Investment Project which investigates the Impact of Federal Tax Policy Variable on Real Estate Investments. The Project is financed by the U.S. Treasury, however the views expressed herein are entirely those of the authors. Since this is an early draft we strongly reserve the right to change equations and conclusions without apology.

Let us turn briefly to taxation goals. According to Pechman,¹ "

Taxation - the method by which a nation implements decisions to transfer resources from the private to the public sector - is a major instrument of social and economic policy. It has two goals; to distribute the cost of government fairly by income classes (vertical equity).and among people in approximately the same economic circumstances (horizontal equity); and to promote economic growth, stability and efficiency.

To further set the stage for our analysis herein we reviewed economic literature on the means of accomplishing growth, stability and full employment and the relationship to these goals of 1) economic depreciation and replacement of capital assets and 2) tax depreciation (and other permitted deductions and 3) investor holding periods. We find as a result of this review that we heartily agree with Dorfman who calls these relationships "one of the ancient unsolved - perhaps unsolvable - problems of capital theory"² Ignoring, at some peril, this advice we proceed to examine various types of single period receipts, annuities and combinations of the two.

Given a before-tax rate of return, our objective is to look at 1) the capital asset life which maximizes the after-tax rate of return, assuming a single holder, 2) the holding period which maximizes the rate of return after tax to the first holder and 3) the succession of holding periods which maximizes the after tax return. We also look at the goal

1 Joseph A Peachman, Federal Tax Policy, Washington, D.C.: The Brookings Institution, 1966 pp 5.

2 Robert Dorfman, "Waiting and the Period of Production"; The Quarterly Journal of Economics, August, 1959, pp 351.

of maximizing present value. The work represents a logical (at least to us) extension of previous independent work of Gaffney³. and of Ricks and Kaplan⁴. To date we are only at the point of discussing with you and soliciting your suggestions for the methodology and the equations.

3 Mason Gaffney, "Tax-Induced Slow Turnover of Capital", Western Economic Journal, Sept. 1967 pp 308-323.

4 R. Bruce Ricks and Donald M. Kaplan, "The Impact of Depreciation Policy on Real Estate Investment Performance", Real Estate Investment Project, University of Calif., Working Paper #4, 59 pp

II. Tax Policy and Returns on Real Assets

Research Design:

The basic approach adopted in this section of the paper is to first construct a general model of real asset investment returns and subsequently to modify this model, for specific alternative types of assets, to reflect the impact of various federal taxation policies.

Two Approaches:

In evaluating investment returns, economists have typically adopted either the internal rate of return or a present value calculation. Both approaches are popular, and the relative merits and disadvantages of each are familiar to both academicians and practitioners. For the purposes of this study it will be appropriate to utilize both approaches in different parts of the analysis.

Internal Rate of Return:

The internal rate of return will be utilized for the evaluation of real assets other than land. Under this approach the effects of all before-tax cash flows, write-offs and taxes will be reflected in the after-tax rate of return realized by an equity investor. Hence, the impact of an earlier cash flow, faster depreciation schedule or reduced tax rate will be observed as a higher internal rate of return.

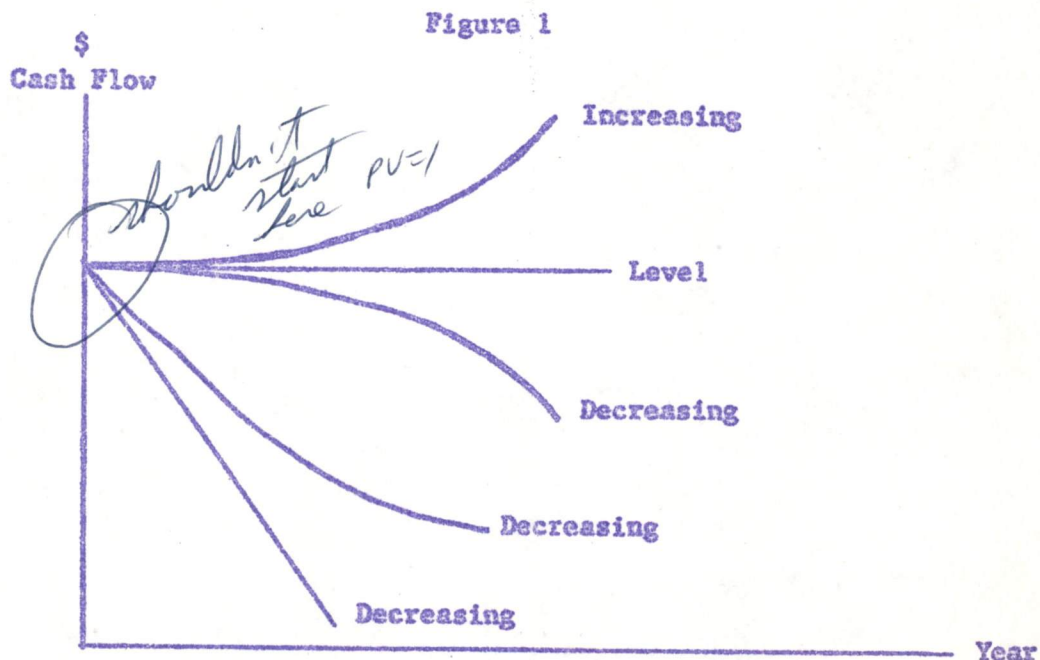
Present Value:

could be deleted

The alternative approach - solving for the present value - is particularly well suited for evaluating land investment performance. Given an original cost, the before-tax rate of return, a particular write-off schedule and taxes, a unique present value can be calculated. Under this approach a shift in any or all of these parameters, instead of being reflected in the after-tax rate of return, will appear as either a higher or lower land value.

Alternative Cash Flow Models:

Like most other classes of investments, real estate (buildings and land) generates annual streams of cash flows. While the inter-temporal pattern of these cash flows will vary between different investment projects, for a particular investment the cash flow can usually be characterized by some basic annuity pattern -- level, increasing or decreasing (at an increasing, decreasing or constant rate of decline). While others do appear, these patterns (see figure 1) are most commonly observed.

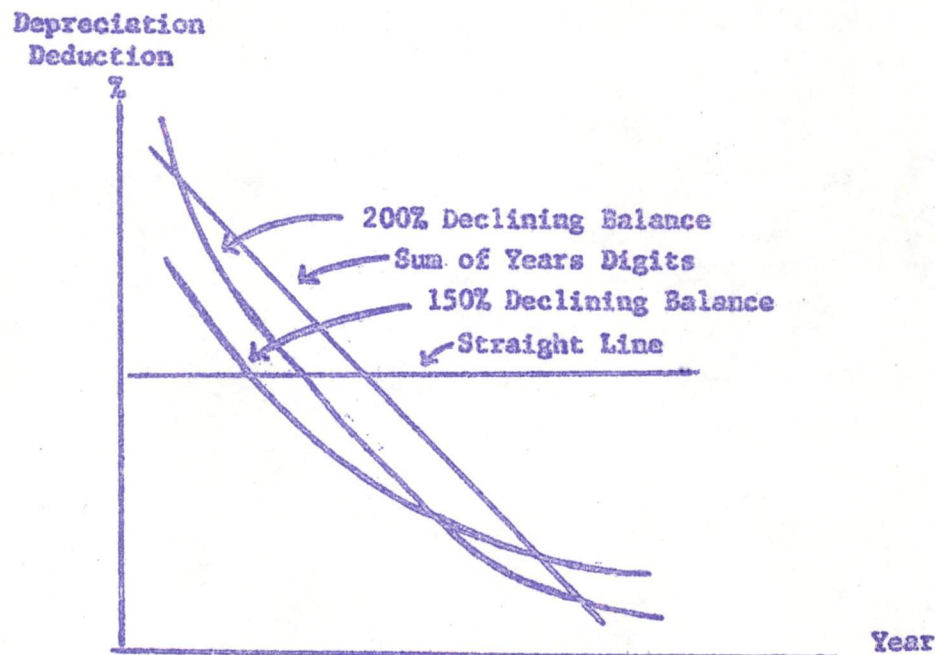


Alternative Capital Recovery Models:

Investment capital recovery exists across a full spectrum. On the one extreme certain investments qualify for full recovery of capital costs at the time the investment is made; i.e. expensing. On the other extreme some investments do not qualify for any write-off, or if they do qualify the write-off is delayed until the investment matures or is turned over.

Between the extremes of expensing and no recovery of costs there exist numerous alternative write-off or depreciation policies. As in the case of cash flows, this paper will consider many alternative depreciation methods but will focus on the more popular ones - straight line, double and 150 percent declining balance, sum of year digits and combined accelerated and straight line (see figure II).

Figure II



Evaluation of Alternative Models:

In order to identify the influence of various federal tax policies on real estate investment performance, a particular investment model (i.e. an inter-temporal cash flow pattern) must be specified. Given this specification, the various tax policies to be examined (depreciation method, tax credit if any, recapture provisions, etc.) are enumerated and appropriately integrated with the cash flow model.

Even with only a small number of cash flow patterns and depreciation alternatives the possible combinations are numerous and complex. Thus, the first two cases examined (appreciating assets and constant value assets) are simple and admittedly artificial. They serve, however, to illustrate the basic points of the analysis and prepare us to move into the more complex and realistic models (depreciating assets and land).

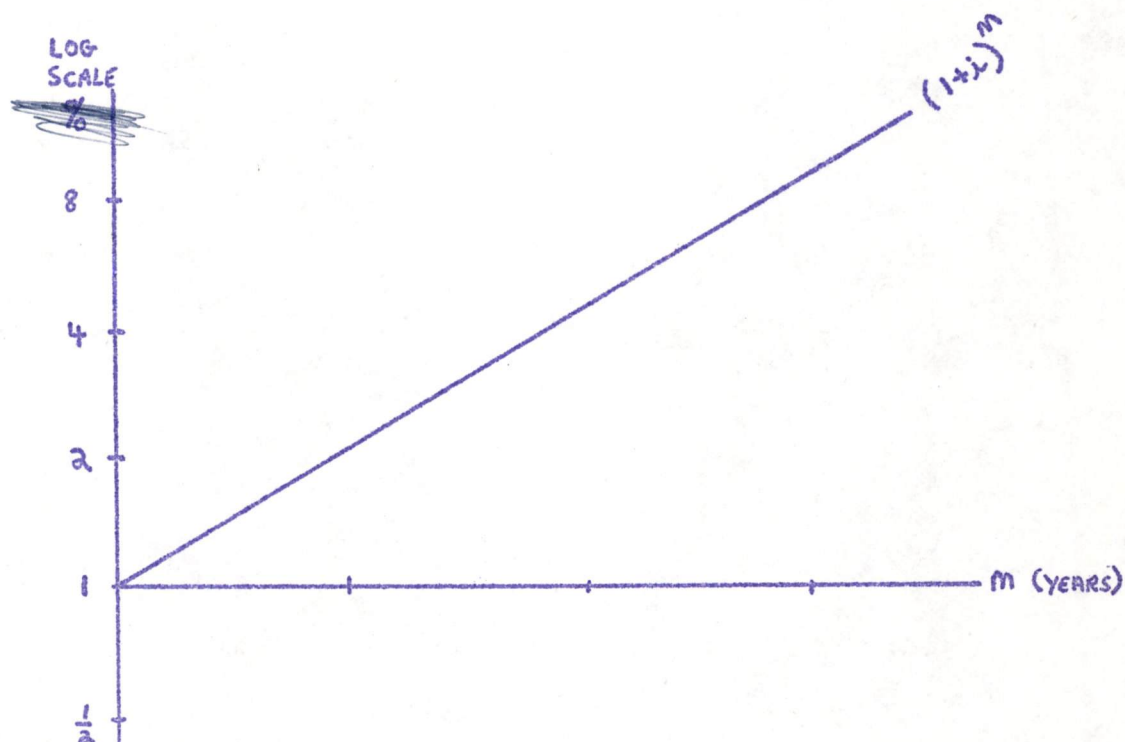
Appreciating Assets:

This simple "point-input point-output" model applies to such assets as trees, wine and cattle. The original investment at time 0, say \$1, is the only explicit cost. It is assumed the asset grows at i percent per year and that the entire return is realized in the year the asset matures, m . Since there are no intermediate cash flows, the ~~effective~~ rate of return, r , realized by an investor over the investment period is given by (1) and (1A) and depicted in Figure III.

$$(1) \quad (1+r)^m = 1(1+i)^m$$

$$(1A) \quad r = i$$

Figure III

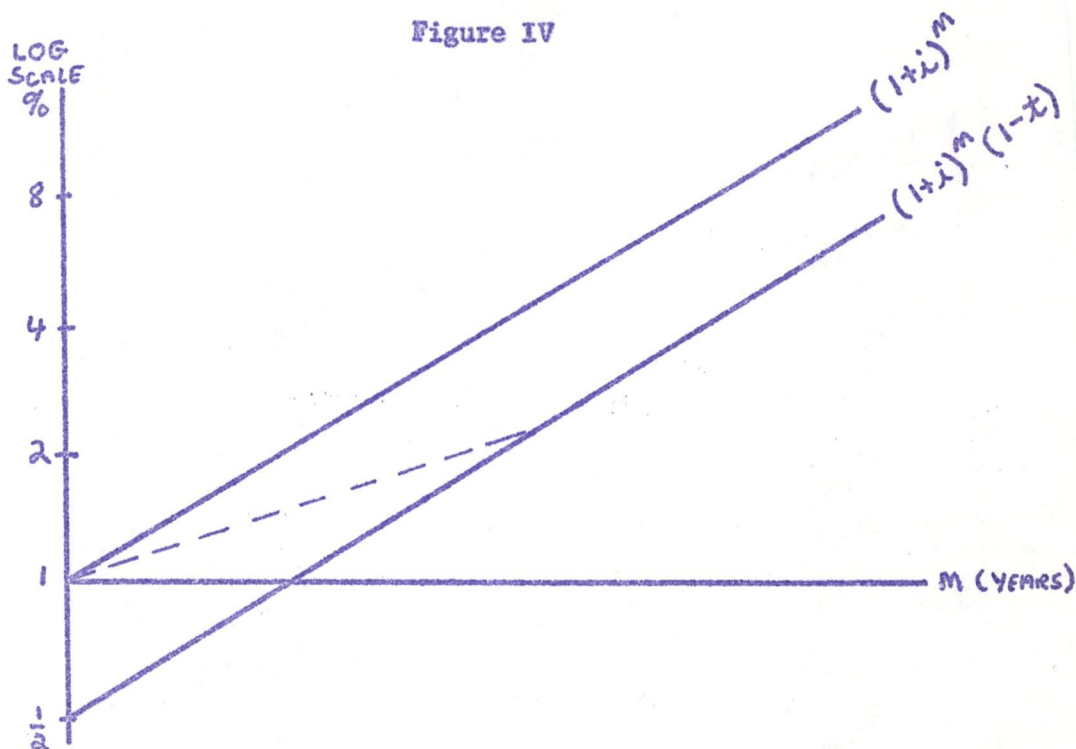


With the addition of a severance tax, t , (1) and (1A) become (2) and (2A) as shown in Figure IV.

$$(2) \quad (1+r)^m = 1(1+i)^m(1-t)$$

$$(2A) \quad r = (1+i)(1-t)^{1/m} - 1$$

Figure IV

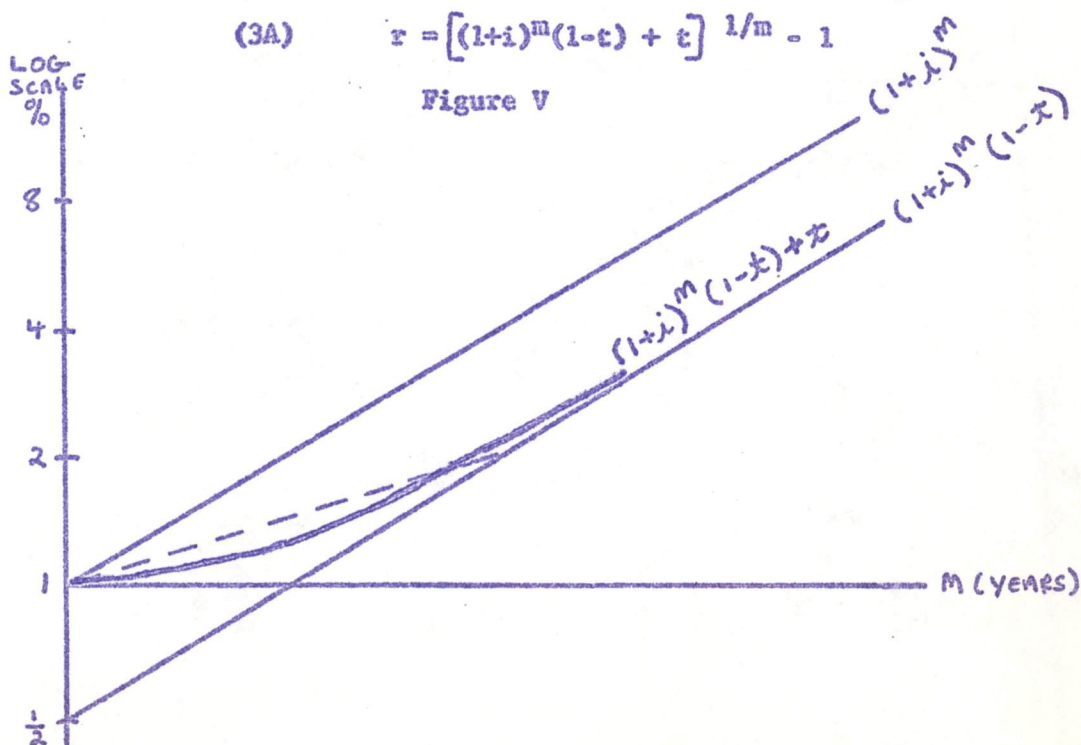


With the addition of a full write-off of the original investment costs in the year of maturity (2) and (2A) become (3) and (3A) as shown in Figure V.

$$(3) \quad (1+r)^m = 1(1+i)^m(1-t) + 1(t)$$

$$(3A) \quad r = [(1+i)^m(1-t) + t]^{1/m} - 1$$

Figure V



Moving the write-off forward to some year earlier than maturity provides a simple proxy for a complete depreciation schedule and (3) becomes (4).

$$(3) \quad (1+r)^m = 1(1+i)^m(1-t) + 1t \quad \text{is equivalent to}$$

$$(1+r)^m = 1(1+i)^m(1-t) + 1t(1+r)^0$$

Moving the write-off forward to year w , where $w \leq m$, yields

$$(4) \quad (1+r)^m = (1+i)^m(1-t) + t(1+r)^{m-w} \quad \text{is equivalent to}$$

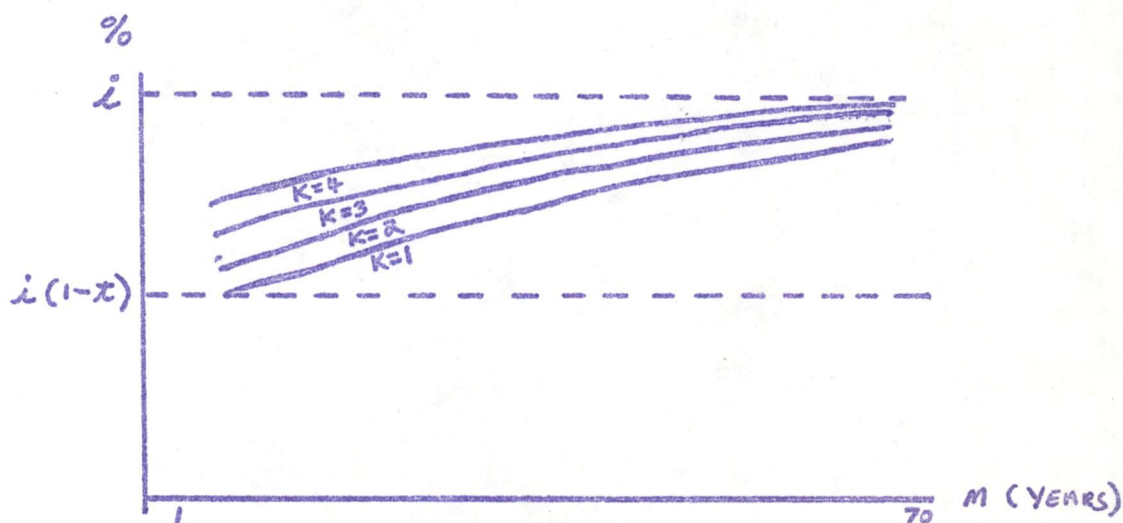
$$(1+r)^m = (1+i)^m(1-t) + t(1+r)^{m-m/k}$$

where $m/k = w$ and $k = 1 \rightarrow \infty$.

Solving equation (4) for the investor's after-tax realized rate of return r provides a simple means of examining the impact of various tax provisions which effectively allow write-off prior to maturity.

Assuming i equals 10 percent, m goes from 1 to 70 years and for simplicity a single tax rate of 50 percent, a value of r can be calculated for each year of the investment period. When k has a value of 1 the last term in equation (4) drops out. This is the case of write-off at maturity. When k has a value of ∞ the write-off takes place instantly; i.e. the investment is expensed. By allowing k to assume a range of increasing values beginning at 1, the effect on r of earlier write-offs can be presented. (See Figure VI).

Figure VI



Constant Value Assets:

Somewhat less artificial, but still simple, is the constant value or full salvage model. Here it is assumed that the investment yields a steady net income during its lifetime, i.e. a level annuity cash flow. It is assumed, furthermore, that the asset neither appreciates nor depreciates. Thus, it has a salvage value at maturity equal to its initial investment cost. Assets such as time deposits and milk cows typify these assumptions. As before, in the initial formulation there is no recovery of capital until the asset reaches maturity.

With a \$1 original investment cost, a level annuity cash flow of i percent before-tax, a tax rate t and the after-tax internal rate of return r , the present value of the after-tax cash flow is given by (5).

$$(5) \quad \sum_{n=1}^m \frac{i(1-t)}{(1+r)^n}$$

Since the salvage value equals the original cost (\$1), the present value of the salvage value after tax in year m is given by (6).

$$(6) \quad \frac{1(1-t)}{(1+r)^m}$$

Full write-off either in maturity or some year prior to maturity is achieved by (7).

$$(7) \quad \frac{1t}{(1+r)^{m/k}}$$

Combining (5), (6) and (7) yields (8).

$$(8) \quad 1 = \sum_{n=1}^m \frac{i(1-t)}{(1+r)^n} + \frac{1(1-t)}{(1+r)^m} + \frac{1t}{(1+r)^{m/k}}$$

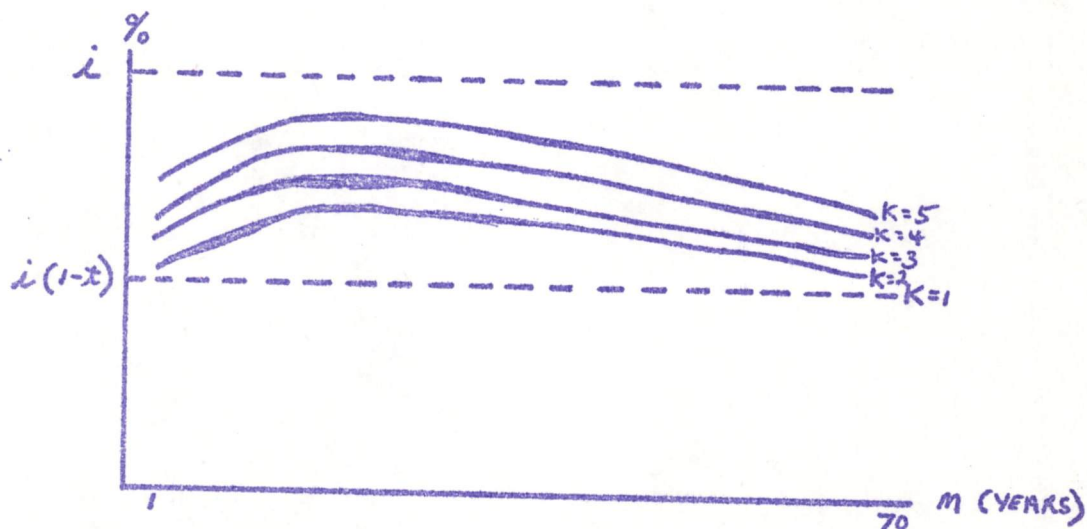
Cost	Cash	Salvage	Arbitrary
	Flow	Value	Write-Off

Rewriting the first term^{IN (%)} as the sum of a finite geometric progression yields (9).

$$(9) \quad 1 = i(1-t) \frac{1-(1+r)^{-m}}{r} + \frac{1(1-t)}{(1+r)^m} + \frac{t}{(1+r)^{m/K}}$$

Using the same set of parameters as specified previously in the point-input point-output model, a distribution of r values over time is achieved as portrayed in Figure VII.

Figure VII



Depreciating Assets:

The essential difference between the two preceding cases and the depreciating assets model is one of generality. In the depreciating assets model the restrictive, simplifying assumptions of the point-input point-output and constant value models are either eliminated or relaxed.

The cash flow generated by the investment is no longer restricted to a level annuity. It can and now will assume the various patterns depicted earlier in Figure I. Also, the assumptions of full salvage value at maturity and an arbitrary full write-off at or before maturity are now dropped. In their place a depreciating salvage value is assumed which in turn implies a scheduled write-off of capital costs according to one of the existing depreciation methods.

Assuming again an initial investment cost of \$1 we define a level annuity cash flow whose present value is 1.

$$1 = \frac{1}{1 - (1+i)^{-m}}$$

With an excise tax t on this annuity the tax payer gets $(1-t)$ times the annuity.

$$1 = \frac{1}{1 - (1+i)^{-m}} (1-t)$$

His rate of return after tax r , is the capitalization rate that will give this reduced annuity a present value of one as shown in (10).

$$(10) \quad 1 = \frac{1}{1 - (1+i)^{-m}} (1-t) \frac{1 - (1+r)^{-m}}{r}$$

Original	Level	Tax	Capitalization
Cost	annuity	Factor	Term
	PV = 1		

In order to provide for alternative increasing and decreasing cash flow annuity patterns (10) is modified as shown in (11).

$$(11) \quad 1 = \frac{1-g}{1 - \left(\frac{1+g}{1+i}\right)^m} (1-t) \frac{1 - \left(\frac{1+g}{1+r}\right)^m}{r-g}$$

where g equals the rate of increase or decrease in the annuity.

At this point all that remains to complete the depreciating assets model is the specification of the write-off of the capital costs. The salvage value term and the arbitrary full write-off term in (9) are now deleted. They are replaced with an annual schedule of depreciation deductions d , where d_w is the deduction against income in year w . The present value of the tax savings due to all the deductions is (12).

$$(12) \quad t \sum_{w=0}^m \left[\frac{d_w}{(1+r)^w} \right]$$

The effect of adding (12) to the right side of (11) is to raise the value of r since now the investor can capitalize his after-tax annuity at a higher rate to make it equal \$1. The complete depreciating assets model is thus expressed by combining (11) and (12) into (13).

$$(13) \quad 1 = \frac{1-g}{1-\left(\frac{1+g}{1+r}\right)^m} (1-t) \frac{1-\left(\frac{1+g}{1+r}\right)^m}{r-g} + t \sum_{w=0}^m \left[\frac{dw}{(1+r)^w} \right] + \frac{\text{RESALE OF LAND}}{(1+r)^m}$$

With specific regard to depreciation, straight line and accelerated (double or 200% declining balance) are two of the most commonly used methods. In the case of straight line the depreciation deduction is the same in any year (See Figure II), and is equal to one over the useful life in years allowed for tax purposes. (The useful life for tax purposes for real assets may be equal to or shorter than the asset's economic or service life.) Substituting into (12) the depreciation term under straight line is given by (14).

$$(14) \quad \left[t \cdot \frac{1}{m} \cdot \frac{1-(1+r)^{-m}}{r} \right]$$

By a similar approach it can be shown that under double declining balance depreciation (DDB) the depreciation term is (15).

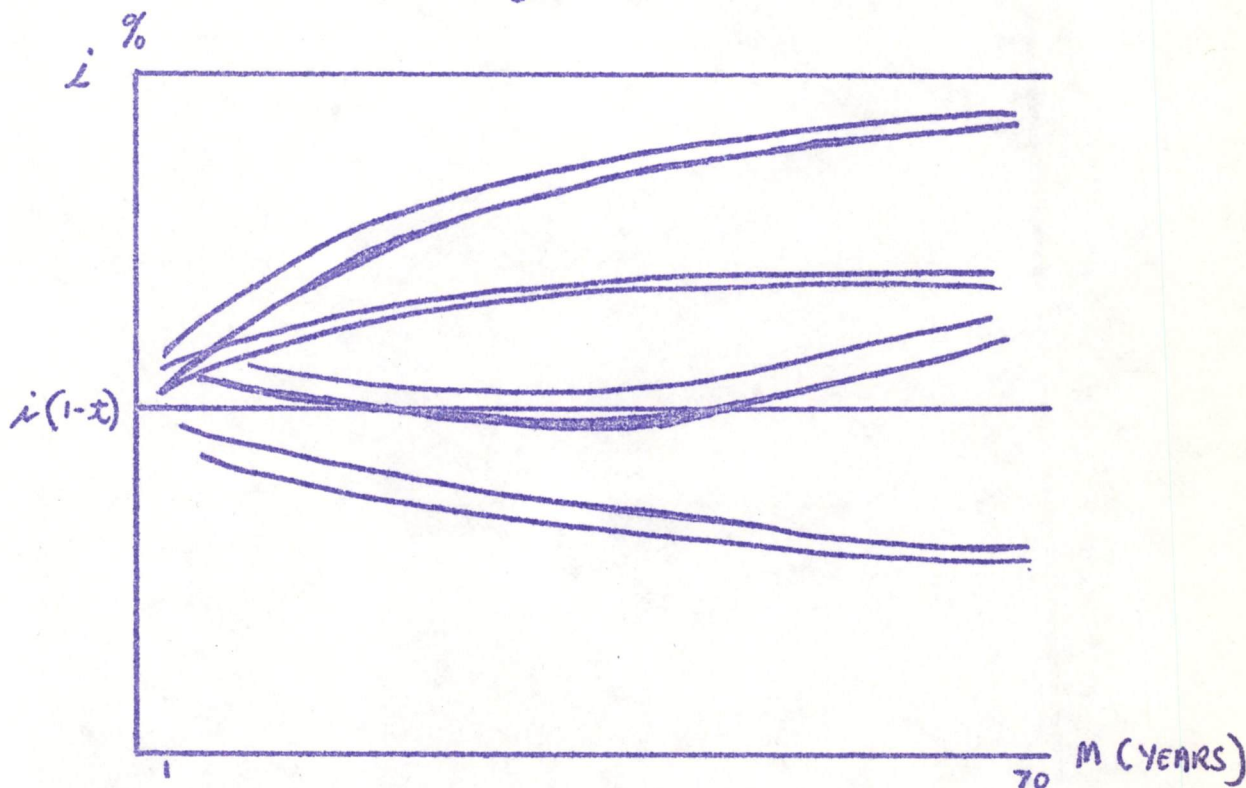
$$(15) \quad \left[t \cdot \frac{2}{m} \cdot \frac{1-\left(\frac{1-2/m}{1+r}\right)^m}{r + 2/m} + \left(\frac{1-2/m}{1+r}\right)^m \right]$$

remaining
undepreciated
book value

As yet similar terms for various other depreciation methods (150% DB, sum of years digits, combined accelerated/straight line, sinking fund, etc.) have not been calculated.

Figure VIII depicts the results of some preliminary results under the depreciating assets model. The results reflect the general shapes and relative positioning over time of values of r for various cash flow annuity patterns and depreciation methods. Again, these results were calculated assuming equal tax and economic lives. Yet to be computed is the sensitivity of these results to changes in i (the before-tax rate of return).

Figure VIII



Land:

The depreciating assets model, when fully evaluated, will adequately handle assets such as buildings. In order to complete the analysis the model need only be extended one additional step to include consideration of land.

On the assumption that no taxes are absorbed as a reduced after-tax rate of return r (or as reduced unit costs of labor and materials), it is now appropriate to switch to a present value formulation. Hence, the effects of taxes, etc. will be reflected directly as higher or lower land values.

The present value of a site in perpetuity and in the absence of taxes is given by (16).

$$(16) \quad PV_0 = \frac{\sum_{n=0}^{\infty} (R_n (1+i)^{-n})}{1-(1+i)^{-\infty}} - C_0$$

where PV_0 is the present value of the site, C_0 is the original investment cost and R_n is the net revenue generated each year. Equation (16) first discounts and sums all future net revenues ^{AND THEN} subtracts the original cost yielding a present value. This present value is then converted to a level annuity by multiplying by the capital recovery factor and this annuity is then capitalized in perpetuity.

With the addition of taxes and the deduction of cost C_0 at time 0 produces (17).

$$(17) \quad PV_0 - T = \frac{\sum_{n=0}^{\infty} (R_n (1+i)^{-n}) (1-t) - C_0 (1-t)}{1-(1+i)^{-\infty}}$$

Alternatively, cost C_0 may be deducted subsequent to time 0 in which case the right term of (17) becomes $C_0(1-t(1+i)^{-w})$ where w is the year of write-off.

Although as yet undone, the land model need only be integrated with the depreciating assets model, under the various cash flow and depreciation alternatives, to complete the formulation.

$$\left(\frac{B}{C}\right) = .50, .85, 0.0$$

$$TL = E \pm H$$

$$TL \neq E \neq H$$

H₁ H₂
SL
DECEL



Present values of assets of different usable lives, n , yielding incomes that decline by a fixed annual percentage, d , beginning from \$1 a year. Table shows effects of using different interest rates.

$$V = \frac{1+i}{1+d} \left[1 - \left(\frac{1-d}{1+i} \right)^n \right] = 1 + \frac{1-d}{1+i} + \left(\frac{1-d}{1+i} \right)^2 + \dots + \left(\frac{1-d}{1+i} \right)^{n-1}$$

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$i \rightarrow$		.00	.01	.03	.05	.10	.20	.50	(1) ÷ (7)
d	n								
↓	↓								
1.0	2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
.9	2	1.10	1.10	1.10	1.10	1.09	1.08	1.07	1.03
.5	4	1.88	1.86	1.83	1.81	1.75	1.66	1.48	1.27
.3	7								
.2	10								
.1	20	8.80	8.28	7.40	6.69	5.40	3.98	2.50	3.52
.07	28	12.2	11.4	9.71	8.45	6.41	4.44	2.63	4.63
.05	40	17.4	15.4	12.4	10.3	7.31	4.80	2.73	6.37
.03	67	28.7	23.6	16.9	13.1	8.45	5.22	2.83	10.14
.02	100	43.5	32.4	20.5	15.0	9.17	5.45	2.88	15.10
.01	200	87.0	49.5	25.7	17.5	10.0	5.71	2.94	29.59
.00	∞	∞	101	34.3	21.0	11.0	6.00	3.00	∞
-.01	∞	∞	∞	51.5	26	12	6.3	3.1	∞

2d row ÷ next
to last row

0 .01 .03 .05 .10 .18 .36

Handwritten signature

Use this to illustrate how i -differences do not much affect valuation, hence allocation, of buildings up to 40 yrs. old. (altho they do discourage hi- i types from buying mgnt Δ s of durability past say 30-40 yrs. Insofar as they are able to write them)

To show how Δt pushes use plans toward present, focus on mgnt years. This is legit. ~~as~~ t affects everyone, discourages bldg. needlessly.

$$V = \frac{1}{(1+i)} \frac{1 - \left(\frac{1-d}{1+i}\right)^n}{1 - \frac{1-d}{1+i}} = \frac{1 - \left(\frac{1-d}{1+i}\right)^n}{i+d} \approx \frac{1 - (1+i+d)^{-n}}{i+d}$$

Adequacy of base. -- math. of increments

If pull in demand, will not destroy outer values, but relocate them. Redistribute base from fast suburbs to

lean centrals. In process, add net value, by Δ synergism.

Moreso, by using better land -- base-point

Adequacy 2 parts

A. Small local open es.

Same as now, except for possible (unlikely) loss of Hamstead

unhappy.

B. Nation

1. No Hamstead concern -- strike that.

2. Some more of Δ s.

— Farm land Δ of \$5-10b a year. These Δ s are inc. -- Havg. Simons inc -- and are part of tax base.

Even if 100% l.v.t. i.e., the Δ in public equity is an income as well as Δ in prt. equity. Public can convert into current cash by borrowing on it if neeq.

— Farming is a declining indy. City l.v. have Δ d much faster, and from higher base.

— Mineral reserves. They Δ ydy. -- it is their nature.

Owner can't ~~deduct~~ ^{expense} leasehold costs. But he can recover them later by depletion. And even if % depletion were removed, and only cost recovery were allowed, The

lapse of time between purchase and tax liability would reduce tax burden greatly. Even if cost recovery never allowed, lapse of time reduces burden.

This brings land back to mind. If appreciating, isn't there a reduced tax burden bcs. of long time? And even if not appreciating?

$$1 = \frac{a}{i} = \sum_{n=0}^{\infty} \frac{a_n}{(1+i)^n}$$

$$(1+t)^n = (1+i)^n (1-t)$$

$$\frac{1}{x} = \frac{a(1-t)}{i}$$

$$1 = \frac{a(1-t)}{r}$$

$$a = i$$

$$r = \frac{i}{a(1-t)}$$

$$r = i(1-t)$$

$$t \rightarrow i \quad \frac{r}{i}$$

$$1 = \frac{a}{i-g}$$

$$1 = \frac{a(1-t)}{r-g}$$

$$a = i-g$$

$$r-g = (i-g)(1-t) = i - i t - g + g t$$

$$r = i(1-t) + g t$$

$$r > i(1-t)$$

$$\begin{aligned} \text{If } i &= 5\% \\ t &= 50\% \\ g &= 2\% \end{aligned}$$

$$r = .65 \times \frac{1}{2} + .01 = .025 + .01 = .035$$

a 40% increase!

Also, a nice, simple demonstration, usable in non-technical article.

In ~~a~~ closed ec., some shift of cap. values to restore common value of r for all invts.

In open ec., entire adjustment is in relative cap. values.
Taxes reduce value of apprais. land more than other.

Model needed where site renewed periodically

$$PV, \text{ or } I, = a + \frac{\cancel{a+g} a(1+g)}{(1+i)^1} + \frac{a(1+g)^2}{(1+i)^2} + \dots$$

$$= a \frac{1}{1 - \frac{1+g}{(1+i)^2}}$$

$$I = \frac{a(1-t)}{1 - \frac{1+g}{(1+i)^2}}$$

$$1 - \frac{1+g}{(1+r)^L} = a(1-t)$$

$$\frac{1+g}{(1+r)^L} = 1 - a(1-t)$$

$$(1+r)^L = \frac{1+g}{1 - a(1-t)}$$

$$r = \left[\frac{1+g}{1 - a(1-t)} \right]^{\frac{1}{L}} - 1$$

$$r \propto \frac{1}{L} \quad ? ?$$

$$a = \sum_{n=0}^L R_n (1+r)^{-n} = \sum_{n=0}^L C_n (1+r)^{-n}$$

$$1 = \frac{\sum_{n=0}^L [R_n - C_n] (1+r)^{-n} (1-t)}{1 - \frac{1+g}{(1+r)^L}} = \frac{R_n \frac{1 - (1+r)^{-L}}{r} (1-t)}{1 - \frac{1+g}{(1+r)^L}}$$

$$(1+r)^L = \frac{1+g}{1 - \dots}$$

If land appreciates, even though never sold, the effective income tax rate may be reduced.

$$\text{Let } S = \frac{a}{i-g}$$

Now impose inc. tax on a.

I. Assume closed ec, so $r < i$, and S remains same.

$$S = \frac{a}{i-g} = \frac{a(1-t)}{r-g}$$

Solve for r

$$r-g = i - it - g + gt$$

$$r = i(1-t) + gt = i + t(g-i)$$

Now the S of const. land must fall relative to S of apprais. land to restore eq. It may be S of apprais. land that rises.

II. Assume open ec. i remains the same, S falls.

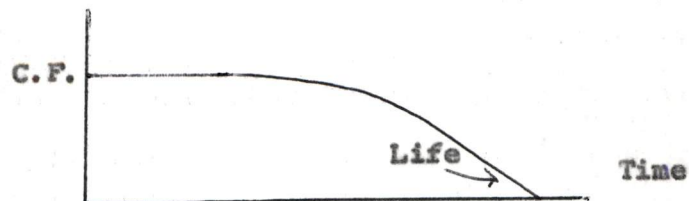
$S' \equiv S$ after taxes.

$$S' = \frac{a(1-t)}{i-g}$$

This makes it seem that tax hurts value of apprais. land as much as of constant-value land. This is not consistent w. previous findings. Reconcile.

Unsolved Problems to Overcome; refinements needed to models;
Project on U.S. Treasury policy and longevity. June 25, 1968

1. I need a neat cash flow model where the flow declines at an increasing rate, so:



I worked one out, but it is clumsy. See if you can make a better one. Here is mine:

$$CF = [1 - (1-g)^L] + [1 - (1-g)^{L-1}] + \dots + [1 - (1-g)^{L-L}]$$

(I shouldn't add the elements of the series above, they would never be summed.)

Where L is Life and g is a small number less than one. --
normally around $g = .05$.

{One problem above is that the definition of CF depends on Life. So in an equation where taxation changed Life, that would change the defined CF. Can solve problem by reserving L as a constant referring to original Life without tax influence; let $\underline{L}$ be life under tax influence.}

The problem is to find a neat, brief, "adroit" expression for the PV of CF. You may remember that these expressions may be changed by manipulating the end points. The best I could do is:

$$\begin{aligned}
 PV &= (1+r)^{-1} [1 - (1-g)^L] + (1+r)^{-2} [1 - (1-g)^{L-1}] + \dots + (1+r)^{-L} [1 - (1-g)^{L-L+1}] \\
 &= \frac{1 - (1-g)^L - \frac{g}{r(1-g)} [1 - (1+r)^{-L}]}{r - \frac{g}{1-g}}
 \end{aligned}$$

I have the work sheets on this if you need them.

An obvious minor improvement is to let $(1-g) \equiv h < 1$. But this is just notational.

Bruce believes this CF pattern is more realistic for income buildings than our others.

2. Introduce a land input and land salvage item into every model we use. (This is preliminary to maximizing PV, which doesn't make much sense without land.)

These models are:

- 1) Point
- 2) Full salvage
- 3) Depreciating
 - a. Constant annuity
 - b. Diminishing annuity
 - i. Constant % decline, $CF=1; (1-g); (1-g)^2; \dots; (1-g)^{L-1}$
 - ii. Constant decline, $CF=1; (1-g); (1-2g); (1-3g); \dots; (1-(L-1)g)$
 - iii. Rising decline--(problem 1, above)
 - c. Rising annuity of limited life

Timing of land salvage must be consistent with determination of economic life. That is, it must always be true that $CF \geq S \cdot i$ as a marginal condition at the year of life, L , without taxes. With taxes, $CF(1-t) \geq S \cdot r$. This is one way to determine the effect of taxes on life.

In previous models, we have assumed a life without taxes, not specifying how life was determined. It was enough for us to show that the introduction of taxes disturbed a no-tax equilibrium.

We can still do that for constant annuities. But for diminishing annuities, land salvage is the basic determinant of life. In point cases, land salvage is an element--remember Faustmann.

In the point case, our model would look so:

$$(C+S)(1+r)^L - S = [(C+S)(1+i)^L - S](1-t) + t \cdot C(1+r)^{L-w}$$

Where $S \equiv$ site value

$w \equiv$ write-off year

Note that expensing ($w=0$) now fails to achieve tax exemption, because $r < i$.

But you could show the tax is all on land income.

3. Show the effect of higher or lower S on economic life.

The answer may not be the obvious one that higher S means shorter life. If Co remains unaffected, yes. But higher S means a need to economize on land; a motive to substitute capital for land. That is done by piling up more capital on each site.

I am quite sure that higher S tends to lengthen life, on balance, therefore. That is for buildings. I am not so sure about trees, etc.

If you are in the dark, we can go over this. I have a good deal on it.

This is more for me than Bruce. The variable here is not t but S. However, a sensitivity analysis would be useful, showing how the basic relationship of L and S is affected by different t; [also by different write-off formulae, but that is too much.]

4. Recast each tax model, maximizing PV of S, instead of r.

$$PVS = \frac{\sum_{n=0}^L [R_n(1+i)^{-n}]}{1 - (1+i)^{-L}} \quad \text{is the basic formula, assuming}$$

perpetual repetition of the life cycle.

- a. Plug in different CF and write-off patterns.
- b. Modify formulas for assumption of rising re-use values of site.

- i. Perpetual constant % rise--this is simplest, analytically.
- ii. One shot rise.
- iii. (Optional--if you want to apply probability theory)
Uncertain future rises.

Note that a tax policy which levels off high returns,

more than low is biased against risk taking. On this basis you can analyze effects on risk, etc. I think it is true that higher k result in greater bias--do a sensitivity analysis.

Here, I have two interest.

(a) For Bruce, effect of Treasury policy on incentive to invest, longevity, etc.

(b) For me, effect of anticipated land value increments on same.

c. Review and organize print-outs from Don K. on parts of this problem.

List unfinished work--empty cells in matrix.

("Lists and files" are the methods of organizing work.

Don't make me do all the organizing--I need your help on that, too.)

d. Look for and make a list of acute biases, and their sources.

E.g., if you find that SL depreciation works tolerably well for a constant annuity, don't rest there. The small bias you do find suggests there are other CF patterns between which there is a greater bias.

Also, note what happens when tax life, TL, is less than economic life, EL. Every analysis could have 4 variants:

i. $TL = EL$

ii. $TL = \frac{EL}{k} \quad (k > 1)$

iii. $TL = EL - k$

iv. $TL = Z \quad (\text{a constant; independent of economic life})$

It would be too much to do each of these, all through, our task would blow up. Instead, from the matrix of all possibilities select those most interesting because:

-- Most realistic

-- Most in need of correction because they introduce most bias.

The most realistic are ii and iv, so stick with them.

In iv, $TL = Z$, Z is independent of EL for broad classes of assets; but it depends on the class.

We have in past analysis neglected the problem of $TL < EL$. But it may be more important than the depreciation pattern.

5. The payoff. The case of ownership turnover. Use all our background to build 2 models: a, max. r ; b, max. PVS.

To discover:

a. By how much does redepreciation increase the incentive to new building by raising r or PVS.

In the second approach, (max. PVS) note that the rise in PVS to redevelopers ("challengers") is offset in part by a higher site value to defenders, who gain by redepreciating old defender buildings. In the first approach, the same effect is seen through higher site costs! lowering r from challenger.

It is entirely possible that the gain in incentive is wholly offset, or more than offset, by higher site costs. That is, defenders may be strengthened more than challengers. This seems

likely because aid to defenders is immediate; aid to challengers is deferred to future when they become defenders.

b. When is the optimal year of ownership turnover?

Finding this is a necessary incident to answering (a), anyway. But this aspect wants separate emphasis for reasons of public relations. Treasury officials are concerned about this angle of policy. High ownership turnover in slums leads to neglect of maintenance; high sales costs, including information costs--an example is painting over rotten wood to help a sale. There is a strong social bias against high ownership turnover.

This is therefore a sensitive point, and one we want to use to begin to get through to Treasury officials.

c. Does multiple depreciation add to a longevity bias?

I surmise it does, because the benefit comes from retaining value in buildings when they are old.

d. Does multiple depreciation raise site values? I surmise yes--cf. (a). This adds to the longevity bias--cf. 3, above.

Also, multiple depreciation makes it possible to depreciate land, if it has appreciated under an old building, by allocating the land value to the building. Does this stimulate new building more than the reverse?

For this analysis, you need to know the rules. They are:

-- Take Double Declining Balance, DDB, on initial owner only.

DDB is double the SL rate, but applied to the declining, undepreciated balance. If $C_0 = 1$, you write off as follows:

<u>Year</u>	<u>Allowable Depreciation</u>
1	$\frac{2}{L}$
2	$\frac{2}{L} (1 - \frac{2}{L})$
3	$\frac{2}{L} (1 - \frac{2}{L})^2$
x	$\frac{2}{L} (1 - \frac{2}{L})^{x-1}$

The PW for this works out to:

$$PW = \frac{2}{L} \frac{1 - \left(\frac{1 - \frac{2}{L}}{1 + r} \right)^L}{r + \frac{2}{L}} + \left(\frac{1 - \frac{2}{L}}{1 + r} \right)^L$$

except in case of sale you cannot write off the last term, the undepreciated book value, directly. Rather, you deduct it from sale proceeds.

-- Pay capital gains tax on sale price less undepreciated book value.

Rate is 1/2 ordinary rate, up to maximum of 25%.

-- Redepreciate from new basis, which is buying price. Take 1 1/2 on declining balance. Switch to SL at will.

-- Repeat until you get tired of it.

Note that the older a building gets, the shorter L you can reasonably choose and get away with. This more than offsets the penalty of having to shift from 2 to 1 1/2 declining balance. Take account of this.

There is a "Recapture" provision you must throw in. I forget the rule--check it out.

Also find out what assets qualify for multiple depreciation. If only real estate, it adds to bias for landowners, and of course longevity.

Your contribution is not just to provide insights and new approaches, but to organize materials:

- make matrixes of all possible combinations of elements of problem.
- select the cells most worth focusing on.
- set up models.
- present output for reader comprehension and easy survey.
- emphasize significant results.

~~"It cannot be"~~

1. Relation of Real Estate Investment to
National Economic Goals

This part of the report contains four sections. First, national economic goals are stated in generalized, oversimplified terms. While some theorists would be quite horrified at the generalizations and would argue with the way they are stated or some of the contradictory implications, most would agree with the general principles indicated. Second, general Treasury policies which are a part of affecting these goals are outlined. This is designed to indicate the obvious point that the Treasury has a number of policy approaches for which real estate is just a part. Further, some changes in methods of real estate taxation raise questions of the need or appropriateness of applying such changes to other assets. Third, the tools the Treasury has for affecting real estate investment are reviewed to orient the reader to the scope of alterations available. Fourth, a brief discussion is presented of the relationship of the real estate profitability variable rate of return to macroeconomic models of the economic role of capital asset formation.

1.1 Economic Goals

Economically, we (as a nation) wish to (1) achieve a high long term growth rate in net national product (NNP).

less "business
taxes" ?

Omitting throughout this analysis the balance of payments component of gross national product (GNP), we adopt a simple definition of NNP as GNP less the decrease in the value of the stock of assets through economic depreciation. While accomplishing this long term growth, we wish to minimize the short term fluctuation in GNP. GNP is used here (rather than NNP) since it is presumed that we want the inevitable decline in the value of a given stock of capital assets to be gradual and smooth rather than to be fluctuating.

Growth which is a result of the expansion of the capital asset base is better than that resulting simply from replacement of depreciating assets. Thus we want to maximize net new capital investment. (This is quite important to Treasury policy which affects asset lives.) While encouraging investment it is also important to minimize short term price fluctuation, and we wish the long term rise in GNP to have either a small price rise component or to average zero price increase. (1)

In order to achieve long term growth, we must maximize productivity of labor and capital. Further, we wish to insure equimarginal productivity of investment in labor and capital so that the greatest increase in NNP is achieved from a given financial commitment to labor and/or capital.

(1) In giving these alternatives we, thus, do not want to enter the argument as to whether a "creeping" price increase is beneficial. Rather, we emphasize that "galloping" price increases are not a goal.

When discussing interest rates in the total economic picture, it is of course quite difficult to speak about "the correct" structure of interest rates. The proper interest rate becomes a question of the proper reward for consumption postponement, since the interest rate equals consumption postponement plus a premium for uncertainty. A high reward favors savers and reduces consumption. It also raises the cost of capital investment. A low reward decreases the cost of capital but also decreases savings. The argument is of course the question whether it is better to stimulate the economy by increased consumption demand or decreased cost of capital investment. Unwilling to enter this fray, the author simply concludes that the interest rate goal is to optimize and achieve as quickly as possible the market clearing equilibrium rate. Thus we wish to optimize MPS, MPC and MPI, the marginal propensities to save, consume and invest. We can say, however, that it is desirable to reduce the uncertainty or risk premium component of the interest rates.

Since economic activity is supported upon an active money supply base, we wish to minimize the portion of total savings which is hoarded in balances and not productively invested. We also wish to reduce the savings which remain committed to economically inefficient assets through unduly harsh tax or other constraints. We also want to maximize the present value of the asset base - both current and

future assets. A factor involved in this is the turnover of real capital. It is unclear from the literature whether this means we should minimize or maximize the economic life of replaceable capital assets (those other than land).

✓ Therefore much of the work herein will be directed at the effect of tax policy on the capital turnover ratio.

There also appears to be agreement that it is better to accomplish economic goals through the private sector to the extent possible rather than through governmental sectors. Because of this it is desired to minimize the necessary corrective governmental activity of a fiscal, monetary and redistributive nature, while accomplishing the preceding goals.

1.2 Treasury Policies

The Treasury, through its influence on fiscal policy, should emphasize those fiscal expenditures and the net difference between expenditures and receipts so as to maximize long term growth in NNP. Specifically its expenditures should have a high multiplier effect on consumption and investment rather than a substitution effect. Its policies should insure that replacement at least equals depreciation and that a proper level of net new capital is produced. It seems reasonable that tax policies⁽¹⁾ should maximize the

(1) Tax policies refer to the form and incidence of tax laws and regulation rather than to the aggregate size of collections relative to expenditures (the deficit or surplus) embodied in fiscal policy.

long term expansion of productive capacity, therefore it should maximize the portion of capital investment which is an addition to, rather than a replacement of, the stock. Treasury tax policy, combined with monetary and fiscal policy, should also minimize fluctuations in GNP.

It is also necessary that prices be somewhat stabilized. Because of this, the Treasury should try to minimize short term price fluctuations; and it should see that its own activities for other purposes have as little destabilizing effect on prices as possible, consistent with accomplishing the goals of the activities. Further, its efforts toward countercyclical price stability should take account of the effects on the other goals. The Treasury should be cognizant of the effect of creeping price rise on the value of the stock of physical assets and on their replacement costs.

In a period of underemployment of labor, Treasury policy which stimulates capital investment should be directed at those investments which are labor-increasing or labor intensive rather than labor-replacing. The method for collecting tax revenues should be one which bears least burden on labor incentives, labor and capital productivity, capital replacement and expansion.

Regarding the reduction of the uncertainty premium in interest rate determination, much of an administrative nature can be done. The Treasury should:

- a) maximize the degree of predictability of Treasury policy and minimize the amount which seems capricious and subject to unpredictable change in the minds of decision makers in the private sector.

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- b) maximize the number of prior Treasury rulings as to the tax qualification of proposed activity prior to investment.
 - c) maximize national uniformity of regulations and enforcement.
 - d) minimize the differences of categories of investors in order to increase the number and equality of bidders for investment opportunities.

Treasury tax incentives should also insure that savings are not committed to economically unproductive uses, when judged on a before-tax basis, simply to achieve beneficial after-tax results. This implies that incentives should reward entrepreneuring and penalize hoarding. The Treasury should provide positive investment incentives for those types of investment which are in consonance with national goals. Funds should be moved from hoarders to consumption and/or investment through changing the propensity of hoarders to hoard or by changing the funds to consumers or investors by redistribution. The Treasury should also reduce uncertainty regarding the resource base and its value by providing incentives to discovery, and to development of the most efficient resources first (those with the highest present value). Thus, unproductive, remote land should not be improved for tax benefit before economically more efficient land.

In minimizing the necessary governmental activity, the Treasury should minimize the following:

- a) tax collection costs per dollar of receipts.
- b) the portion of equimarginal allocation of resources

between labor and capital and between consumption and investment which must flow through the Treasury for purposes of reallocation or redistribution.

- c) the amount of bias in private economic decisions which is motivated by tax reduction strategy.
- d) inducements to retain rather than distribute earnings for tax reducing rather than economic reasons --economically inefficient accumulation of surplus.

1.3 Treasury Real Estate Tools

The Treasury real estate tools are defined as those which apply specifically to real property investment. These tools may be classified according to the time at which the investor realizes their effects. The time periods are:

- 1) Beginning: when a new improvement is constructed for a first user.
- 2) Entrance or purchase: when an investor acquires a new or used asset.
- 3) Exit or Sale: when the investor sells or disposes of the asset.
- 4) End: the end of the economic life of the improvement asset.
- 5) Transaction Life: the holding period that the investor owns the asset; the time between entrance and exit or purchase and sale.
- 6) Economic Life: the time between beginning and end; the capital turnover period.
- 7) Tax Life: the asset life that the Treasury and/or investor sets for tax depreciation.

Although the allocation of incidence of a given Treasury real estate tool is somewhat arbitrary, it has been done to categorize thinking:

- Beginning: . investment tax credit on new improvements.
 . accelerated depreciation on new improvements by first user.

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- . expensing or capitalization and depreciation of site preparation costs.
- Entrance:
- . deduction of acquisition costs.
 - . prepaid interest.
 - . additional first year's depreciation.
 - . type of taxpayer entity to hold title.
- Life:
- . selection and substantiation of:
 - depreciation method
 - useful life for depreciation
 - salvage value
 - depreciable cost basis
 - . redepreciation of an asset which has been depreciated by a previous user.
 - . expensing or capitalizing and depreciating of maintenance and improvements to assets.
 - . holding period choice of investor.
- Sale:
- . method of taxation of disposition:
 - capital gains
 - exchange or trade
 - recapture of accelerated depreciation
 - recapture of investment credit
 - . classification of seller as investor versus dealer.
- End:
- . effect of actual resale value of improvement versus the disposition of land with and without structure.

These tools, and others, affect the type of investor, the type, location and quality of property, the economic life of the structure and the holding periods or transaction lives of investors. Through this they affect the type and

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timing of initial and additional expenditures or capital investments in an important component of the economy. These expenditures have multiplier and substitution effects on other sectors of the economy. Therefore, it is important to look carefully at the effects of alternative tax policies (applied through Treasury real estate tools) on real estate investment performance.

Again it is mentioned that it is public consensus that it is appropriate to minimize the corrective governmental activity necessary to accomplish national goals. Pechman disagrees and says,⁽¹⁾

"Taxation - the method by which a nation implements decisions to transfer resources from the private to the public sector - is a major instrument of social and economic policy. It has two goals: to distribute the cost of government fairly by income classes (vertical equity) and among people in approximately the same economic circumstances (horizontal equity); and to promote economic growth, stability and efficiency."

This section outlined national economic goals and Treasury policy that would be co-ordinated with these goals. We also briefly outlined real estate tools that are available to the Treasury. These tools or tax variables effecting the individual investor in his decision process are next stated in model form. Later in the report the relative impact of these tools are evaluated and recommendations are made for revision and updating.

(1) Joseph A. Pechman, Federal Tax Policy, Washington, D.C., The Brookings Institution, 1966, p. 5.

1.4 Econometric Models and the Rate of Return Model

This study is conducted on a micro-theory level, but it is, of course, closely related to certain problems which are usually classed with macro-theory.

This relationship between our models and the macro models can be brought into focus by examining some macro models of investment behavior and noting that such models give an important place to many of the same variables which appear in our model.

An investment model developed by Klein includes the following equation¹ for expenditures on non-farm residential construction:

$$I_h = -11.3 + .0764 \frac{Y - T}{g_h} - .776 i_L + .0011 F + .00812 (I_h^s)_{-1}$$

I_h = Expenditures on non-farm residential construction

$Y - T$ = Disposable personal income

g_h = Implicit deflator, non-farm residential construction

i_L = Average yield, corporate bonds, per cent

F_s = Number of marriages

I_h^s = Number of housing starts

Hall and Jorgenson use the following equation in their model²:

$$I_t = \sum_{s=0}^{\infty} \mu_s \Delta K_{t-s}^+ + \delta K_t$$

Gross investment in period t , I_t , is the sum of a weighted average of past changes in desired capital and replacement investment. The change in desired capital in period $t-s$ is ΔK_{t-s}^+ ; the parameter μ_s is the proportion of the

(1) Lawrence P. Klein, "A Postwar Quarterly Model: Description and Applications", *Models of Income Determination*, 1963, pp. 11-57.

(2) Robert E. Hall and Dale W. Jorgenson, "Tax Policy and Investment Behavior", *American Economic Review* (June, 1967), pp. 391-414.

change in desired capital in period $t-s$ that results in investment expenditures in period t . Replacement investment is proportional to capital stock K_t ; the constant of proportionality is δ , the rate of replacement.

For a Cobb-Douglas production function, the desired level of capital K^+ is:

$$K^+ = \alpha \frac{pQ}{c}$$

where p is the price of output, Q is its quantity, c the rental price of capital, and α the elasticity of output with respect to capital.

Sherman Maisel has developed three equations of interest:

$$(1) \quad HU_{STS} = f \left[\Delta HH, HU_{REM}, (HU_{VAC})^{-1}, \left(\frac{CPI_{SR}}{P_{ICNFR}} \right)^{-1}, \sum_{i=4}^{-2} (RM_{GBS3})^i, (HU_{STS})^{-1}, (HU_{STS})^{-3} \right]$$

$$(2) \quad I_{CO}^{54} = f \left[\sum_{i=4}^0 w_i IP_{CO_i}^{54}, \sum_{i=4}^{-2} \left(\frac{Y_d}{P_{GNP}} \right)^{-i} \right]$$

$$(3) \quad I_{CNFRO}^{54} = f \left[HU_{AVL}, \left(\frac{P_{ICNFR}}{P_{GNP}} \right)^{-2} \right]$$

HU_{STS} = number of private housing units started

I_{CNFRO}^{54} = current level expenditures on residential non-farm construction other than new dwelling units

I_{CO}^{54} = private nonresidential, nonbusiness construction, quarterly rate

HH = number of households

HU_{REM} = net removals of housing units

HU_{VAC} = adjusted vacant available housing units

CPI_{SR} = rent component of consumer price index

RM_{GBS3} = average market yield of three month US Treasury bills

P_{ICNFR} = implicit price deflator for GNP expenditures on non-farm residential construction, 1954 = 1.00

IP_{CO}^{54} = value of building permits for private, non-residential, non-business construction, billions of 1954 dollars

Y_d = disposable personal income, billions of dollars

P_{GNP} = implicit price deflator for GNP, 1954 = 1.00

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HU = total available housing units, thousands
AVL

Some of the variables in these models cannot be directly influenced by the Treasury, for example, the number of marriages. Others are important in one area of real estate investment but not for real estate investment in general. The housing variables of the Maisel model⁽¹⁾ are an example. Variables falling into these two categories have been excluded from the micro-model.

The variable which appears in all of these models is the opportunity cost of capital. In Klein it appears as i_L , the average yield on corporate bonds, expressed as a percentage. In Hall and Jorgenson the desired level of capital is inversely related to the rental price of capital. Maisel's equations include the average market yield of three month US Treasury bills. In the micro model attention is focused on the rate of return to the investor expressed as the internal rate of return. The internal rate of return is one measure of the cost of not making a particular investment, and is therefore analagous to the measures used in the macro models.

A general expression for the internal rate of return would be:

$$V_0 = \sum_{j=1}^{\infty} \frac{R_j}{(1+r)^j}$$

where

V_0 = initial cost

R_j = return in year j

r = internal rate of return

In the micro-model used in this study this general formula becomes much more complicated. All of the following variables are included:

(1) Sherman Maisel, "Nonbusiness Construction," in J. Duesenberry, G. Fromm, L. Klein, and E. Kuh (eds.) The Brookings Quarterly Econometric Model of the United States, 1965, pp. 179-203.

Formulas for Rate of Return and Depreciation⁽¹⁾

V_0 = initial property value = total cost

c = portion of property cost which qualifies for the 7% investment credit

V_n = property value in year n ; assumed herein to equal selling price net of selling costs

n = holding period

L_0 = initial loan amount

O_0 = initial owner's equity amount = down payment = $V_0 - L_0$

E_j = operating earnings after operating expenses including property tax but before debt payment, depreciation and income tax

I_j = interest amount in a given year, j

A_j = amortization amount on the loan in a given year, j

K_j = total loan payment = $I_j + A_j$

D_j = depreciation expense in a given year, j

d = portion of the property cost which qualifies for depreciation or the depreciable cost basis

u = useful life for depreciation purposes

s = portion of the depreciable cost basis estimated to be salvage value

t_j = income tax rate in a given year, j

ct_j = capital gains tax rate in a given year, j ; with a maximum value of .25 if $n > .5$ (over 6 months)

r = internal rate of return on equity

(1) The reader may wish to refer to similar formulations developed by R. Bruce Ricks and Sui Wong which appear in Paul F. Wendt and Sui Wong, "Investment Performance: Common Stocks versus Apartment Houses", The Journal of Finance, December, 1965, pp. 638-640. Those in this appendix utilize different symbols and concentrate on more specification of the depreciation variables.

The expression for the internal rate of return is:

$$V_0 - L = \frac{cV_0(.07)}{(1+r)} \quad (1a)$$

$$+ \sum_{j=1}^n \frac{E_j - t_j(E_j - I_j - D_j) - I_j - A_j}{(1+r)^j} \quad (1b)$$

$$+ \frac{V_n - ct_j[V_n - (V_0 - \sum_{j=1}^n D_j) - (L_0 - \sum_{j=1}^n A_j)]}{(1+r)^n} \quad (1c)$$

$$- \frac{(t_r - ct_n) \left[\sum_{j=1}^n D_j - \left(\frac{n}{u} \right) (1-s) (dV_0) \left(1 - \left(\frac{n \times 12 - 20}{100} \right) \right) \right]}{(1+r)^n} \quad (1d)$$

where

the numerators of the right hand side are as follows:

1a = investment credit

1b = cash flow in a given year, j

1c = resale proceeds after tax and after loan repayment

1d = the recapture amount subject to the following limits:

if $F_n \equiv 1 - \left(\frac{12n-20}{100} \right)$ then F_n has the limits $1 \geq F_n = 1.20 - .12n \geq 0$.

if $E, t, ct,$ are constants and c is zero (no investment credit), and the loan is a level payment amortizing one ($K = I_j + A_j = \text{constant}$) and if as assumed in this paper $V_n = V_0$ (resale value remains constant)

then (1) becomes

$$O_o = \sum_{j=1}^n \frac{E(1-t) + t(I_j) + t(D_j) - K}{(1+r)^j} \quad (2a)$$

$$+ V_n - ct \left(\sum_{j=1}^n D_j \right) - (L_o - \sum_{j=1}^n A_j) \quad (2b)$$

$$- (t-ct) \left[\sum_{j=1}^n D_j \left(\frac{n}{u} \right)^{1-s} (dV_o) \left(1 - \left(\frac{n \times 12 - 20}{100} \right) \right) \right] \quad (2c)$$

where

$$K = L_o \left(\frac{i}{1 - (1+i)^{-m}} \right)$$

$$A_j = A_1 (1+i)^{j-1}$$

$$A_1 = K - (iL_o)$$

For straight line depreciation:

$$D_j = \frac{1}{u} (1-s) dV_o \quad (3)$$

For double declining balance depreciation:

$$D_j = \frac{2}{u} (dV_o - \sum_{j=1}^{j-1} D_j) \quad (4)$$

For 150% declining balance replace 2 with 1.5 in (4)

For sum of the years digits depreciation

$$D_j = \frac{u - (j-1)}{1/2(u(u+1))} (1-s) (dV_o) \quad (5)$$

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Solving (1) for r gives the internal rate of return on equity after tax and after financing.

- if $t = 0$ and $ct = 0$ (1) becomes the before-tax rate of return
- if $d = 0$ (1) becomes the before-depreciation rate of return
- if (1d) is omitted or if the straight line method is used (1) becomes the before-recapture rate of return

For simplicity, this model can be restated in terms of the simple general model above. R is now a function of operating income, taxes, and debt service; and taxes are a function of depreciation method, depreciable cost basis, salvage value, holding period, useful life, recapture, and interest on debt.

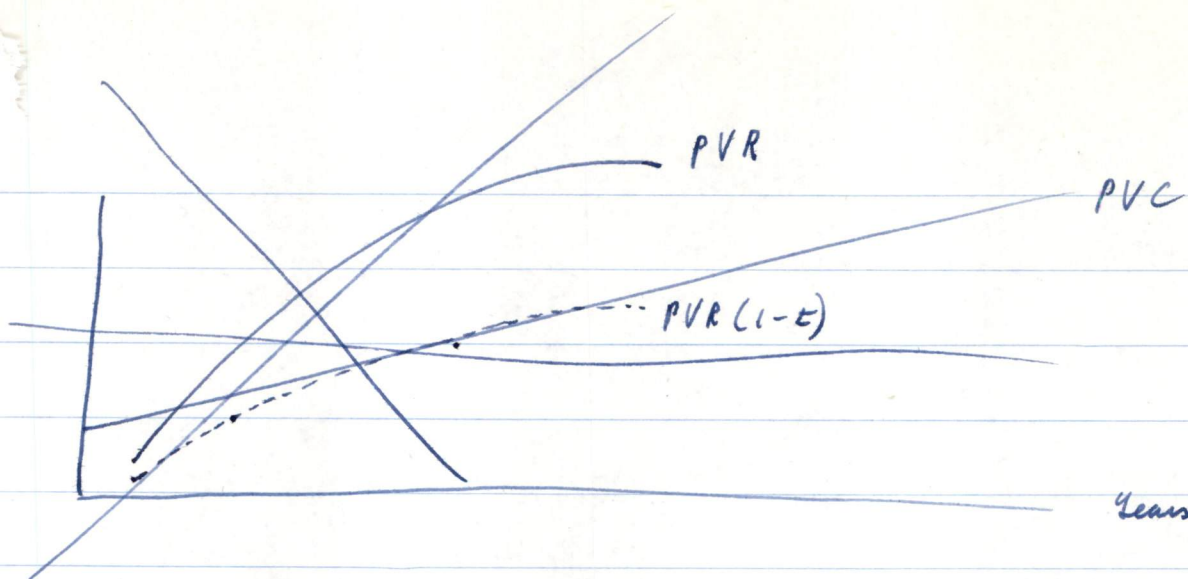
It is possible to give the internal rate of return an important position in a macro-model by summing this micro-model across property type and then summing across investor type, but this study remains at the micro level. The construction of a macro model is the responsibility of Professor Graybeal.

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Appendix

For further explanation of economic goals see:

- (1) Paul A. Samuelson, Economics; An Introductory Analysis, 7th ed., (New York, McGraw-Hill, 1967)
- (2) Edward Shapiro, Macroeconomic Analysis, (New York, Harcourt, Brace & World, 1966)
- (3) Myron H. Ross, Income: Analysis and Policy, (New York, McGraw-Hill, 1964)
- (4) Daniel Hamberg, Principles of a Growing Economy, (New York, W. W. Norton & Co., 1961)
- (5) John P. Lewis and Robert C. Turner, Business Conditions Analysis, 2nd ed/. (New York, McGraw-Hill, 1967)
- (6) Thomas F. Dernburg and Duncan M. McDougall, Macroeconomics, 2nd ed., (New York, McGraw-Hill, 1963)
- (7) Milton Friedman, "A Monetary and Fiscal Framework for Economic Stability," The American Economic Review, vol. 38, (June 1948) pp. 245-264.



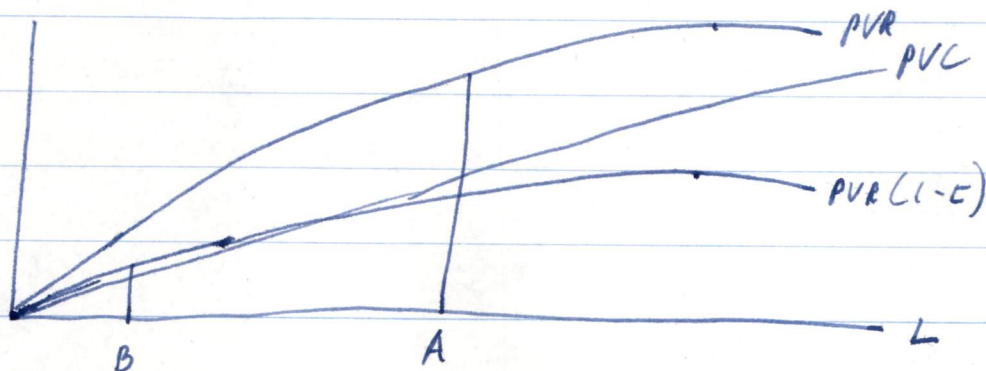
Leverage of ^{gross-income} tax on ~~gross income~~ ^{site value.}

General case. Max S. Repetition cycles

in perp.

$$S = \frac{\sum_0^L [(CR_n - C_n) (1+i)^{-n}]}{1 - (1+i)^{-L}}$$

Landowner faced w. spectrum of options. Assume Life of discrete cycle of inv. - biz. is indep. vble. At each L, do your best. Trade higher on PVC for higher PVR. Competition pulls land use into the popular things, so ~~at the margin, the~~ the unpopular ones become reasonable alternation for any one individual, poised on margin of market. ^{So R curve partly conforms to C curve.} So above Fig. is unrealistic. In fact, there is a $PVR > PVC$ for every age. This adds to leverage of tax bias.



A is max $PVR - PVC$

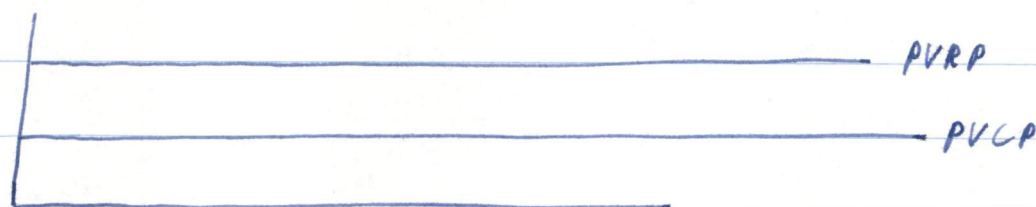
B is max $PVR(1-E) - PVC$

Whoops!

$$\frac{\Delta PVR - \Delta PVC}{PVR - PVC} \stackrel{?}{<} \frac{\Delta PVR(1-t) - \Delta PVC}{PVR(1-t) - PVC}$$

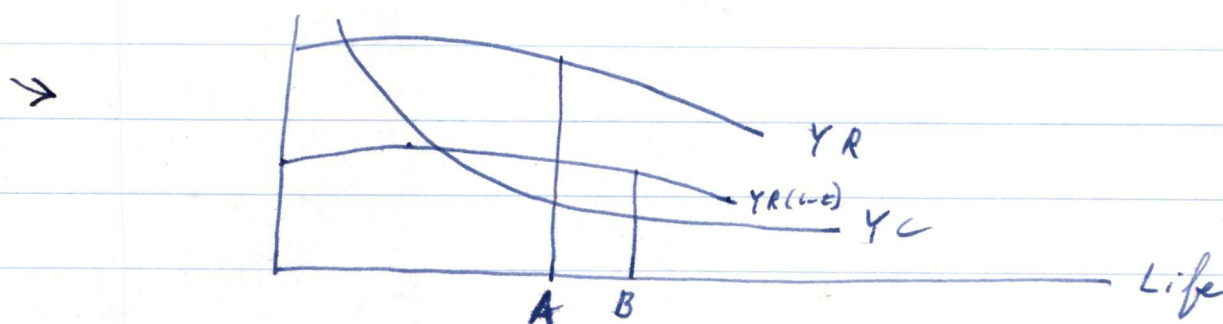
Graph that which is to be used: $\frac{PVR}{1-(1+i)^{-L}} - \frac{PVC}{1-(1+i)^{-L}}$

Say before taxes there are horiz., sep. complete indiff. among lives. Call them PVRP (PVR in Perpetuity.)



No intertemp. bias here! Only bias vs. intensity. I have to identify short life and intensity.

Graph annualized capt. input vs. rev. YR & YC



A is max. w/o taxes

B after taxes.

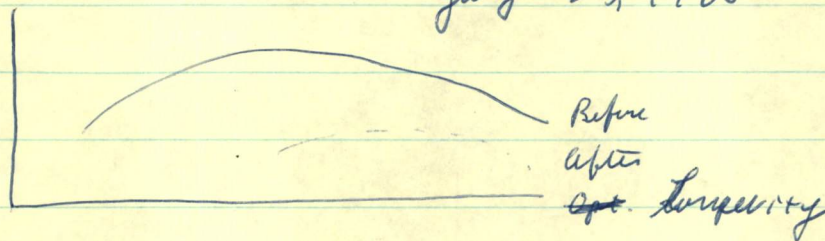
YR has same shape as PVRP, etc.

M+g, w. Ricks, Kaplan, Harris Friedman,
George Boutin, and Mark Told (hug?)

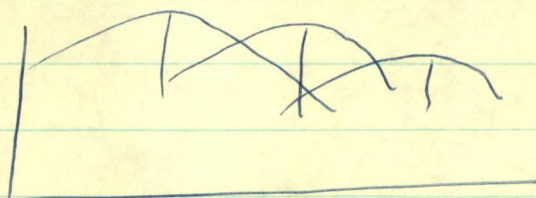
Stan Surrey, U.S. Treas.

July 23, 1968

PV of
land



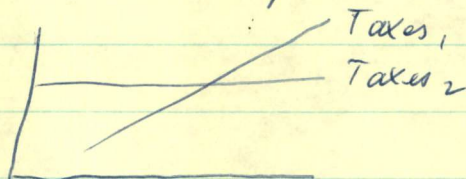
Before
after
opt. Surrender



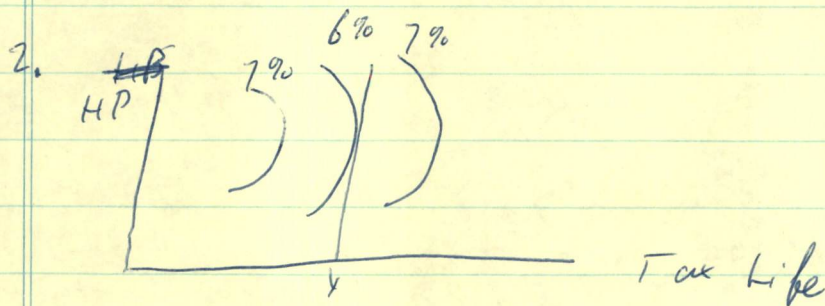
Succession of holders

Each successive holder holds to max r .
But all get different r .

People object to negative cash flow. Rule of thumb,
you sell when cash flow turns negative



So they prefer level cash flow. So prefer level write-off.



Given tax life x , best HP is a tangency.

Assumed these nos.

R. of R 10% (i)

Tax bracket 50%

Tax life 45 yrs.

Tax method SYD, SL

Holding pd. 1-70 (ave. 11)

Most investors conc. on rate of return method, not PV
" " use great leverage. Even rich ones. They
use all they can, reduce taxes to zero.

~~With~~ With hi leverage, Δ l.v. is very important to
investor's equity return. (Add to IDA report)

1. Stimulate constr.

Subject to, a) Min. trees. cost.

b) Max. holding pd.

[(Demolition & land clearing ^{makes} is an asset that lasts
forever.)
Demolition cost is treated as land cost, now.]

Sources on op. inc. from old bldgs -- Grebley Case,
etc.

Is now standard for lender to demand a piece of the
equity.

As my contribution, we op's in. decays exponentially,
thru rest of challenge. See if small changes can
have big effect on timing

Check-off Matrix - ~~annuity only~~ Sum of digits

Point input of \$1. time 0.

Deprecn. Pattern
Cash Flow Pattern
↓

S. L.			Acc			Skg. Fund			Acc-SL		
TL=EL	$TL = \frac{EL}{k}$	$TL = k$	$TL = EL$	$TL = \frac{EL}{k}$	$TL = k$	$TL = EL$	$TL = \frac{EL}{k}$	$TL = k$			

1. Point Output ①

1	2
3	4
5	6

2. Full Salvage

~~2. Full Salvage~~

3. Const. annuity

4. Dim. annuity
a. Const. %

b. S.L.

c. Accel.
drop-off

5. Growing annuity
a. Const. %

b. S.L.

① In this case, ~~then~~ in practice one would seldom actually use S.L. or other gradual write-off. However, the ~~at~~ results of write-off at one point of time have been recorded separately. The value of the present worth is to show the separable effect of a high salvage value at the end of an annuity.

**SCANNED
AS
FOUND**

S. L.

Acc.

S. F.

6. Rising Salvage
a. Const. ann.

b. Dim. ann.
(Const. % only)

c. Growing ann.

For every point on the grid there are two basic coordinates, and a module with six cells, numbered thus:

1	2
3	4
5	6

They refer to the ff., which should be done ~~for each~~ (or explicitly blanked out) for each point.

1. The basic equation, with $i = .10$, $t = .50$, $L = 1 - 70$.
2. $i = .02, .04, .08, .16, .32$.
3. The basic equation under conditions of: ~~risk~~
 - a. Predictable deviation of C.F. from basic pattern.
 - b. Uncertain " " " " " "

- A. → 4. Series of owners (especially re case 6)
- B. 5. Max. P.V. instead of $\leq$, assuming const. $i = t$.
Select ec. life to max. PV
Note effects on opt. ec. life.

6. $L < 1$

7. In tabulating, compare a short-lived annuity of declining yield (like a truck), ^{and $TL \geq EL$} against a long-lifer of const. or rising yield, ^{and $TL < EL$} after the full contract.

9:30

Comparing Accel. vs. Strt. Line, which has greater bias to longevity?

See App. by Richter & Kaplan.

Treas. Pol.

Ownership Turnover

Capl. Turnover

Capl. stock of nation.

1. Accel. vs. S.L.

4/a. Recapture

1b. Inability to change from accel. to SL

2 Double Deprec.

3 Life

a. Extended Life

b. Change in useful life --
sensitivity to real life

4 Inv. Credit

4/a Recapture

4b Expensing

5 200% vs. 150%

6 Reduction of Property Tax

SL deprec.

$$PV = \sum_{t=1}^L \left(\frac{d_t}{(1+r)^t} \right) = \frac{1}{L} \cdot \frac{1 - (1+r)^{-L}}{r} \quad (\text{Assume as mine problem!})$$

$$1 = \frac{i}{1 - (1+i)^{-L}} (1-t) \frac{1 - (1+r)^{-L}}{r} + t \frac{1}{L} \frac{1 - (1+r)^{-L}}{r}$$

$$1 = \left[\frac{i(1-t)}{1 - (1+i)^{-L}} + \frac{t}{L} \right] \frac{1 - (1+r)^{-L}}{r}$$

$$r = \left[\frac{i(1-t)}{1 - (1+i)^{-L}} + \frac{t}{L} \right] [1 - (1+r)^{-L}]$$

Problem: In analyzing SL, should I assume d_t begins at time 0 or time 1? If time 0, it becomes much more favorable to short lives, and so loses much intertemporal bias.

Answer: In practice, my first deprec. is taken at end of yr. when I buy the asset, or build it. This increases the bias against short assets. (Indeed, it may even be a main cause of it.)

~~Expensing~~ Expensing, in practice, has a similar bias bcs. you reduce yr. tax liability at the end of the year. To be sure, expenses laid out in last mo. of the yr. get deducted soon. But Δ inventory is treated as income -- curious how Eisner v. Macomber lets this kind of unrealized income be taxed!

Expensing, on other hand, may encourage waste. that Δ s life for 2-4 yrs. - So here is at least one bias to brevity.

But in general, all write-off systems are biased to longevity, bec. none of them let you write-off at time zero.

And SL , Σ is from 1 to m . Thus, the first write-off is at end of year; and for a one-year asset there is no benefit at all.

a. Let $L=1$

$$1 = (1+i)(1-t) \frac{1}{1+r} + t \cdot \frac{1}{1+r}$$

$$1+r = 1+i - t - ti + t =$$

$$r = i(1-t) \text{ ~~not a twist~~ }$$

[If write off were at 0, there would be great benefit, if $L=1$: $r = i(1-t) + tr$; $r = i$]

b. But also, for an infinite asset, there is no benefit at all

Let $L = \infty$

$$1 = i(1-t) \frac{1}{r} + t \cdot 0 \cdot \frac{1}{r}$$

$$r = i(1-t)$$

c. So note that SL can be given a large ~~twist~~ twist against longevity

simply by advancing all write-offs one year. That might be a good practical recommendation to make.

d. Let $L=14$

$$1 = \frac{.05}{1 - \frac{1}{2}} \cdot \frac{1}{2} \cdot \frac{1 - (1+t)^{-14}}{r} + \frac{1}{2} \cdot \frac{1}{14} \cdot \frac{1 - (1+t)^{-14}}{r}$$

$$\frac{r}{1 - (1+t)^{-14}} = .05 + \frac{1}{28}, \quad \text{Answer requires tables or computer. Use computer.}$$

Another way to group terms:
 $L \geq 1$

$$\frac{x}{1-(1+i)^{-L}} = \frac{i}{1-(1+i)^{-L}} - t \left[\frac{i}{1-(1+i)^{-L}} - \frac{i}{L} \right]$$

L	$\frac{i}{1-(1+i)^{-L}}$	$\frac{i}{L}$	$\frac{i}{1-(1+i)^{-L}} - \frac{i}{L}$
1	$1+i$	1	i
2	$\frac{i}{2i+i^2} = \frac{1+2i+i^2}{2+i} = \frac{1}{2+i} + i$	$\frac{1}{2}$	$\frac{3i+2i^2}{4+2i}$
3	$\frac{i(1+i)^3}{3i+3i^2+i^3} = \frac{1+3i+3i^2+i^3}{3+3i+i^2} \left[\frac{1}{3} + i \right]$	$\frac{1}{3}$	

Rate of return after SL & acc. deprecn.: a) different ~~time~~ total lives, L
 b) different deprecn. paths & hence different mean lives.

I, SL

a. Const. annuity.

$$1 = \frac{i}{1-(1+i)^{-L}} (1-t) \frac{1-(1+r)^{-L}}{r} + \frac{t}{L} \frac{1-(1+r)^{-L}}{r}$$

b. Full Salvage

$$1 = i(1-t) \frac{1-(1+r)^{-L}}{r} + \frac{(1-t)t}{(1+r)^L} : \text{if write-off at } L$$

SL:

$$1 = i(1-t) \frac{1-(1+r)^{-L}}{r} + \cancel{\frac{(1-t)t}{(1+r)^L}} + \frac{1-t}{(1+r)^L} + t \frac{1-(1+r)^{-L}}{L \cdot r}$$

Rewrite above two, to clarify their difference:

i. Deduct all at L

$$1 = i(1-t) \frac{1-(1+r)^{-L}}{r} + \frac{1-t}{(1+r)^L} + \frac{t}{(1+r)^L}$$

ii. SL

$$1 = i(1-t) \frac{1-(1+r)^{-L}}{r} + \frac{1-t}{(1+r)^L} + t \cdot \frac{1}{L} \cdot \frac{1-(1+r)^{-L}}{r}$$

c. Declining "annuity"

$$1 = \frac{i+g}{1-\left(\frac{1-g}{1+i}\right)^L} (1-t) \frac{1-\left(\frac{1-g}{1+r}\right)^L}{r+g} + \frac{t}{L} \frac{1-(1+r)^{-L}}{r}$$

For proof, see p. 2

Problem: define a declining annuity whose $PV = 1$.

Method: find annuity beginning with 1 whose $PV = X$; then divide by X .

$$\begin{aligned}
 PV = X &= \frac{1}{1+i} + \frac{1(1-g)}{(1+i)^2} + \dots + \frac{(1-g)^{n-1}}{(1+i)^n} \\
 &= \frac{1}{(1+i)} \frac{1 - \left(\frac{1-g}{1+i}\right)^n}{1 - \frac{1-g}{1+i}} = \frac{1 - \left(\frac{1-g}{1+i}\right)^n}{i+g}
 \end{aligned}$$

$$1 = \frac{i+g}{1 - \left(\frac{1-g}{1+i}\right)^n} \cdot \frac{1 - \left(\frac{1-g}{1+i}\right)^n}{i+g}$$

Problems & approaches

1. Advancing w in point case helps short investor more than long.
[Bias to brevity? No, it offsets the bias to longevity that comes from deferring tax.

Short run bias to longevity remains, except write-off.] This is partly just b.c. short investor needs more help — long investor is already tax exempt.

Reconcile w. fact that neutral tax calls for slower write-off than s.l.; & in point case, calls for $w = m$.

It is because there are two sources of bias: deferred tax; and advanced write-off. Both lower the effective t , i.e. Δr . Deferred tax also has bias to longevity, b.c. it makes the payment date recede with L .

Advance of w — has it also a bias to longevity?

In the ^{neutral} timber case, the gr. of write-off never comes up, b.c. there is just no tax on sale at m . How did I then bring it into

TISTOC? Re taxes on severance or realized income.

- a. — 1st approach: if I have a neutral tax on tree, then allow early w , what bias appears? This case sort of resembles accel. deprecn.

- b. — 2d approach: con case,

$$1 = i \frac{1 - (1+r)^{-L}}{r} (1-t) + \frac{1(1-t)}{(1+r)^L} + \frac{1 \times t}{(1+r)^w}$$

If $w = L$, then

$$1 - (1+r)^{-L} = \frac{i}{r} (1-t) [1 - (1+r)^{-L}] \quad \text{and} \quad r = i(1-t)$$

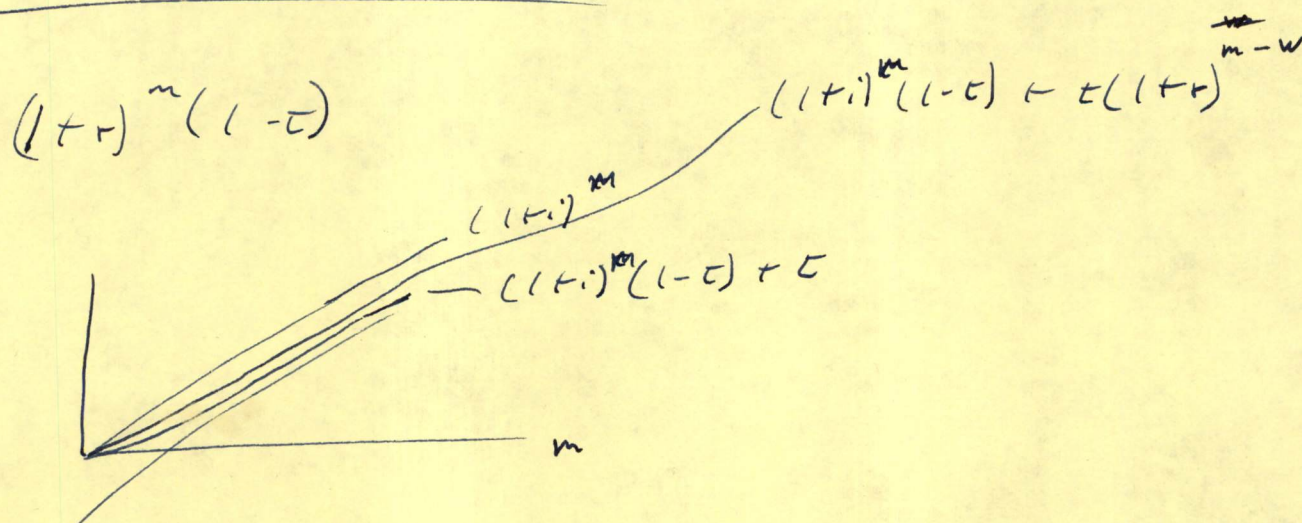
But if $w < L$, then

$$r = i(1-t) \frac{1 - (1+r)^{-L}}{1 - \frac{1-t}{(1+r)^L} - \frac{t}{(1+r)^{L-(L-w)}}$$

a. $w = \frac{m}{k}$

i) If $k = \infty$, $w = 0$, no bias, ~~no~~ but no tax. So if we assumed same tax ~~levy~~ levy, τ wd. have $\tau = \infty$ also, and the lack of bias is dubious. So the qn. must be rephrased. as w gets smaller, what happens to bias if we $\Delta \tau$ to keep same levy.

$$1 = \frac{i}{1 - (1+i)^{-n}} (1-\tau) \frac{1 - (1+i)^{-n}}{i}$$



What happens, as w gets smaller, to the bias?

Fast write-off of deprec. assets is like writ'g off point asset before maturity. It is like $w = \frac{m}{k}$; (or $w = k$, for assets in specified classes.)

To begin analysis, answer 2 qns.

a. If $w = \frac{m}{k}$, how is bias affected when k rises, so w falls

b. If $w = k$, " " " " " " k falls, so w falls.

A new approach to TISOC can tie everything important together

a. ^{inc.} Taxes capd. into lower l.v., partly.

b. Opt. timing is one that max's l.v., at given $\bar{i}$

c. Every asset yields cash flow plus salvage value (mostly land.)

d. Life ~~ends~~ ^{ends} when $S.i >$ cash flow. This solves problem in models of when life ends, if annuity falls at const. r_0 , e.g.

e. Mgmt. land becomes submgmt; all land used less intensively

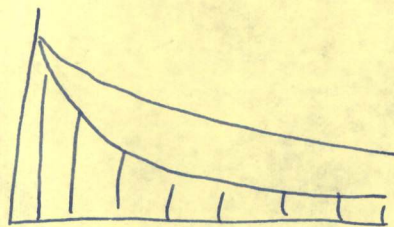
f. Make 2 models, for open & closed economy

~~But~~ Or even 3 -- the 3d is the U.S., closed to labor but open to capital.

But labor & capt. have elastic internal supply, even in closed economy, bcs. of unemployment, leisure, dissaving, slower turnover, etc. So, the open economy model is best. That means $\bar{i} = i_B$ (the Base i , before taxes) so taxes show up in lower l.v. Thus, the TISOC approach is only a foot in the door -- to reach people who would be repelled by the land value max. approach.

So, beginning next article, transit from ~~the~~ max. i approach to max. S .

$$1 = \frac{i + g}{1 - \left(\frac{1-g}{1+i}\right)^L}$$



$$S = \frac{1}{1+i} + \frac{1-g}{(1+i)^2} + \frac{(1-g)^2}{(1+i)^3} + \dots + \frac{(1-g)^{n-1}}{(1+i)^n}$$

$$= (1+i)^{-1} \frac{1 - \left(\frac{1-g}{1+i}\right)^L}{1 - \frac{1-g}{1+i}} = \frac{1 - \left(\frac{1-g}{1+i}\right)^L}{i+g}$$

$$1 = \frac{S}{S} = \left[\frac{1}{1+i} + \frac{1-g}{(1+i)^2} + \dots + \frac{(1-g)^{n-1}}{(1+i)^n} \right] \frac{i+g}{1 - \left(\frac{1-g}{1+i}\right)^L}$$

~~As~~ To make my declining inv. have a PV = 1, I let the first term be $\div$ by S ; and every term be divided by S . I still have a declining ~~annuity~~ cash flow.

~~Now every term is reduced by $\times (1-t)$~~

The current cash flow is:

$$CF = [1 + (1-g) + (1-g)^2 + \dots + (1-g)^{n-1}] \frac{i+g}{1 - \left(\frac{1-g}{1+i}\right)^L}$$

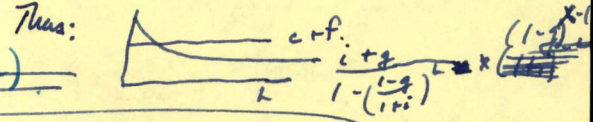
I reduce this to a P.V. by discounting each term by $(1+i)^n$.

$$PV = \left[\frac{1}{1+i} + \frac{1-g}{(1+i)^2} + \dots + \frac{(1-g)^{n-1}}{(1+i)^n} \right] \frac{i+g}{1 - \left(\frac{1-g}{1+i}\right)^n}$$

$$= \frac{1 - \left(\frac{1-g}{1+i}\right)^n}{i+g} \times \frac{i+g}{1 - \left(\frac{1-g}{1+i}\right)^n} = 1$$

The CF begins from a higher start than if constant, to compensate for the fall.

~~Now, I reduce the CF by $\times(1-t)$.~~



This CF yields a return $= i$, for whatever the Life. In practice equilibrium, the inv. of \$1 for different Lives will yield these C.F.s.

Now, I reduce the CF by taking a fixed %, t , from each periodic payment.

The investment of \$1.00 still yields the same gross C.F. But it corresponds now to lower r .

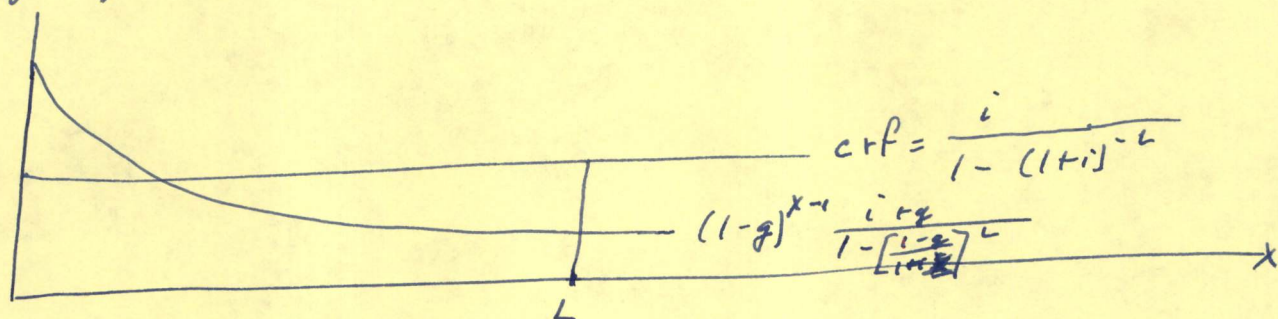
To find the t , we ~~discount~~ find that t which makes the reduced CF have $PV = 1$. Naturally it will be lower than i .

Also, the more weighted is the CF. toward the future, the higher we expect r to be

In const. case, $CF = \frac{i}{1 - (1+i)^{-L}}$

In declining case, $CF = \frac{i+g}{1 - \left(\frac{1+g}{1+i}\right)^L} (1+g)^{x-1}$

where x is the given year, and L is life. Maybe our program was wrong for failing to distinguish x and L .



So declining case starts out higher. But is it, when $x=1$

$$(1+g)^0 \frac{i+g}{1 - \left(\frac{1+g}{1+i}\right)^L} \stackrel{?}{>} \frac{i}{1 - \left(\frac{1}{1+i}\right)^L}$$

When L is large, yes.

When $L=1$, $\frac{i+g}{1+i - (1+g)} = 1+g = \frac{i}{1+i-1} = \frac{i}{1+i}$

$$(1-g)(1-g) = 1-g-g+g^2$$

1p.4

When $L = 2$

$$\frac{i+g}{(1+i)^2 - (1-g)^2} = \frac{(i+g)(1+i)^2}{1+2i+i^2 - [1-2g+g^2]} = \frac{(i+g)(1+i)^2}{2(i+g)+i^2-g^2}$$

$$\frac{i}{1-(1+i)^{-2}} = \frac{i(1+i)^2}{(1+i)^2 - 1} = i$$

Try numbers. $i = .05$
 $g = .03$

$$\frac{1}{1.05} = .9524$$

$$\left(\frac{1}{1.05}\right)^2 = .9070$$

$$1-g = .97$$

$$\frac{i}{1-(1+i)^{-2}} = \frac{.05}{1-}$$

$$.97^2 = .943$$

$$\frac{1-g}{1+i} = \frac{.97}{1.05} = .92$$

$$.92^2 = .857$$

~~Probably~~ the summation formula doesn't work when $r > 1$, as Don has been suspecting? Because num. becomes negative. But so does denom., so it seems OK.

$$S = 1 + r + r^2 + \dots + r^{n-1}$$

$$rS = r + r^2 + \dots + r^{n-1} + r^n$$

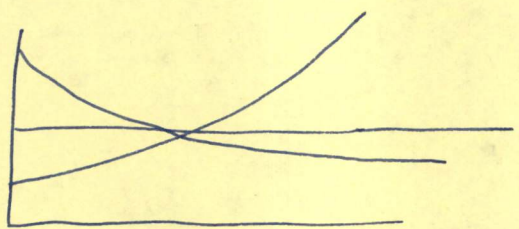
$$S - rS = 1 - r^n$$

$$S = \frac{1 - r^n}{1 - r} = \frac{r^n - 1}{r - 1} > 1 \quad \text{when } r > 1$$

When $r=1$, $S = \frac{0}{0}$, but that shouldn't be a general problem. $S=n$, very simple.

(In my appraisig land cases, I always had $r < 1$)

How about selection of the reciprocal, $\frac{1}{S}$, when $r > 1$. Is that affected? The CF pattern of given PV will begin lower



I see no reason why same reasoning won't hold.

We shouldn't expect any of these annuity cases to give us such striking results as the point case, because they contract less.

So, settle for small differences in r , as shown by computer.

But do new models for everything, based on max. PV of land, and w. salvage value.

Check by hand some of the computer results, too.

Side qn. Show that $c + f = \frac{i}{1 - (1+i)^{-L}}$ & i

$$\text{Num: } \frac{di}{di} = 1$$

$$\text{Denom: } \frac{d}{di} = -1 - L(1+i)^{-L-1} = L \frac{1}{(1+i)^{L+1}}$$

~~$\frac{1}{(1+i)^{L+1}}$~~ To show that ratio $d i$, must show that
% Δ num. > % Δ denom.

$$\text{i.e., } \frac{1}{i} > \frac{L(1+i)^{-L-1}}{1 - (1+i)^{-L}} = \frac{L}{(1+i)^{L+1} - (1+i)^0}$$

$$(1+i)^{L+1} \cancel{-1} - i > Li$$

$$(1+i)^{L+1} > 1 + (L+1)i$$

Yes

$$L = 2$$

$$1 + 3i + 3i^2 + i^3 - 1 - i > 2i$$

$$i + 3i^2 + i^3 > 0$$

It follows that when we ~~add g~~ subtract g,

$$\text{that } \frac{i+g}{1 - \left(\frac{1-g}{1+i}\right)^L} > \frac{i}{1 - (1+i)^{-L}}$$

And conversely when we add g.

Summing up:

So we find with annuities, ~~the~~ only moderate differences in r:

a. For different lives

b. For different patterns

c. There is no doubt a big difference between declining annuities of short life, & rising ones of long life.

The big ~~contrast gains~~ tax avoidance maneuver is to defer tax altogether, by first write-off followed by sale at cap. gain.

Next, try tax life arbitrary; or, shorter than ec. life; etc.

Ricks

$$PVS = \frac{\sum_{n=0}^L \left[(R_n - C_n) (1+r)^{-n} \right] (1-t) + tPVW}{1 - (1+r)^{-L}}$$

9me. flow patterns

1. Steady flow, a

$$PV = a \frac{1 - (1+r)^{-n}}{i} (1-t) + tPVW$$

(a) Solving for r , Cost = 1, $a = \frac{i}{1 - (1+i)^{-n}}$

$$1 = \frac{1 - (1+r)^{-n}}{r} \frac{i}{1 - (1+i)^{-n}} (1-t) + tPVW$$

$$r = i \frac{1 - (1+r)^{-n}}{1 - (1+i)^{-n}} (1-t) + t \cdot PVW$$

(b) Solving for PV. Assume $i=r$, tax borne by fixed input whose value you are ~~maxing~~ to max. at $n=0$. This doesn't work out smoothly, but in maxing r , above, there was no rate factor used

$$PV = \frac{1 - (1+i)^{-n}}{i} \frac{i}{1 - (1+i)^{-n}} (1-t) + \frac{tPVW - C_0}{1 - (1+i)^{-n}}$$

$$= \frac{1}{i} - t(1 - PVW)$$

$$PVS = \frac{1}{i}$$

Point Case

1. Max r

$$(1+r)^n = (1+i)^n (1-t) + t \cancel{PWR} (1+r)^{n-w}$$

or, on PV basis

$$\frac{(1+r)^n}{(1+r)^n} = \frac{(1+i)^n}{(1+r)^n} (1-t) + t \frac{(1+r)^{n-w}}{(1+r)^n}$$

i.e. $1 = (1+i)^n (1-\tau) (1+r)^{-n} + \tau (1+r)^{-w}$

Note that method (α) is the same as (β) , but the original expository method was modified to a PV basis.

2. Max. PV

Here it must be assumed there is an input at time = 0 which is residual.

- a. Try one-gen. case, all C_0 is resid.
- b. One-gen, some C_0 resid., some fixed, (Mining)
- c. Multi-gen., " " " " This

+ the general case.

This is

1. Without land

$$(1+r)^m = (1+i)^m (1-t) + \cancel{PVA} t (1+r)^{m-w}$$

2. Add land, S .

$$(C+S)(1+r)^m - S = [(C+S)(1+i)^m - S](1-t) + t \cdot C (1+r)^{m-w}$$

One comparison of (2) and (1) is easy to make: let $w=0$.

In (1), this means $r=i$.

In (2):

$$(C+S)(1+r)^m - S - t C (1+r)^m = [(C+S)(1+i)^m - S](1-t)$$

$$C(1+r)^m(1-t) + S[(1+r)^m - 1] = C(1+i)^m(1-t) + S[(1+i)^m - 1](1-t)$$

$$r < i$$

So expensing ceases to afford tax exemption. Land income is still being taxed.

But almost certainly, a sensitivity analysis would show that

$$\frac{dr}{dC} = 0, \text{ indicating there is no tax on } C \text{ as such.}$$

2) Transit to study of bldgs. & land rent.

Bldg. tax is neutral, ^{as to longevity of bldg.} being on V. Lowers all returns,
+ has compensates for appearing to tax time.

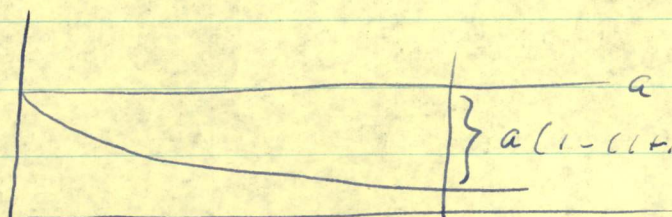
But defers site renewal. — an apparent contradiction, but
probably reconcilable -- e.g. by saying last x yrs. of "life" are a
"life after death" of empty shell of bldgs.

Land yields income under old bldgs. in life-after-death.

✓ Jobs for MC, in Mre. study.

My method of deprec. also accords w. princ. that inc. = $i \times V$, & residual is deprec.

It makes the inc. tax a property tax



$$a(1 - (1+i)^{-n}) = V_i, \text{ or } V = a \frac{1 - (1+i)^{-n}}{i}$$

So rule is simplicity itself. In any year, income is V_i .

Deprec. is $a - V_i = \frac{a}{(1+i)^{m-n}}$

(After V_0 is fully recovered, any further income is land rent, or a windfall. That shd. be determined by indep. appraisal of land, and not by assuming any excess is land rent, else you discourage risk, etc, and also you imply as corollary that any deficit stems from lower land rent & your reward error)

The bldg. tax is on V , and is therefore like the above proposed income tax

Now add one year to asset life. ΔV is $\frac{a}{(1+i)^{m+1}}$, and that should justify ΔC of same amount.

Now this resembles Wickseil's staggered case where m years interest on ΔC = one year's interest on m units of ΔC which have grown at comp. int. over $m-n$ years.

In this case I am increasing life by one year, & I am asking

what level of a justifies r ? Alternatively, $a = \Delta C (1+i)^{m+1}$
 or $a = \Delta C (1+i) + i \cdot V_0 (1+i)^{-1}$, where yr. 0 is the
 year initially 0. These are probably equal:

$$a = \Delta C + i(\Delta C + V_0 (1+i)^{-1}) \cong \Delta C + i V_0$$

In absence of taxes, unqual. longevity ~~is~~ (ML) is ec. if ~~at~~
 $a \geq \Delta C + Vi$.

If tax base is $Vi (= a(1-(1+i)^{-m}) = a - \Delta C)$ then there is no
 special advantage -- what is unqual. remains so, at $r = i(1-t)$

If tax base is ~~even~~ smaller, there is a gain.

Another way of putting it is, if taxpayer gains in year one, he can
 gain more by adding a year in front of it. He gets int. on $V_1 + \Delta C(1+i)$
 and if tax $< t \cdot Vi$ he gets a year's int. on V w/o paying a tax.

I have 2 alt. ways of conceiving the m -yr. asset.

a. Annulose. Then a in yr. 1 is 95% recovery, but
 I then add to the 5% of a which is income the 5% of V
 which is accrued, so most of a is income

b. View as a whole, then deprec. is $-\Delta V = \frac{a}{(1+i)^m}$.

Either way, most of a in yr. one is inc.

✓ It needs to be formalized; but I am sure.

2 tasks remain: 1) Order of exposition
 2) Transmit to study of bldgs. & l.v.

• Fallacy in App. B, Ricks

Effect of the Holding Period on the Rate
the Rate of Return from Depreciation
(without recapture)

Table XVIII showed that, in general, with accelerated depreciation the highest interval rate of return from depreciation is for a one year holding period. However in ~~the~~ certain cases under the sum of the years digits method the maximum rate of return was for a holding period longer than one year but shorter than the useful life of the asset for tax purposes. ~~Under~~ Under the straight line method the maximum rate of return from depreciation occurs when the holding period equals the useful life. This appendix analyzes mathematically the relationship between ~~and~~ holding period, ~~and~~ useful life, depreciation method and rate of return.

Buy 10 get 10 - 6½
Sell 10
Dep 6

The general statement is that it pays (in terms of increasing the rate of return over the holding period) to hold the asset through year $n+1$ if:

$$\sum_{j=1}^{n+1} \frac{tD_j}{(1+r)^j} + \frac{P_{n+1}}{(1+r)^{n+1}} > \sum_{j=1}^n \frac{tD_j}{(1+r)^j} + \frac{P_n}{(1+r)^n} \quad (1)$$

$$\text{or} \quad \sum_{j=1}^n \frac{tD_j}{(1+r)^j} + \frac{tD_{n+1}}{(1+r)^{n+1}} + \frac{P_{n+1}}{(1+r)^{n+1}} > \sum_{j=1}^n \frac{tD_j}{(1+r)^j} + \frac{P_n}{(1+r)^n} \quad (2)$$

* where t = the investor marginal tax bracket; D_j is the depreciation deduction in any given year, j ; P_n = the resale proceeds at the end of year n (see Appendix A)

(2)

since $\sum_{j=1}^n \frac{\frac{t}{2} D_j}{(1+r)^j}$ appears on both sides (2) becomes

$$\frac{\frac{t}{2} D_{n+1}}{(1+r)^{n+1}} + \frac{P_{n+1}}{(1+r)^{n+1}} \stackrel{?}{=} \frac{P_n}{(1+r)^n} \quad (3)$$

in other words it pays to hold another year ^(through $n+1$) if the additional year's tax saving from depreciation plus ~~that~~ ^{the value} proceeds if ~~it is~~ the property is sold a year later exceed the value of the proceeds if sold a year earlier (at n)

However, P_{n+1} and P_n are related in a ^{very} specific way if depreciation is the only parameter (other than holding period) affecting the rate of return. Since in the absence of recapture the reduction in cost basis is taxed at capital gains rates (presumed herein to be $\frac{t}{2}$) on resale and since the cost basis is reduced in year n by D_n then,

$$P_{n+1} = P_n - \frac{t}{2} D_{n+1} \quad (4)$$

and (3) becomes:

$$\frac{\frac{t}{2} D_{n+1} + \left(P_n - \frac{t}{2} D_{n+1} \right)}{(1+r)^{n+1}} \stackrel{?}{=} \frac{P_n}{(1+r)^n} \quad (5)$$

We can simplify (5) to:

$$\frac{\frac{I}{2} D_{n+1} + P_n}{(1+r)^{n+1}} > \frac{P_n}{(1+r)^n} \quad (6)$$

and since if (6) is true at the beginning of the holding period (year 0) it is also true at year n , (6) can be simplified to

$$\frac{\frac{I}{2} D_{n+1} + P_n}{(1+r)} > P_n \quad (7)$$

Equation (7) says, is the ^{rate of} return produced by holding the property another year as great as the internal rate of return if the asset is sold at the end of year n ; is the value at n of $\frac{I}{2} D_{n+1} + P_n$ greater than P_n .

Another way of expressing this is: Is there an increment to the rate of return from a delay in the sale? Is:

$$\frac{\frac{I}{2} D_{n+1} + P_n}{P_n} > 1 + r_n \quad (8)$$

or, since $\frac{\frac{I}{2} D_{n+1} + P_n}{P_n} = \frac{\frac{I}{2} D_{n+1}}{P_n} + \frac{P_n}{P_n} = 1 + \frac{\frac{I}{2} D_{n+1}}{P_n}$

(8) becomes

$$1 + \frac{\frac{t}{2} D_{n+1}}{P_n} \stackrel{?}{>} 1 + r_n \quad (9)$$

Thus, the question becomes is:

$$\frac{\frac{t}{2} D_{n+1}}{P_n} \stackrel{?}{>} r_n \quad (10)$$

where r_n is the internal rate of return from holding the asset through year n ; does an additional year holding period increase the rate of return?

Now, let us turn to an investigation of the relationship between P_{n+1} and P_n and between

D_{n+1} and D_n by considering the choice between a one year and a two year holding period.

Referring back to equation (1) or (2) we see that:

the question of whether a two year holding period produces a greater rate of return than a one year holding period

$$\frac{\frac{t}{2} D_1}{(1+r)^1} + \frac{\frac{t}{2} D_2}{(1+r)^2} + \frac{P_2}{(1+r)^2} \stackrel{?}{>} \frac{\frac{t}{2} D_1}{(1+r)^1} + \frac{P_1}{(1+r)^1} \quad (11)$$

and from (3) we see that (11) becomes;

$$\frac{\frac{t}{2} D_1}{(1+r)^1} + \frac{\frac{t}{2} D_2}{(1+r)^2} + \frac{P_1 - \frac{t}{2} D_2}{(1+r)^2} \geq \frac{\frac{t}{2} D_1}{(1+r)^1} + \frac{P_1}{(1+r)^1} \quad (12)$$

By cancelling the common term $\frac{\frac{t}{2} D_1}{(1+r)^1}$ and multiplying through by $(1+r)^1$ to simply shift the reference date of comparison to the end of year one rather than at the beginning, equation (12) becomes:

$$\frac{\frac{t}{2} D_2}{(1+r)^1} + \frac{P_1 - \frac{t}{2} D_2}{(1+r)^1} \geq P_1 \quad (13)$$

which by collecting terms becomes;

$$\frac{\frac{t}{2} D_2 + P_1}{(1+r)^1} \geq P_1 \quad (14)$$

So, by the previously used technique (14) can be rearranged to ask the question is:

$$\frac{\frac{t}{2} D_2 + P_1}{P_1} \geq r_1 \quad (15)$$

And since r_1 is the rate of return produced by a one-year holding period it can be looked

upon as the rate of return which established the following equality:

$$V_0 = \frac{\frac{1}{2} D_1}{(1+r)'} + \frac{P_1}{(1+r)'} \quad (16)$$

where V_0 is the investment at the beginning of the year. (V_0 might also be looked upon as the proceeds from some previous sale reinvested for a year or P_0).

Therefore:

$$(1+r_1)' = \frac{\frac{1}{2} D_1}{V_0} + \frac{P_1}{V_0} \quad (17)$$

$$(1+r_1)' = \frac{\frac{1}{2} D_1 + V_0 - \frac{1}{2} D_1}{V_0} \quad (18)$$

$$(1+r_1)' = \frac{\frac{1}{2} D_1 + V_0}{V_0} \quad (19)$$

$$r_1 = \frac{\frac{1}{2} D_1}{V_0} \quad (20)$$

Therefore, the question of whether the rate of return over the holding period is increased by a two year holding period becomes (from (20) and (15)) is:

$$\frac{\frac{1}{2} D_2}{P_1} > \frac{\frac{1}{2} D_1}{V_0} \quad (21)$$

And, the fraction $\frac{\frac{t}{2} D_2}{P_1}$ is larger if:

generalization: hold if + only if

$$\frac{D_2}{D_1} > \frac{P_1}{V_0}$$

$$\frac{D_{N+1}}{D_N} > \frac{P_N}{P_{N-1}}$$

(22)

In other words is the ratio of the second year's depreciation ^{that of} to the first year greater than the ratio of the ~~resale~~ resale proceeds forgone ~~by~~ in the first year by holding another year to the investment at the beginning of the first year? Or is the rate of decrease in depreciation less than the rate of decrease in proceeds? The latter can be expressed is:

$$\frac{D_1 - D_2}{D_1} ? < \frac{V_0 - P_1}{V_0}$$

(23)

The relationship of P_1 to V_0 has been defined in (4) as:

$$P_1 = V_0 - \frac{t}{2} D_1 \quad (24)$$

as long as the tax rate and depreciation are positive (and other factors are zero) So, (23) ($P_1 < V_0$ and (22)) can be expressed as:

$$\frac{D_2}{D_1} ? > \frac{V_0 - \frac{t}{2} D_1}{V_0}$$

(25)

Now we are ready to look mathematically at the alternative depreciation methods (as defined in Appendix A).

The straight line method provides equal annual deductions during the useful life for tax purposes.

$\leftarrow$ $D_2 = D_1$ and $D_{n+1} = D_n$ and $\frac{D_2}{D_1} = 1$

while $\frac{V_0 - \frac{1}{2}D_1}{V_0} < 1$, so it pays to hold through year

two. In fact $D_{n+1} = D_n$ while $\frac{P_n - \frac{1}{2}D_{n+1}}{P_{n-1}} < 1$

so it pays (in the form of higher internal rate of return just considering depreciation) to hold until the end of the useful life for tax purposes, as was shown in Table XVIII

The double declining balance method provides decreasing annual depreciation deductions. The formula is $D_n = \frac{2}{U} (d(V_0 - \sum_{i=1}^{n-1} D_i))$ where d = the portion of original investment which is depreciable and U = useful life. (See Appendix A). Since d is a constant let us assume it is 1 and we can thus ignore it.

$$D_1 = \frac{2}{U} V_0 \quad (26)$$

$$D_2 = \frac{2}{U} (V_0 - D_1) \quad (27)$$

$$D_2 = \frac{2}{U} (V_0 - \frac{2}{U} V_0) \quad (28)$$

$$D_2 = \frac{2}{v} V_0 \left(1 - \frac{2}{v}\right)$$

(29)

$$\frac{D_2}{D_1} = \frac{\frac{2}{v} V_0 \left(1 - \frac{2}{v}\right)}{\frac{2}{v} V_0}$$

(30)

$$\frac{D_2}{D_1} = \frac{1 - \frac{2}{v}}{1}$$

(31)

For example, if the useful life is 20 years $\frac{1}{v}$ (the straight line rate) is $\frac{1}{20} = .05$ and $\frac{2}{v} = .10$.

$$\frac{D_2}{D_1} = \frac{1 - .10}{1} = .9$$

and, since the rate $\frac{2}{v}$ is applied to a declining balance

$$D_n = \frac{2}{v} \left(V_0 - \sum_{n=1}^{n-1} D_n \right)$$

$$D_n = \frac{2}{v} V_0 - \frac{2}{v} \sum_{n=1}^{n-1} D_n$$

each year's depreciation is $\frac{2}{v}$ less than that of the year before and

$$\frac{D_{n+1}}{D_n} = \frac{1 - \frac{2}{v}}{1}$$

(32)

Thus, while the absolute dollar amount of the depreciation decreases each year, the ratio of each year's deduction to that of the previous year is constant at $\frac{1 - \frac{2}{v}}{1}$.

therefore, if $\frac{D_2}{D_1} < \frac{V_0 - \frac{t}{2} D_1}{V_0}$ then

$$\frac{D_{n+1}}{D_n} < \frac{P_n - \frac{t}{2} D_{n+1}}{P_n} \text{ since } P_n > D_n > D_{n+1} > \frac{t}{2} D_{n+1}.$$

If we substitute $dV_0 \frac{2}{v}$ for D_1 in (25) we have:

$$\frac{D_2}{D_1} > \frac{V_0 - dV_0 \frac{2}{v} \frac{t}{2}}{V_0} \quad (33)$$

$$\frac{1 - \frac{2}{v}}{1} > \frac{V_0 (1 - d \frac{2}{v} \frac{t}{2})}{V_0} \quad (34)$$

$$\frac{1 - \frac{2}{v}}{1} > \frac{1 - d \frac{2}{v} \frac{t}{2}}{1}$$

and since d has a maximum value of 1, ^{and $\frac{t}{2} \leq 1$} then
it is true that

$$\frac{1 - \frac{2}{v}}{1} < \frac{1 - d \frac{2}{v} \frac{t}{2}}{1} \quad (35)$$

Therefore the depreciation declines at a faster rate than resale proceeds and for the double declining balance method (21) is answered that the rate of return is higher for a one year holding period than any period longer than one year. Table XVIII is confirmed for the ^{double} declining balance method and

~~we can~~ equation (35) shows that if the ~~2~~ $\frac{2}{U}$ is replaced by $\frac{1.5}{U}$ for the 150% declining balance method the rate of return from depreciation is still maximized from a one year hold.

The sum-of-the-years-digits method is not as clear cut. The ratio $\frac{D_{n+1}}{D_n}$ is constant for the other two methods but not for the SYD method.

$$D_n = \frac{U - (n-1)}{\frac{1}{2} U (U+1)} \quad (36)$$

where the numerator is the years of useful life remaining in ~~beginning~~ in the n^{th} year and the denominator is the sum of the digits from one to U .

$$\frac{D_{n+1}}{D_n} = \frac{\frac{U-n}{\frac{1}{2} U (U+1)}}{\frac{U-(n-1)}{\frac{1}{2} U (U+1)}} \quad (37)$$

$$\frac{D_{n+1}}{D_n} = \frac{U-n}{U-(n-1)} \quad (38)$$

which decreases over n and approaches $\frac{1}{2}$ as n approaches $U-1$.

$$C = \frac{z D_1}{(1+r)^5} + C - \frac{z D_2}{(1+r)^n}$$

$$r = \frac{z}{l} (1-z) + (r \cdot z) (1 - \frac{z}{l} \cdot \frac{n}{2})$$

where $z = (1+r)^{-n}$

$$\frac{dz}{dn} = (-z) \left(\frac{n}{1+r} \frac{dr}{dn} + \ln(1+r) \right)$$

$$\frac{dr}{dn} = -\frac{z}{l} \cdot \frac{dz}{dn} - (r \cdot z) \cdot \frac{z}{2l} + (1 - \frac{z}{l} \cdot \frac{n}{2}) (r \frac{dz}{dn} + z \frac{dr}{dn})$$

$$\frac{dr}{dn} \left(1 - z + \frac{z \cdot \frac{n}{2}}{2l} \right) = -\frac{z}{l} \frac{dz}{dn} - \frac{r \cdot z \cdot z}{2l} + (1 - \frac{z}{l} \cdot \frac{n}{2}) (r) \left(\frac{dz}{dn} \right)$$

$$\frac{dr}{dn} \left(1 - z + \frac{z \cdot \frac{n}{2}}{2l} \right) = \left(-\frac{r \cdot z \cdot z}{2l} \right) + \left(\frac{dz}{dn} \right) \left[r - \frac{r \cdot \frac{n}{2}}{2l} - \frac{z}{l} \right]$$

$$\frac{dr}{dn} \left[1 - z + \frac{z \cdot \frac{n}{2}}{2l} + \left(\frac{z \cdot n}{1+r} \right) \left(r - \frac{r \cdot \frac{n}{2}}{2l} - \frac{z}{l} \right) \right] = -\frac{r \cdot z \cdot z}{2l} - \left(r - \frac{r \cdot \frac{n}{2}}{2l} - \frac{z}{l} \right) (z \ln(1+r))$$

$$\frac{dr}{dn} = (-z) \left[\frac{r \cdot z}{2l} + \left(\ln(1+r) \right) \left(r - \frac{r \cdot \frac{n}{2}}{2l} - \frac{z}{l} \right) \right]$$

$$z \left[-\frac{1}{2} + 1 - \frac{z \cdot n}{2l} - \left(\frac{n}{1+r} \right) \left(r - \frac{r \cdot \frac{n}{2}}{2l} - \frac{z}{l} \right) \right]$$

If U , the useful life, is very long $\frac{D_{n+1}}{D_n} = \frac{U-n}{U-(n-1)}$ is close to unity in the first few years (where n is small).

Therefore the question of whether

$$\frac{D_2}{D_1} > \frac{V_0 - \frac{1}{2} D_1}{V_0}$$

depends on the value for U . This explains why when the portion depreciable was high and the useful life ~~was short~~ we saw the rate of return in Table VIII (with positive values for the other parameters) was maximized at a holding period between one year and the end of the useful life.¹⁴ Looking at Table XVIII where depreciation is examined alone we see that with $d = 100\%$ and $U = 20$ the ratio $\frac{r_2}{r_1} = \frac{2.54}{2.65} = .958$ while for an increased U of 50 the ratio $\frac{r_2}{r_1}$ is only $\frac{1.09}{1.09} = .991$. Not shown in Table XVIII is that the ratio of $\frac{r_2}{r_1} =$ when $U = 50$

14. The impact in Table VIII of useful life is dual. On the one hand as explained above, the shorter U , the smaller the ratio $\frac{D_2}{D_1}$ and the shorter the optimum holding period. On the other hand, the shorter U the more important is depreciation relative to the other parameters.

$$y = (1+y)^x$$

$$\frac{dy}{dx} = (u^x)' =$$

$$y = u(x)$$

$$\frac{dy}{dx} = \frac{x}{u} \frac{du}{dx} + \ln u \frac{du}{dx}$$

$$\frac{dy}{dx} = \left(\frac{x}{1+y} \right) \frac{dy}{dx} + \ln(1+y)$$

$$y = u^v$$

$$\ln \frac{y}{x} = v \ln u$$

$$\frac{1}{y} y' = \frac{v}{u} \frac{du}{dx} + \ln u \frac{dv}{dx}$$

Conclusion

Looking only at the depreciation parameter without recapture and without transfer costs, the holding period which optimizes the rate of return from the tax savings generated by depreciation is the following:

For straight line: $n = V$ - hold until the end of the useful life

For declining balance: $n = 1$ - hold for one year

For sum of years digits: $n = f(V)$ - it depends on the useful life

$$y = f(x, y)$$

find dr/dn

$$r = [f(n)]^x = u$$

$$\partial r / \partial n = dr / du$$

$$y = (x^2 + 1) = u^v$$

$$\frac{\partial y}{\partial x} =$$

$$y = (1+y)(x) =$$

$$y = u \cdot x$$

$$\frac{\partial y}{\partial x} = u \frac{\partial x}{\partial x} + x \frac{\partial u}{\partial x} = (1+y) \cdot 1 + x \frac{\partial y}{\partial x} = \frac{\partial y}{\partial x}$$

$$\frac{1+y}{1-x} = \frac{\partial y}{\partial x}$$

$$y = x + xy$$

$$y(1-x) = x$$

$$y = \frac{x}{1-x}$$

$$\frac{d}{dx} \frac{x}{1-x} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\frac{\partial y}{\partial x} = \frac{(1-x)(1 - (x)(-1))}{(1-x)^2} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\left(\frac{2 \cdot r}{L} + \frac{2 \cdot r}{L} \cdot r + \left(\frac{2 \cdot r}{L} \right) \cdot \left(- \frac{\ln(1+r)}{(1+r)^n} \right) \right)$$

$$P = \frac{2}{r \cdot L} + \frac{2 \cdot \ln(1+r)}{(1+r)^n}$$

$$L = \frac{1}{P} + \frac{1}{r} \cdot n - 2 \frac{\ln(1+r)}{(1+r)^n}$$

$$P \cdot r = \frac{2}{L} (1 + n \cdot r)$$

$$P \cdot r = \frac{2}{L} + \frac{2 \cdot n \cdot r}{L} + \left(\frac{2 \cdot r}{L} \right) \cdot \left(- \frac{\ln(1+r)}{(1+r)^n} \right)$$

$$P = \frac{2}{r \cdot L} + \frac{2 \cdot \ln(1+r)}{(1+r)^n}$$

$$L = \frac{1}{P} + \frac{1}{r} \cdot n - 2 \frac{\ln(1+r)}{(1+r)^n}$$

$$\frac{\partial z}{\partial n} = - (1+r)^{-n} \ln(1+r)$$

$$\frac{\partial z}{\partial n} = -n \frac{\partial}{\partial n} (1+r)^{-n} + \ln(1+r) \cdot (-1)$$

$$\ln z = -n \ln(1+r)$$

$$\text{where } r = f(n)$$

$$z = (1+r)^{-n}$$

Sinking fund deprec.

$$a = \frac{i}{(1+i)^n - 1}, \text{ so } a \frac{(1+i)^n - 1}{i} = 1$$

$B \equiv \text{Book Value}$

$$B_x = 1 - \left[\cancel{a} \frac{(1+i)^x - 1}{i} \right] = 1 - \frac{(1+i)^x - 1}{(1+i)^n - 1} = \frac{(1+i)^n - (1+i)^x}{(1+i)^n - 1} < 1$$

True deprec.

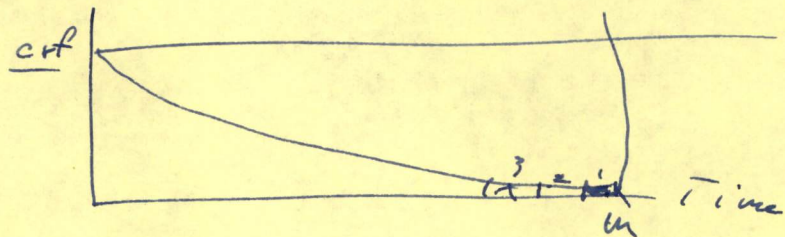
$$B = 1 - \sum_{t=1}^x (1+i)^{-(L-x)} = 1 - (1+i)^{-L} [(1+i)^1 + (1+i)^2 + \dots + (1+i)^x]$$

$$= 1 - (1+i)^{-L} \frac{1 - (1+i)^x}{1 - 1 - i} = 1 - \frac{(1+i)^x - 1}{i(1+i)^L} = \frac{i(1+i)^L - [(1+i)^x - 1]}{i(1+i)^L}$$

$$B = 1 - \frac{i}{1 - (1+i)^{-L}} \sum_{t=0}^{n-1} (1+i)^{-(n-x)} = 1 - \frac{i}{1 - (1+i)^{-L}} (1+i)^{-L} [1 + (1+i) + \dots + (1+i)^{x-1}]$$

$$= 1 - \frac{i}{(1+i)^L - 1} \frac{(1+i)^x - 1}{i}$$

$$\frac{1 - (1+i)^n}{1 - 1 - i}$$



$$1 = \cancel{\text{MIRR}} (1-t) \left[\frac{i}{r} \frac{1-(1+r)^{-L}}{r} + (1+r)^{-L} \right] + \frac{t}{(1+r)^{L-(L-w)}}$$

$$\frac{1-t}{(1+r)^L} + \frac{t}{(1+r)^{L-x}} \quad \text{where } x = L-w$$

$$= \frac{1-t}{(1+r)^L} + \frac{t(1+r)^{+x}}{(1+r)^L} = \frac{1-t(1-(1+r)^x)}{(1+r)^L}$$

$$r = i(1-t) \frac{1-(1+r)^{-L}}{1-(1+r)^{-L} [1-t(1-(1+r)^x)]}$$

When $x = 0$, $r = i(1-t)$

When $x > 0$, $r > i(1-t)$

Let $x = L - \frac{L}{k} = \frac{L}{k}$ (In general, $x = L - \frac{L}{k} = L(1 - \frac{1}{k})$)

Could there be a first-term error?

S of const. annuity, P.V.

$$S = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^L}$$
$$= \frac{1}{1+i} \frac{1 - (1+i)^{-L}}{1 - \frac{1}{1+i}} = \frac{1 - (1+i)^{-L}}{i}$$

$$S = a + ar + \dots + ar^{n-1} = a \frac{1-r^n}{1-r}$$

If $a=r$, then $S = r + r^2 + \dots + r^n$

$$rS = r^2 + \dots + r^n + r^{n+1}$$

$$S(1-r) = r - r^{n+1}$$

$$S = \frac{r - r^{n+1}}{1-r} = \frac{1 - r^n}{\frac{1-r}{r}}$$

In growth case, $S = \frac{1}{1+i} + \frac{1+g}{(1+i)^2} + \dots + \frac{(1+g)^{L-1}}{(1+i)^L}$

$$a = \frac{1}{1+i}$$

$$r = \frac{1+g}{1+i}$$

$$S = a \frac{1-r^n}{1-r} = \frac{1}{1+i} \frac{1 - \left(\frac{1+g}{1+i}\right)^L}{1 - \frac{1+g}{1+i}} = \frac{1 - \left(\frac{1+g}{1+i}\right)^L}{i+g}$$

Seems OK.

Case of growing finite income.

P.V. rises, then falls. So it is an appreciating asset before it is a depreciating asset.

It is not very realistic, to have inc. grow & suddenly stop.

$$\frac{1}{w - (1+1) - 1} (2-1) =$$

$$\frac{\frac{1+1}{1} - 1}{w \left(\frac{1+1}{1} \right) - 1} \cdot \frac{1+1}{(2-1)}$$